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Stochastically independent randomization and uncertainty aversion*

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Summary. This paper proposes a preference-based condition for stochastic independence of a randomizing device in a product state space. This condition is applied to investigate some classes of preferences that allow for both independent randomization and uncertainty or ambiguity aversion (a la Ellsberg). For example, when imposed on Choquet Expected Utility (CEU) preferences in a Savage framework displaying uncertainty aversion in the spirit of Schmeidler [27], it results in a collapse to Expected Utility (EU). This shows that CEU preferences that are uncertainty averse in the sense of Schmeidler should not be used in settings where independent randomization is to be allowed. In contrast, Maxmin EU with multiple priors preferences continue to allow for a very wide variety of uncertainty averse preferences when stochastic independence is imposed. Additionally, these points are used to reexamine some recent arguments against preference for randomization with uncertainty averse preferences. In particular, these arguments are shown to rely on preferences that do not treat randomization as a stochastically independent event.

Keywords and Phrases: Uncertainty aversion, Stochastic independence, Preference for randomization, Ambiguity aversion.

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1 Introduction

An example seminal to interest in uncertainty (or ambiguity) aversion is Ellsberg's [9] "two-color" problem. There is a "known urn" which contains 50 red balls and 50 black balls, and an "unknown urn" which contains a mix of red and black balls, totaling 100, about which no information is given. Ellsberg observed (as did many afterwards, more carefully) that a substantial fraction of individuals were indifferent between the colors in both urns, but preferred to bet on either color in the "known urn" rather than the corresponding color in the "unknown urn". This violates not only expected utility, but probabilistically sophisticated behavior more generally. One contemporary criticism of the displayed behavior was put forward by Raiffa [23] who pointed out that flipping a coin to decide which color to bet on in the unknown urn should be viewed as equivalent to betting on the "known" 50-50 urn. One can think of such preferences as displaying a preference for randomization.

Jumping ahead to more recent work, there is a burgeoning literature attempting to model uncertainty (or ambiguity) aversion in decision makers using representations with non-additive probabilities or sets of probabilities. Some of this work (e.g. Lo [20], Klibanoff [18]) accepts this preference for mixture or randomization as a facet of uncertainty aversion, while other work (e.g. Dow and Werlang [6], Eichberger and Kelsey [8]) does not. This has led to several papers, most directly Eichberger and Kelsey [7], but also Ghirardato [11] and Sarin and Wakker [24], related to this difference. In particular, all three papers observe that the choice of a "one-stage" or Savage model as opposed to a "two-stage" or Anscombe-Aumann model can lead to different preferences when modeling uncertainty aversion. In Eichberger and Kelsey [7] the authors set out to "show that while individuals with non-additive beliefs may display a strict preference for randomization in an Anscombe-Aumann framework they will not do so in a Savage-style decision theory."[1]

This paper was motivated in part by the intuition that the one-stage/two-stage modeling distinction is largely a red herring, at least as it relates to preference for randomization. In particular, while appreciating that there can be differences between the frameworks, one goal of this paper is to relate these differences to violations of stochastic independence and to point out that they have essentially no role to play in the debate over preference for randomization in uncertainty aversion. In making this point, the related finding of the restrictiveness of Choquet expected utility preferences in allowing for randomizing devices is key.

An additional contribution of the paper is to provide preference based conditions to describe a stochastically independent randomizing device in a non-Bayesian environment. Section 2 sets out some preliminaries and notation. Section 3 describes two frameworks in which a randomizing device can be modeled. Section 4 provides the key preference conditions and contains the main results on the restrictiveness of Choquet expected utility when stochastic independence

[1] [7, Abstract].

is required and the relative flexibility of Maxmin expected utility with multiple priors. Section 5 concludes.

2 Preliminaries and notation

We will consider two representations of preferences, each of which generalizes expected utility and allows for uncertainty aversion. The first model is Choquet Expected Utility (CEU). CEU was axiomatized first in an Anscombe-Aumann framework by Schmeidler [27], and then in a Savage framework by Gilboa [14] and Sarin and Wakker [24]. In a Savage framework, but assuming a rich set of consequences and a finite state space, Wakker [29], Nakamura [21], and Chew and Karni [5] have axiomatized CEU. The second model is Maxmin Expected Utility with non-unique prior (MMEU). MMEU was first axiomatized in an Anscombe-Aumann framework by Gilboa and Schmeidler [15]. In a Savage framework, but assuming a rich set of consequences and allowing a finite or infinite state space, MMEU has been axiomatized by Casadesus-Masanell, Klibanoff, and Ozdenoren [4].

Consider a finite set of *states of the world* S. Let X be a set of consequences. An *act* f is a function from S to X. Denote the set of acts by F. A function $v : 2^S \to [0,1]$ is a *capacity* or *non-additive probability* if it satisfies,

(i) $v(\emptyset) = 0$,
(ii) $v(S) = 1$, and
(iii) $A \subseteq B$ implies $v(A) \leq v(B)$.
It is *convex* if, in addition,
(iv) For all $A, B \subseteq S$, $v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$.

Now define the (finite) *Choquet integral* of a real-valued function a to be: $\int a dv = \alpha_1 v(E_1) + \Sigma_{i=2}^{n} \alpha_i [v(\bigcup_{j=1}^{i} E_j) - v(\bigcup_{j=1}^{i-1} E_j)]$, where α_i is the ith largest value that a takes on, and $E_i = a^{-1}(\alpha_i)$.

Let $\succeq$ be a binary relation on acts, F, that represents (weak) preferences. A decision maker is said to have CEU preferences if there exists a utility function $u : X \to \Re$ and a non-additive probability $v : 2^S \to \Re$ such that, for all $f, g \in F$, $f \succeq g$ if and only if $\int u \circ f dv \geq \int u \circ g dv$. CEU preferences are said to display *uncertainty aversion* if v is convex.[2] A decision maker is said to have MMEU

[2] This characterization of uncertainty aversion for the CEU model stems from an axiom of Schmeidler's [27] of the same name in an Anscombe-Aumann framework. Casadesus-Masanell, Klibanoff, and Ozdenoren ([4], [3]) develop analogous axioms for a Savage-style framework. This notion of uncertainty aversion has been by far the most common in the literature. However, recently, Epstein [10] and Ghirardato and Marinacci [13] have proposed alternative notions of uncertainty aversion. In particular, for the case of CEU, Ghirardato and Marinacci's characterization requires the capacity v to be balanced. All convex capacities are balanced, but the converse is not true. Epstein's notion neither implies nor is implied by convexity of v. However the reason for this is that he uses a set of preferences larger than expected utility as an uncertainty neutral benchmark. If (as is the philosophy in this paper and in Ghirardato and Marinacci) expected utility is the uncertainty neutral benchmark, then Epstein's notion also requires v to be balanced. The reason these notions are weaker than Schmeidler's is that they are based solely on preference comparisons for which at least one of the

preferences if there exists a utility function $u : X \to \Re$ and a non-empty, closed and convex set B of additive probability measures on S such that, for all $f, g \in F$, $f \succeq g$ if and only if $\min_{p \in B} \int u \circ f dp \geq \min_{p \in B} \int u \circ g dp$. All MMEU preferences display *uncertainty aversion*.[3] Finally, note that the set of MMEU preferences strictly contains the set of CEU preferences with convex capacities.

3 Modeling a randomizing device

Corresponding to the two standard frameworks for modeling uncertainty (Anscombe-Aumann and Savage) there are at least two alternative ways to model a randomizing device. In an Anscombe-Aumann setting, a randomizing device is incorporated in the structure of the consequence space. Specifically the "consequences" X, are often taken to be the set of all simple probability distributions over some more primitive set of outcomes, Z. In this set-up, a randomization over two acts f and g with probabilities p and $1 - p$ respectively is modeled by an act h where $h(s)(z) = pf(s)(z) + (1 - p)g(s)(z)$, for all $s \in S, z \in Z$. Observe that h is, indeed, a well-defined act because the set of simple probability distributions is closed under mixture.

Returning to the "unknown urn" of the introduction, Table 1 shows the three acts (a) "bet on red," (b) "bet on black," and (c) "randomize 50-50 over betting on red or on black" as modeled in this setting.

Table 1. Unknown urn with randomization in the consequence space (Anscombe-Aumann)

	R(ed)	B(lack)
a	\$100	\$0
b	\$0	\$100
c	$\frac{1}{2}$\$100 $\oplus$ $\frac{1}{2}$\$0	$\frac{1}{2}$\$100 $\oplus$ $\frac{1}{2}$\$0

Alternatively, consider a Savage-style setting with a finite state space (e.g., Wakker [28], Nakamura [21], or Gul [16]). Here a convex combination of two elements of the consequence space X need not be an element of X (and need not even be defined). Therefore, to model a randomization, we may instead expand the original state space, S, by forming the cross product of S with the possible outcomes (or "states") of the randomizing device. For example, Table 2 shows the acts (a) "bet on red," (b) "bet on black," (c) "bet on red if heads, black if tails," and (d) "bet on black if heads, red if tails" in the case of the unknown urn with a coin used to randomize.

two acts being compared is "unambiguous." In contrast, Schmeidler's approach relies, in addition, on comparisons between certain pairs of "ambiguous" acts that are implicitly ranked as more or less ambiguous by his axiom (or its Savage counterpart). See Casadesus-Masanell, Klibanoff, and Ozdenoren [4] for a more detailed discussion along these lines.

[3] This is true using the approach of either Schmeidler [27], Casadesus-Masanell, Klibanoff, and Ozdenoren ([4], [3]), or Ghirardato and Marinacci [13]. Under the assumption that preferences over "unambiguous" acts are expected utility, it is true in Epstein's [10] approach as well.

Table 2. Unknown urn with randomization in the state space only (Savage)

	R(ed), H(eads)	B(lack), H(eads)	R(ed), T(ails)	B(lack), T(ails)
a	\$100	\$0	\$100	\$0
b	\$0	\$100	\$0	\$100
c	\$100	\$0	\$0	\$100
d	\$0	\$100	\$100	\$0

In comparing the two models, observe that the Anscombe-Aumann setting builds in several key properties that a randomizing device should satisfy while the Savage setting does not. In particular, the probabilities attached to the outcomes of the randomizing device should be unambiguous and the device should be stochastically independent from the (rest of the) state space. Arguably these two properties capture the essence of what is meant by a randomizing device. Both properties are automatically satisfied in an Anscombe-Aumann setting. In a Savage setting, as we will see below, these properties require additional restrictions on preferences.[4]

Several recent papers (including Eichberger and Kelsey [7], Ghirardato [11], and Sarin and Wakker [24]), have noted that CEU need not give identical results in the two frameworks. Specifically, they suggest that the choice of a one-stage (Savage) or two-stage (Anscombe-Aumann) model can lead to different behavior. To see this in the unknown urn example, consider the case where the decision maker's marginal capacity over the colors is $v(R) = v(B) = \frac{1}{3}$. In the Anscombe-Aumann setting this is enough to pin down preferences as $c \succ a \sim b$, (i.e., the Raiffa preferences or preference for randomization).

In the Savage setting, consider the capacity given by

$$
\begin{aligned}
v(R \times \{H, T\}) = v(B \times \{H, T\}) &= \frac{1}{3}, \\
v(\{R, B\} \times H) = v(\{R, B\} \times T) &= \frac{1}{2}, \\
v(R \times H) = v(R \times T) = v(B \times H) = v(B \times T) &= \frac{1}{6}, \\
v((R \times H) \cup (B \times T)) = v((R \times T) \cup (B \times H)) &= \frac{1}{3}, \\
v(\text{any 3 states}) &= \frac{2}{3}.
\end{aligned}
$$

This capacity yields the preferences $a \sim b \sim c \sim d$, and thus does not provide a preference for randomization as in the Anscombe-Aumann setting. Why can this occur despite the fact that the marginals are identical in the two cases and the product capacity is equal to the product of the marginals on all

[4] A randomizing device *could* be modeled in an Anscombe-Aumann setting by expanding the state space in exactly the same way as illustrated for the Savage setting. In this case, the same additional restrictions on preferences as in the latter setting would be required to ensure that the randomizing device was unambiguous and stochastically independent.

rectangles? Mathematically, as Ghirardato [11] explains, the source is a failure of the usual Fubini Theorem to hold for Choquet integrals. Intuitively, however, it is not clear what is going “wrong” in the example.

To gain some insight, it is useful to examine the weights applied to each state when evaluating the randomized acts using the Choquet integral. For example, as Table 3 shows, “Bet on Red if Heads, Black if Tails” is evaluated using *non-product* weights. The fact that such non-product weights can be applied suggests that the CEU preferences with the capacity above reflect ambiguity not only about the color of the ball drawn from the urn but also about the correlation between the randomizing device and the color of the ball. This can also be seen by noting that $v(\{R, B\} \times H) > v((R \times H) \cup (B \times T))$, in contrast to the equality one might expect if H and T are really produced by a symmetric, independent randomization. While such ambiguity is certainly possible, it runs directly counter to the stochastic independence we would expect of a randomizing device. In the next section, therefore, I propose conditions on preferences that ensure this independence.

Table 3. Non-product weights for randomized act

	R, H	B, H	R, T	B, T
c	\$100	\$0	\$0	\$100
weights	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

4 Stochastically independent randomization and preferences

Here I propose conditions on preferences that are designed to reflect two properties of a randomizing device: unambiguous probabilities and stochastic independence. These two properties are essential to what is meant by a randomizing device.

Formally, consider preferences, $\succeq$, over acts, $F : S \to X$, on a finite product state space, $S = S_1 \times S_2 \times \ldots \times S_N$. Let S_{-i} denote the product of all ordinates other than i. Denote by F_{S_i} the subset of acts for which outcomes are determined entirely by the ith ordinate. This means that $f \in F_{S_i}$ implies $f(s_i, s_{-i}) = f(s_i, \hat{s}_{-i})$ for all $s_{-i}, \hat{s}_{-i} \in S_{-i}$ and $s_i \in S_i$. For $f, g \in F$ and $A \subseteq S$, denote by $f_A g$ the act which equals $f(s)$ for $s \in A$ and equals $g(s)$ for $s \notin A$. We now state some useful definitions concerning preferences.

Definition 1 *$\succeq$ satisfies* **solvability** *on S_i if, for $f \in F_{S_i}$, $x, y, z \in X$ and $A_i \subseteq S_i$, $x_{A_i \times S_{-i}} z \succeq f \succeq y_{A_i \times S_{-i}} z$ implies $f \sim w_{A_i \times S_{-i}} z$ for some $w \in X$.*

Solvability should be seen as a joint richness condition on $\succeq$ and X. It is satisfied in all axiomatizations of which I am aware of EU, CEU, or MMEU over Savage acts on a finite state-space. For example, Nakamura [21] imposes solvability directly, while Wakker ([28], [29]), Gul [16] and Casadesus-Masanell, Klibanoff, and Ozdenoren ([4], [3]) ensure it is satisfied through topological assumptions on X and continuity assumptions on $\succeq$.

Definition 2 *$\succeq$ satisfies* **expected utility (EU)** *on S_i if $\succeq$ restricted to F_{S_i} can be represented by expected utility where the utility function is unique up to a positive affine transformation and the probability measure on the set of all subsets of S_i is unique.*

While the definition is intentionally stated somewhat flexibly, it could easily be made more primitive/rigorous by assuming that preferences restricted to F_{S_i} satisfy the axioms in one of the existing axiomatizations of expected utility over Savage acts on a finite state space such as Wakker [28], Nakamura [21], Gul [16], or Chew and Karni [5]. This definition is intended to capture the fact that the decision-maker associates a unique probability distribution with S_i and uses that distribution to weight outcomes. Note that the uniqueness requirement on the probability measure entails the existence of consequences $x, y \in X$ such that $x \succ y$ (where preferences over X are derived from preferences over the associated constant-consequence acts in the usual way). Furthermore, any of the axiomatizations cited will imply solvability on S_i as well.

Definition 3 *$s_i \in S_i$ is* **null** *if $f_{s_i \times S_{-i}} h \sim g_{s_i \times S_{-i}} h$ for all $f, g, h \in F_{S_i}$.*

Note that given EU, a state is null if and only if it is assigned zero probability.

Definition 4 *S_i is* **stochastically independent** *of S_{-i} if, for all $\hat{s}_{-i} \in S_{-i}$, $f \in F_{S_i}$ and $w \in X$,*

$$f \sim w \tag{1}$$

implies,

$$f_{S_i \times \hat{s}_{-i}} w \sim w. \tag{2}$$

While this is formulated as a general definition of stochastic independence of an ordinate, this paper will focus only on independence of a randomizing device. For this purpose, the main definition is the following:

Definition 5 *S_i is a* **stochastically independent randomizing device (SIRD)** *if S_i is stochastically independent and contains at least two non-null states, and $\succeq$ satisfies solvability and EU on S_i.*

This condition is designed to differentiate between EU ordinates that are stochastically independent from the rest of the state space and those that are dependent, while still allowing for possible uncertainty aversion on other ordinates. A useful way to understand this definition is as follows: There are several potential reasons why (1) could hold while (2) is violated. First, it might be that uncertainty aversion over S_i leads a different marginal probability measure over S_i to be used when evaluating the acts in (2) than when evaluating acts in (1). This is ruled out by the assumption that preferences satisfy EU on S_i. Second, it might be that the marginal over S_i conditional on $\hat{s}_{-i}$ is different than the unconditional marginal over S_i due to some stochastic dependence (or uncertainty about stochastic independence) between S_i and S_{-i}. Since we want to model an independent randomizing device, it is proper that the SIRD condition does not allow for such dependence.

Also supporting the idea that this definition reflects stochastic independence is the observation that if preferences are EU and non-trivial, then S_i an SIRD is equivalent to requiring that the representing probability measure be a product measure on $S_i \times S_{-i}$. Note also that all of the results that follow will also hold true if we additionally impose that S_{-i} is stochastically independent of S_i (by switching the role of i and $-i$ in the definition of stochastically independent). Thus this concept shares the symmetry that a notion of stochastic independence should intuitively possess.

In the next two sections we develop the implications of SIRD for some common classes of uncertainty averse preferences.

4.1 MMEU and randomizing devices

This section develops the implications for MMEU preferences of one ordinate of the state space being a SIRD. MMEU will be found to be flexible enough to easily incorporate both a SIRD and uncertainty aversion.

Theorem 1 *Assume $\succeq$ are MMEU preferences satisfying solvability for some S_i that contains at least two non-null states. Then the following are equivalent:*

(i) S_i is a SIRD;
(ii) There exists a probability measure on 2^{S_i}, $\hat{p}$, such that all probability measures, p, in the closed, convex set of measures, B, of the MMEU representation satisfy $p(s) = \hat{p}(s_i)p(S_i \times s_{-i})$, for all $s \in S$.

Proof. ($(i) \Rightarrow (ii)$) We first show that all $p \in B$ must have the same marginal on S_i. Fix outcomes $x, y \in X$ such that $x \succ y$. EU on S_i implies that $\succeq$ restricted to F_{S_i} may be represented by $\sum_{s_i \in S_i} u(f(s_i))\hat{p}(s_i)$ where $\hat{p}$ is the unique representing probability measure on 2^{S_i}, and u is unique up to a positive affine transformation. Using the MMEU representation of $\succeq$ yields a utility function $\tilde{u}$ and a set of measures B such that, for all $f, g \in F_{S_i}$,

$$\begin{array}{ccc} \min\limits_{p \in B} \sum\limits_{s_i \in S_i} \tilde{u}(f(s_i))p(s_i \times S_{-i}) & \geq & \min\limits_{p \in B} \sum\limits_{s_i \in S_i} \tilde{u}(g(s_i))p(s_i \times S_{-i}) \\ & \Longleftrightarrow & \\ \sum\limits_{s_i \in S_i} u(f(s_i))\hat{p}(s_i) & \geq & \sum\limits_{s_i \in S_i} u(g(s_i))\hat{p}(s_i). \end{array}$$

Without loss of generality, set $u(x) = \tilde{u}(x) = 1$ and $u(y) = \tilde{u}(y) = 0$. Using the fact that S_i satisfies EU and solvability, combined with the MMEU representation, allows one to apply Nakamura ([21], Lemma 3) and conclude that, given the normalization, the two utility functions must be the same (i.e., $\tilde{u}(x) = u(x)$ for all $x \in X$). Therefore,

$$\begin{array}{ccc} \min_{p\in B}\sum_{s_i\in S_i} u(f(s_i))p(s_i \times S_{-i}) & \geq & \min_{p\in B}\sum_{s_i\in S_i} u(g(s_i))p(s_i \times S_{-i}) \\ & \Longleftrightarrow & \\ \sum_{s_i\in S_i} u(f(s_i))\hat{p}(s_i) & \geq & \sum_{s_i\in S_i} u(g(s_i))\hat{p}(s_i). \end{array}$$

Suppose there is some $p' \in B$ such that $p'(s_i \times S_{-i}) \neq \hat{p}(s_i)$ for some $s_i \in S_i$. Without loss of generality, assume that $\hat{p}(s_i') > p'(s_i' \times S_{-i})$ for an $s_i' \in S_i$. Consider the act $f = x_{s_i' \times S_{-i}}y$. Solvability guarantees that there exists a $z \in X$ such that $z \sim f$. Thus,

$$\begin{aligned} u(z) &= \min_{p\in B}\sum_{s_i\in S_i} u(f(s_i))p(s_i \times S_{-i}) \\ &\leq p'(s_i' \times S_{-i}) \\ &< \hat{p}(s_i') \\ &= \sum_{s_i\in S_i} u(f(s_i))\hat{p}(s_i) \\ &= u(z), \end{aligned}$$

a contradiction. Therefore, it must be that $p \in B$ implies $p(s_i \times S_{-i}) = \hat{p}(s_i)$ for all $s_i \in S_i$. In other words, all the marginals on S_i agree.

Now we show that each $p \in B$ is a product measure on $S_i \times S_{-i}$. This part of the argument proceeds by contradiction. Suppose that $p \in B$ does not imply that $p(s) = \hat{p}(s_i)p(S_i \times s_{-i})$, for all $s \in S$. Then there must exist a $p_0 \in B$ and a $\hat{s} \in S$ such that

$$p_0(\hat{s}) < \hat{p}(\hat{s}_i)p_0(S_i \times \hat{s}_{-i}). \tag{3}$$

According to p_0, the probability of $\hat{s}_i$ and $\hat{s}_{-i}$ occuring together is less than the product of the respective marginal probabilities. We now show that this is inconsistent with the assumption that S_i is stochastically independent. Consider the act $f \in F_{S_i}$ such that $f = x_{\hat{s}_i \times S_{-i}}y$. Since $x \succeq f \succeq y$, solvability on S_i implies there exists a $w \in X$ such that $w \sim f$. Observe that our normalization of u and the preference representation imply $u(w) = \hat{p}(\hat{s}_i)u(x) + (1 - \hat{p}(\hat{s}_i))u(y) = \hat{p}(\hat{s}_i)$.

Define the act $h = f_{S_i \times \hat{s}_{-i}}w$. By SIRD, $f \sim w$ implies $h \sim w$. We have the following contradiction:

$$\begin{aligned} u(w) &= \min_{p\in B}\sum_{s\in S} u(h(s))p(s) \\ &\leq \sum_{s\in S} u(h(s))p_0(s) \\ &= \sum_{s\in S_i\times\hat{s}_{-i}} u(f(s))p_0(s) + \sum_{s\notin S_i\times\hat{s}_{-i}} u(w)p_0(s) \\ &= p_0(\hat{s}) + u(w)(1 - p_0(S_i \times \hat{s}_{-i})) \end{aligned}$$

$$
\begin{aligned}
&= u(w) + (p_0(\hat{s}) - \hat{p}(\hat{s}_i)p_0(S_i \times \hat{s}_{-i})) \\
&< u(w).
\end{aligned}
$$

(Note that the last inequality follows from (3).) Therefore each $p \in B$ must in fact be a product measure on $S_i \times S_{-i}$ and (ii) is proved.

($(ii) \Rightarrow (i)$) That (ii) implies EU is satisfied on S_i is clear because $\hat{p}$ is the unique representing probability measure. To see that (1) implies (2) is satisfied on S_i, consider any $f \in F_{S_i}$ and $w \in X$ such that $f \sim w$. Fix an $\hat{s}_{-i} \in S_{-i}$ and define $h = f_{S_i \times \hat{s}_{-i}} w$. By (ii),

$$\min_{p \in B} \sum_{s \in S} u(h(s))p(s) = \min_{p \in B} \sum_{s_{-i} \in S_{-i}} p(S_i \times s_{-i}) \left[\sum_{s_i \in S_i} u(h(s_i, s_{-i}))\hat{p}(s_i) \right]$$

and,

$$\min_{p \in B} \sum_{s \in S} u(w)p(s) = \min_{p \in B} \sum_{s_{-i} \in S_{-i}} p(S_i \times s_{-i}) \left[\sum_{s_i \in S_i} u(w)\hat{p}(s_i) \right].$$

Since $f \sim w$,

$$\sum_{s_i \in S_i} u(h(s_i, s_{-i}))\hat{p}(s_i) = \sum_{s_i \in S_i} u(w)\hat{p}(s_i) \text{ for all } s_{-i} \in S_i.$$

Therefore the two minimization problems are the same and $h \sim w$. □

Thus, we get quite a natural representation in the MMEU framework:

- All the marginals on the randomizing device agree, reflecting the lack of ambiguity about the device.
- All the measures in B are product measures on $S_i \times S_{-i}$, reflecting the independence of the randomizing device.

Remark 1. It is not hard to see from the theorem that, in the Ellsberg "unknown urn" example, if "bet on red" is indifferent to "bet on black" then *any* MMEU preferences that are not EU and for which the coin is a SIRD lead to the "Raiffa" preference for randomization. As a concrete example, consider the MMEU preferences with set of measures

$$
\begin{aligned}
\{p \mid p(R \times H) &= p(R \times T) = \frac{1}{2}x,\ p(B \times H) = p(B \times T) \\
&= \frac{1}{2}(1-x),\ \frac{1}{3} \leq x \leq \frac{2}{3} \right\}.
\end{aligned}
$$

By the theorem, these preferences make $\{H, T\}$ a SIRD and it is easy to verify that they exhibit the "Raiffa" preference for randomization.

Remark 2. The set of product measures that emerges from the MMEU characterization is consistent with a notion of independent product of two sets of measures proposed by Gilboa and Schmeidler [15]. Specifically, the set B is trivially the independent product (in their sense) of the (unique) marginal on S_i and the set of marginals on S_{-i} used in representing preferences over $F_{S_{-i}}$. It is worth noting that no purely preference based justification for their broader notion (when neither of the sets is a singleton) is known.

4.2 CEU, uncertainty aversion, and randomizing devices

This section examines uncertainty averse CEU preferences on a product state space where one of the ordinates is assumed to be a candidate randomizing device. In stark contrast to the results of the previous section, this class is shown to include only expected utility (EU) preferences. This suggests that CEU preferences with a convex capacity are incapable of modeling both a randomizing device and uncertainty aversion simultaneously.

Theorem 2 *If CEU preferences, $\succeq$, display uncertainty aversion and, for some i, S_i is a SIRD then $\succeq$ must be EU preferences.*

Proof. Recall that the state space is $S = S_1 \times S_2 \times \ldots \times S_N$. Without loss of generality, let S_1 be a SIRD. Uncertainty aversion implies that the capacity v in the CEU representation is convex. The *core* of a capacity v is the set of probability measures that pointwise dominate v (i.e., $\{p \mid p(A) \geq v(A), \text{for all } A \subseteq S; p \text{ a probability measure.}\}$) Any CEU preferences with a convex v are also MMEU preferences with the set of measures, B, equal to the core of v (Schmeidler [26]). It follows that $v(A) = \min_{p \in \text{core}(v)} p(A)$ for all $A \subseteq S$ (i.e., v is the lower envelope of its core). Since preferences are MMEU and S_1 is a SIRD, Theorem 1 implies that there exists a probability measure on 2^{S_1}, $\hat{p}$, such that all probability measures, p, in the core of v satisfy $p(s) = \hat{p}(s_1)p(S_1 \times s_{-1})$, for all $s \in S$. Thus the core of v must be of a very special form. The remainder of the proof is devoted to showing that convexity of v and a core of this form are only compatible when preferences are EU.

First I derive a key equality implied by convexity together with the form of the core. To this end, fix any $s_1 \in S_1$ and $A_{-1}, B_{-1} \subseteq S_{-1}$. Denote the complement of s_{-1} relative to S_{-1} by s^c_{-1} and the complement of s_1 relative to S_1 by s^c_1. Define the sets $C = s_1 \times S_{-1}$ and $D = (s_1 \times A_{-1}) \cup (s^c_1 \times B_{-1})$. Convexity of v implies that

$$v(C) + v(D) \leq v(C \cup D) + v(C \cap D).$$

Using the structure of the core of v and the fact that v is the lower envelope of its core yields the opposite inequality:

$$\begin{aligned}
&v(C) + v(D) = \hat{p}(s_1) + \min_{p \in \text{core}(v)} \left[\hat{p}(s_1)p(S_1 \times A_{-1}) + (1 - \hat{p}(s_1))p(S_1 \times B_{-1})\right] \\
&\geq \hat{p}(s_1) + \hat{p}(s_1) \min_{p \in \text{core}(v)} p(S_1 \times A_{-1}) + (1 - \hat{p}(s_1)) \min_{p \in \text{core}(v)} p(S_1 \times B_{-1}) \\
&= v(C \cup D) + v(C \cap D).
\end{aligned}$$

Combining the two inequalities, it must be that, for all $s_1 \in S_1$ and all $A_{-1}, B_{-1} \in S_{-1}$,

$$\begin{aligned} &\min_{p \in \mathrm{core}(v)} \left[\hat{p}(s_1) p(S_1 \times A_{-1}) + (1 - \hat{p}(s_1)) p(S_1 \times B_{-1})\right] \\ = \;& \hat{p}(s_1) \min_{p \in \mathrm{core}(v)} p(S_1 \times A_{-1}) + (1 - \hat{p}(s_1)) \min_{p \in \mathrm{core}(v)} p(S_1 \times B_{-1}). \end{aligned} \tag{4}$$

Now, using equation (4), an argument by contradiction shows that the core of v cannot contain more than one measure. Suppose core(v) contains more than one probability measure. Then there exists an $\bar{s} \in S$ such that $\arg\min_{p \in \mathrm{core}(v)} p(\bar{s}) \subset \mathrm{core}(v)$. Since all the measures in the core are of the form $p(s) = \hat{p}(s_1) p(S_1 \times s_{-1})$, it must be that $\arg\min_{p \in \mathrm{core}(v)} p(S_1 \times \bar{s}_{-1}) = \arg\min_{p \in \mathrm{core}(v)} p(\bar{s})$. Since $p(S_1 \times \bar{s}_{-1}) = 1 - p(S_1 \times \bar{s}^c_{-1})$ for any $p \in \mathrm{core}(v)$,

$$\arg \min_{p \in \mathrm{core}(v)} p(S_1 \times \bar{s}_{-1}) \cap \arg \min_{p \in \mathrm{core}(v)} p(S_1 \times \bar{s}^c_{-1}) = \emptyset.$$

Thus, for any non-null $s_1 \in S_1$,

$$\begin{aligned} &\min_{p \in \mathrm{core}(v)} \left[\hat{p}(s_1) p(S_1 \times \bar{s}_{-1}) + (1 - \hat{p}(s_1)) p(S_1 \times \bar{s}^c_{-1})\right] \\ > \;& \hat{p}(s_1) \min_{p \in \mathrm{core}(v)} p(S_1 \times \bar{s}_{-1}) + (1 - \hat{p}(s_1)) \min_{p \in \mathrm{core}(v)} p(S_1 \times \bar{s}^c_{-1}), \end{aligned}$$

in violation of equation (4).

Therefore, the core of v must be a singleton and, since v is the lower envelope of its core, v must be a probability measure and preferences are EU. □

Remark 3. This theorem shows that CEU with a convex capacity is a *very* restrictive class of preferences in a Savage-like setting. In particular a decision maker with such preferences must be either uncertainty neutral (i.e., an expected utility maximizer) or must not view any ordinate of the state space as a stochastically independent randomizing device. Note that this fact is disguised in an Anscombe-Aumann setting because there the randomizing device is built into the outcome space and thus automatically separated from the uncertainty over the rest of the world.

Remark 4. The theorem allows us to better understand the result of Eichberger and Kelsey [7], who find that convexity of v, a symmetric additive marginal on S_1, and a requirement that relabeling the states in S_1 not affect preference, together imply no preference for randomization. The result shown here makes clear that the lack of preference for randomization in their paper comes from the fact that decision makers having preferences in this class (with v somewhere strictly convex) *cannot* act as if the randomizing device is stochastically independent in the sense of SIRD. In other words, the uncertainty averse preferences they consider rule out *a priori* the possibility of a stochastically independent device and thus of true randomization. In this light, their result arises because all of the non-EU preferences they consider *force* a range of possible correlations (which are then viewed pessimistically since they are another source of uncertainty) between the device and the rest of the state space. Once they admit preferences like MMEU, which, as shown above, can reflect a proper randomizing device (a SIRD) as well as uncertainty aversion, preference for randomization reappears.

Remark 5. If convexity of v is replaced by the weaker requirement of v balanced (or, equivalently, the core of v non-empty), as advocated by Ghirardato and Marinacci [13], preferences do not collapse to EU. For example, if the capacity used in section 3 is modified by setting $v((R \times H) \cup (B \times T)) = v((R \times T) \cup (B \times H)) = \frac{1}{2}$ rather than $\frac{1}{3}$, the resulting preferences make $\{H, T\}$ a SIRD and are not EU. This capacity has a non-empty core, but is not convex. Note that these preferences still display a preference for randomization. To the extent that one is willing to accept this weaker characterization of uncertainty aversion and wants to use CEU preferences in a Savage-like setting, these findings suggest that the class of capacities with non-empty cores that are not convex may be of particular interest.

4.3 Further discussion of the SIRD condition

The key to these results is the definition of an SIRD, in particular the assumption that (1) implies (2). I argued above that given preferences satisfying EU on S_i and given the restriction of the acts in (1) to be S_i-measurable, it is quite natural to accept (1) implies (2) as reflecting the stochastic independence of S_i from the rest of the state space.

It is worth elaborating a bit on why SIRD is appropriate for a randomizing device. Since stochastic independence does concern *independence*, and uncertainty aversion fundamentally involves violations of the independence axiom/sure thing principle of subjective expected utility theory, it is fair to ask whether imposing SIRD unnecessarily restricts uncertainty aversion. Does SIRD confound stochastic independence with the violations of independence inherent in uncertainty aversion? Theorem 1 answers this question in the negative and suggests that uncertainty aversion is not restricted at all by imposing SIRD. Specifically, *any* MMEU preferences[5] over $F_{S_{-i}}$ are compatible with S_i being a SIRD. Notice that it is exactly and only uncertainty aversion over S_{-i} that is unrestricted. This is appropriate, since any other uncertainty aversion must be either over S_i (ruled out by EU) or over the correlation between S_i and S_{-i} (incompatible with the presumption of stochastic independence). There simply is nothing else to be uncertain about.

It seems that SIRD strikes a reasonable balance – enforcing stochastic independence of a randomizing device while allowing uncertainty aversion on the other ordinates of the state space.

5 Conclusion

This paper has provided preference-based conditions that a randomizing device should satisfy. When these conditions are applied to the class of CEU preferences

[5] Recall that this includes any CEU preferences with a convex capacity as well.

with convex capacities in a product state-space model a collapse to expected utility results. This does not occur with MMEU preferences in the same setting. In particular, it appears that some previous results on the absence of preference for randomization were driven not by some deep difference in Anscombe-Aumann and Savage style models as they relate to uncertainty aversion, but by the restrictiveness, as it relates to stochastic independence, of the CEU functional form with a convex capacity which is exacerbated in Savage style models. When stochastic independence is properly accounted for, preference for randomization by uncertainty averse decision makers arises in both one- and two- stage models.

To my knowledge, Blume, Brandenburger and Dekel [2] are the only others to have developed a preference axiom for stochastic independence. Their work is in the context of preferences satisfying the decision-theoretic independence axiom. This leads their condition to be unsatisfactory in the setting of this paper. In particular, their axiom asks more of conditional preferences than is reasonable in the presence of uncertainty aversion and does not need to address the consistency of conditional with unconditional preferences.

There have been several functional (i.e., non-preference axiom based) notions of stochastically independent product that have been proposed for the MMEU or CEU models. For the case where one marginal is additive in the MMEU model, as was mentioned following Theorem 1, the results of the approach taken here agree with the notion proposed in Gilboa and Schmeidler [15]. Approaches specific to the CEU model have been suggested by Ghirardato [11] and Hendon, Jacobsen, Sloth, and Tranaes [17]. If one marginal is additive and the product capacity is convex (as in Theorem 2), these approaches are weaker than the one advocated here. Specifically, preferences that are independent in the sense of Ghirardato [11] or Hendon *et al.* [17] may violate SIRD.

Some other recent work on shortcomings of the CEU model in capturing probabilistic features is Nehring [22]. An analysis relating separability of events in the CEU model to expected utility is given in Sarin and Wakker [25]. In the context of inequality measurement under uncertainty, Ben-Porath, Gilboa and Schmeidler [1] advocate MMEU type functionals and show that they are closed under iterated application while CEU functionals are not. Differences between CEU and MMEU are also discussed in Klibanoff [19] and Ghirardato, Klibanoff and Marinacci [12].

Any discussion of behavior, such as preference for randomization, that departs from what is considered standard raises some natural questions. First, descriptively, do actual decision makers behave in this way? Unfortunately, there are no studies that I am aware of that examine this issue. To do so properly would require: (1) taking some device like a coin and verifying that the decision maker viewed it as a SIRD; and (2) making it explicit to the decision maker that it is possible to choose acts that depend, not only on the main feature of interest (e.g., the color of ball drawn) but simultaneously on the realization of the coin flip. It is worth noting that many standard Ellsberg-style experiments do not offer an opportunity to examine preference for randomization because they tend to ask only questions such as “Do you prefer betting on red (black) in urn I or red (black) in urn II?" rather than allowing a fuller range of choices that provide a role for randomization.

Second, normatively, is the preference for randomization described here "reasonable" or "rational" behavior or is it normatively unacceptable? In examples where uncertainty averse behavior seems reasonable, I find preference for randomization to be just as reasonable. Randomization acts to limit the negative influence of uncertainty on expected payoffs. In a nutshell, if one is afraid that the distribution over colors will be an unfavorable one, if one does not suffer this fear regarding the outcome of a coin flip, and if one is sure that the realization of the coin is independent, then the fact that any joint distribution must respect this independence limits the extent to which acts that pay based on the coin as well as the color ("randomizations") can be hurt by uncertainty over the colors. This paper has shown that to reject this argument, one must either (1) reject uncertainty aversion as defined here or (2) reject the possibility of committing to acts that are contingent on a SIRD (i.e., reject the static, Savage-like model of independent randomization). To argue the former, as in Raiffa [23], one may invoke reasoning based on the decision-theoretic independence axiom/sure thing principle to reject Ellsberg-type behavior as irrational. However, at the very least, the normative force of the independence axiom/sure-thing principle is a topic on which there are a wide range of opinions. Arguments relying on an inability to commit to a randomized action bring in an explicit dynamic component that is beyond the scope of this paper to fully address. Such arguments are not, in my view, particularly satisfying since they leave open the question of why such acts would not be introduced, by third-parties if necessary, given that the decision-maker desires them.

Finally, it is important to emphasize that, although the language of this paper has been in terms of independent randomization, fundamentally a SIRD is simply an ordinate of a product state space over which preferences are expected utility and which is viewed as stochastically independent of the rest of the state space. In many situations where an individual faces a number of uncertainties, it may be useful to be able to assume that the individual is expected utility on some dimensions but uncertainty averse on others, and that these facets of the uncertainty are stochastically independent of others. In this regard, Theorem 2 has shown that CEU with a convex capacity is not an appropriate class of preferences, while, by Theorem 1, MMEU preferences are capable of reflecting these features.

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