Perceived Ambiguity and Relevant Measures*

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Abstract

We axiomatize preferences that can be represented by a monotonic aggregation of subjective expected utilities generated by a utility function and some set of i.i.d. probability measures over a product state space, S^∞. For such preferences, we define *relevant measures*, show that they are treated as if they were the only marginals possibly governing the state space and connect them with the measures appearing in the aforementioned representation. These results allow us to interpret relevant measures as reflecting part of *perceived ambiguity*, meaning subjective uncertainty about probabilities over states. Under mild conditions, we show that increases or decreases in ambiguity aversion cannot affect the relevant measures. This property, necessary for the conclusion that these measures reflect *only* perceived ambiguity, distinguishes the set of relevant measures from the leading alternative in the literature. We apply our findings to a number of well-known models of ambiguity-sensitive preferences. For each model, we identify the set of relevant measures and the implications of comparative ambiguity aversion.

Keywords: Symmetry, beliefs, ambiguity, ambiguity aversion, sets of probabilities

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1 Introduction

In Savage's subjective expected utility (SEU) theory [39], an individual's preference over acts (maps from states of the world to outcomes) can be described using two arguments: a subjective probability over states that enables her to identify each act with a distribution over outcomes, and a von Neumann-Morgenstern (vNM) utility function that describes her risk attitude (i.e., preference over distributions over outcomes). Subsequent work has developed models that permit a richer description of uncertainty about states and attitudes toward this uncertainty. In particular, this richness is useful for describing behavior under *ambiguity*.[1] Perceived ambiguity, meaning subjective uncertainty about probabilities over states, induces uncertainty concerning the distribution over outcomes an act generates. Attitude towards ambiguity describes how averse or attracted the individual is to this induced subjective uncertainty. For many models allowing ambiguity to affect behavior there is, just as in SEU, a component of the model that describes preferences over distributions over outcomes (i.e., risk attitude). The remaining components of these models do two things: (1) map each act into a *set* of distributions over outcomes, and (2) *aggregate* the evaluations of each of these distributions.

One might be tempted to assert that a map as in (1) must reflect perceived ambiguity, as opposed to ambiguity aversion. However, a given preference typically has many representations, each using a different (map, aggregator) pair. The main goal of this paper is to define through preferences and identify in representations a particular map (called the relevant measures) such that behavior is as if relevant measures were the possible resolutions of ambiguity. In this sense, the relevant measures will be shown to be a part of perceived ambiguity. Furthermore, if a map is supposed to reflect *only* perceived ambiguity, then it is necessary that it is not affected by increases or decreases in ambiguity aversion. Hence, a complementary goal is to identify conditions under which increases or decreases in ambiguity aversion cannot change the relevant measures.

We accomplish these goals in a setting that encompasses a large set of preference models provided that the state space has an infinite product structure and preferences over Anscombe-Aumann acts (functions from the state space to lotteries over outcomes) satisfy a type of symmetry with respect to that structure. This symmetry requires preferences to treat bets on an event identically to bets on any other event that differs only in permuting the role of some ordinates, S, of the state space, S^∞. We provide preference axioms (collectively referred to as Continuous Symmetry) and show (Theorem 3.1) that they imply representation by a monotonic aggregation of subjective expected utilities generated using a single utility function and a set of i.i.d. probability measures. Aside from symmetry, the main substantive requirement of our axioms is state-independent, expected utility preferences over lotteries. Though restrictive, symmetry requirements are essentially cross-ordinate requirements only, and allow great freedom in specifying

[1] This word is used in the sense of the decision theory literature following Ellsberg [13]. See e.g., Ghirardato [21] who states ". . . 'ambiguity' corresponds to situations in which some events do not have an obvious, unanimously agreeable, probability assignment."

preferences over acts depending on any single ordinate (for example, enough freedom to embed popular models from the ambiguity literature applied to acts measurable with respect to S).

This symmetric environment allows us to define a *relevant measure* as a marginal distribution, ℓ, on S that matters for preferences in the following sense: For each open set of marginal distributions, L, containing ℓ, we can find two acts, f and g, that yield the same distribution over outcomes as each other under all i.i.d. distributions generated by marginals not in L and yet the individual strictly prefers f over g. We show (Theorem 3.3) that the set of relevant measures is the unique closed subset of marginals that are necessary and sufficient for the set of measures appearing in the representation of Continuous Symmetric preferences given in Theorem 3.1. We then provide results highlighting two properties showing that relevant measures satisfy our goals:

First, we show (Theorem 3.2) that a marginal is a relevant measure if and only if, for each open neighborhood containing it, the corresponding limiting frequency event is non-null. In this sense, relevant measures are the only marginals treated as possibly governing the state space. They describe part of perceived ambiguity – specifically, *which* common marginals are viewed as possible resolutions of the ambiguity. Note the role of Continuous Symmetry in the result: it is what allows probabilities over S to be identified with (limiting frequency) events in S^∞.

Second, Theorem 3.4 provides sufficient conditions under which one preference (weakly) more ambiguity averse than another implies that the two preferences have the same set of relevant measures.[2] For example, this will be true for all Continuous Symmetric preferences that are strictly monotonic on non-null events. The result uses the standard definition of comparative ambiguity aversion (see e.g., Gilboa and Marinacci [27, Definition 16]) which says one preference is more ambiguity averse than another if, for any act and any lottery, the former ranks (resp. strictly ranks) the act above the lottery, then so does the the latter. Under the conditions of Theorem 3.4, a preference change coming from a change in relevant measures is never the same as an increase or decrease in ambiguity aversion. Specifically, the set of preferences that are (weakly) more or less ambiguity averse than a given $\succsim$ is disjoint from the set of preferences that have a different set of relevant measures than $\succsim$.

In Section 4, we specialize to Continuous Symmetric versions of two well-known models of ambiguity-sensitive preferences: the α-MEU model (see e.g., Ghirardato, Maccheroni and Marinacci [22]) and the smooth ambiguity model (see e.g., Klibanoff, Marinacci and Mukerji [30], Nau [34], Seo [40]). For each representation, we both identify the set of relevant measures and describe the implications of comparative ambiguity aversion in terms of the representation.[3] We also identify the relevant measures

[2]This is reminiscent of Yaari's [45] result that, under sufficient differentiability, SEU preferences can be ranked in terms of risk aversion only if they share a common subjective probability measure.

[3]The same is done for additional models in the online supplement (Klibanoff, Mukerji and Seo [32]): the extended MEU with contraction model (see e.g., Gajdos et. al. [20], Gajdos, Tallon and Vergnaud [19], Kopylov [33], Tapking [43]), the vector expected utility model (see Siniscalchi [42])

in a Bewley-style representation of incomplete preferences and use this identification to draw comparisons with other notions of revealed sets of measures in the literature.

What if not all ordinates are considered symmetric, but only symmetric *conditional on* some set of observables? In Appendix B, we show that our findings extend when replacing our overall symmetry assumption with symmetry conditional on descriptions (vectors of observable characteristics). In the extended results, i.i.d. measures are replaced by functions mapping descriptions to i.i.d. measures. A standard linear regression model is an example of such a function: given a description, ξ, the i.i.d. measure is Normal with mean $\beta\xi$ and variance σ^2. The analogue of a set of relevant measures is a set of pairs (β, σ) denoting a corresponding set of regression models.

What types of questions do the relevant measures allow us to address? Two examples are the following: First, in economic modeling one may want to impose constraints on preferences so that they reflect either some type of calibration of perceived ambiguity to external data or some equilibrium/internal consistency conditions on perceived ambiguity. Our theory shows why, if such constraints were to be imposed, it might be reasonable to do so through constraints on the relevant measures. A simple example of such a constraint might be the requirement that the empirical frequency distribution be considered one of the possible resolutions of ambiguity, and thus one of the relevant measures. Second, relevant measures provide a test for differences in perceived ambiguity. If two preferences differ in their relevant measures, then they must differ in perceived ambiguity.

1.1 Related literature

There is an alternative preference-based approach to identifying a unique map from acts into sets of distributions over outcomes (see Ghirardato, Maccheroni and Marinacci [22], Nehring ([35],[36]), Ghirardato and Siniscalchi [24], Siniscalchi [41]). Loosely, this approach uses marginal rates of substitution in utility space to identify the distributions over states that generate this mapping. A brief comparison with our approach is in order. An advantage of the alternative approach is that it does not require a product state space or symmetry conditions on preferences. For Continuous Symmetric preferences, Theorem 4.5 shows that the set identified by the alternative approach consists of *some* convex combinations of the i.i.d. measures generated by the relevant measures. Which particular convex combinations appear can be affected by increases or decreases in ambiguity aversion (Section 4.3 contains examples demonstrating this). For our goal of finding a map reflecting *only* perceived ambiguity, this is a disadvantage of the alternative approach. Ghirardato and Siniscalchi [24, p.3] emphasize that the distributions identified in their approach are those that "identify candidate solutions to optimization problems." This conceptually explains the dependence of their set on ambiguity aversion, as one would expect the solution to an optimization problem under ambiguity to depend on both perceived ambiguity and ambiguity aversion.

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and the second-order Choquet representation (see Amarante [4]) of invariant biseparable preferences (defined by Ghirardato, Maccheroni and Marinacci [22]).

Another approach simply takes sets of probability distributions over the state space as an objective primitive. Such models include those in Gajdos et al. [20], Gajdos, Tallon and Vergnaud [19], Kopylov [33], Wang [44], and Cerreia-Vioglio et al. [8]. Our theory provides a way of examining the connection between this objective primitive and perceived ambiguity. One illustration of this is our Theorem D.1 which shows that when the objectively given set in the extended MEU with contraction model of Gajdos et. al. [20] consists of i.i.d. measures, these are exactly the i.i.d. measures generated by the relevant measures.[4]

Our paper imposes a symmetry property on preferences. In doing so, we are following the work of de Finetti [9] and Hewitt and Savage [29] in the context of expected utility and recent extensions of this work to larger classes of preferences and various notions of symmetry by Epstein and Seo ([15],[16],[17],[18]), Al-Najjar and De Castro [3] and Cerreia-Vioglio et. al. [8]. In fact, our Theorem 4.5 may be viewed as a generalization of de Finetti's theorem. Ours is the only paper to use any of these "symmetries" to explore the concept of which i.i.d. measures (or generalizations thereof) are relevant and the implications of this relevance for modeling perceived ambiguity. The relationship between our symmetry axiom (Event Symmetry) and similar preference-based notions in the literature is detailed in Klibanoff, Mukerji and Seo [31].

2 Setting and Notation

Let S be a compact metric space and $\Omega = S^\infty$ the state space with generic element $\omega = (\omega_1, \omega_2, ...)$. The state space Ω is also compact metric (Aliprantis and Border [2, Theorems 2.61 and 3.36]). Denote by Σ_i the Borel σ-algebra on the i-th copy of S, and by Σ the product σ-algebra on S^∞. An act is a simple Anscombe-Aumann act, a measurable $f : S^\infty \to X$ having finite range (i.e., $f(S^\infty)$ is finite) where X is the set of lotteries (i.e., finite support probability measures on an outcome space Z). The set of acts is denoted by $\mathcal{F}$, and $\succsim$ is a binary relation on $\mathcal{F} \times \mathcal{F}$. As usual, we identify a constant act (an act yielding the same element of X on all of S^∞) with the element of X it yields.

Denote by Π the set of all finite permutations on $\{1, 2, ...\}$ i.e., all one-to-one and onto functions $\pi : \{1, 2, ...\} \to \{1, 2, ...\}$ such that $\pi(i) = i$ for all but finitely many $i \in \{1, 2, ...\}$. For $\pi \in \Pi$, let $\pi\omega = \left(\omega_{\pi(1)}, \omega_{\pi(2)}, ...\right)$ and $(\pi f)(\omega) = f(\pi\omega)$.

For any topological space Y, $\Delta(Y)$ denotes the set of (countably additive) Borel probability measures on Y. Unless stated otherwise, a measure is understood as a countably additive Borel measure. For later use, $ba(Y)$ is the set of finitely additive bounded real-valued set functions on Y, and $ba^1_+(Y)$ the set of non-negative probability charges in $ba(Y)$. A measure $p \in \Delta(S^\infty)$ is called symmetric if the order doesn't matter, i.e., $p(A) = p(\pi A)$ for all $\pi \in \Pi$, where $\pi A = \{\pi\omega : \omega \in A\}$. Denote by ℓ^∞ the

[4] Less related are models of preferences over sets of lotteries as in Olszewski [37] and Ahn [1]. As these models lack acts and a state space, the question of which probabilities are relevant in evaluating acts doesn't arise.

i.i.d. measure with the marginal $\ell \in \Delta(S)$. Define $\int_{S^\infty} f dp \in X$ by $\left(\int_{S^\infty} f dp\right)(B) = \left(\int_{S^\infty} f(\omega)(B)\, dp(\omega)\right)$. (Since f is simple, this is well-defined.)

Fix $x_*, x^* \in X$ such that $x^* \succ x_*$. For any event $A \in \Sigma$, 1_A denotes the act giving x^* on A and x_* otherwise. Informally, this is a bet on A. More generally, for $x, y \in X$, xAy denotes the act giving x on A and y otherwise. A finite cylinder event $A \in \Sigma$ is any event of the form $\{\omega : \omega_i \in A_i \text{ for } i = 1, ..., n\}$ for $A_i \in \Sigma_i$ and some finite n.

Endow $\Delta(S)$, $\Delta(\Delta(S))$ and $\Delta(S^\infty)$ with the relative weak* topology. To see what this is, consider, for example, $\Delta(S)$. The relative weak* topology on $\Delta(S)$ is the collection of sets $V \cap \Delta(S)$ for weak* open $V \subseteq ba(S)$, where the weak* topology on $ba(S)$ is the weakest topology for which all functions $\ell \longmapsto \int \psi d\ell$ are continuous for all bounded measurable ψ on S. Also note that a net $\ell_\alpha \in ba(S)$ converges to $\ell \in ba(S)$ under the weak* topology if and only if $\int \psi d\ell_\alpha \to \int \psi d\ell$ for all bounded measurable ψ on S. For a set $D \subseteq \Delta(S)$, denote the closure of D in the relative weak* topology by $\overline{D}$.

The support of a probability measure $m \in \Delta(\Delta(S))$, denoted $\operatorname{supp} m$, is a relative weak* closed set such that $m((\operatorname{supp} m)^c) = 0$ and if $L \cap \operatorname{supp} m \neq \emptyset$ for relative weak* open L, $m(L \cap \operatorname{supp} m) > 0$. (See e.g., Aliprantis and Border [2, p.441].)

Let $\Psi_n(\omega) \in \Delta(S)$ denote the empirical frequency operator $\Psi_n(\omega)(A) = \frac{1}{n}\sum_{t=1}^{n} I(\omega_t \in A)$ for each event A in S. Define the limiting frequency operator Ψ by $\Psi(\omega)(A) = \lim_n \Psi_n(\omega)(A)$ if the limit exists and 0 otherwise. Also, to map given limiting frequencies or sets of limiting frequencies to events in S^∞, we consider the natural inverses $\Psi^{-1}(\ell) = \{\omega : \Psi(\omega) = \ell\}$ and $\Psi^{-1}(L) = \{\omega : \Psi(\omega) \in L\}$ for $\ell \in \Delta(S)$ and $L \subseteq \Delta(S)$.

For $f \in \mathcal{F}$, $u : X \to \mathbb{R}$ and $D \subseteq \Delta(S)$, let $\tilde{f} : D \to \mathbb{R}$ be the function defined by $\tilde{f}(\ell) = \int u(f)\, d\ell^\infty$ for each $\ell \in D$. Let $\tilde{F} = \left\{\tilde{f} : f \in \mathcal{F}\right\}$. $G : \tilde{F} \to \mathbb{R}$ is *increasing* if $\tilde{f} \geq \tilde{g}$ implies $G\left(\tilde{f}\right) \geq G(\tilde{g})$. $G : \tilde{F} \to \mathbb{R}$ is *isotonic* if $\alpha, \beta \in u(X)$, $\alpha > \beta$ implies $G(\alpha) > G(\beta)$. $G : \tilde{F} \to \mathbb{R}$ is *mixture continuous* if for all $\tilde{f}, \tilde{g}, \tilde{h} \in \tilde{F}$ the sets $\{\lambda \in [0,1] : G(\lambda \tilde{f} + (1-\lambda)\tilde{g}) \geq G(\tilde{h})\}$ and $\{\lambda \in [0,1] : G(\tilde{h}) \geq G(\lambda \tilde{f} + (1-\lambda)\tilde{g})\}$ are closed.

3 Continuous Symmetric Preferences and Relevant Measures

3.1 Continuous Symmetric Preferences

We start by delineating the scope of our theory of relevant measures. The theory will apply to preferences $\succsim$ satisfying the following axioms.

Axiom 1 (Weak Order)**.** *$\succsim$ is complete and transitive.*

Axiom 2 (Monotonicity)**.** *If $f(\omega) \succsim g(\omega)$ for all $\omega \in S^\infty$, $f \succsim g$.*

Monotonicity rules out state-dependence of preferences over X. This allows us to focus on states purely as specifying the resolution of acts.

Axiom 3 (Risk Independence)**.** *For all $x, x', x'' \in X$ and $\alpha \in (0,1)$, $x \succsim x'$ if and only if $\alpha x + (1-\alpha)\, x'' \succsim \alpha x' + (1-\alpha)\, x''$.*

This is the standard vNM independence axiom on lotteries. This rules out non-expected utility preferences over lotteries. It allows us to separate attitudes toward risk from other aspects of preferences in a simple way, using a familiar vNM utility function.

Axiom 4 (Non-triviality)**.** *There exist $x, y \in X$ such that $x \succ y$.*

The key axiom is Event Symmetry which implies that the ordinates of S^∞ are viewed as interchangeable.

Axiom 5 (Event Symmetry)**.** *For all finite cylinder events $A \in \Sigma$ and finite permutations $\pi \in \Pi$,*

$$\alpha 1_A + (1-\alpha)h \sim \alpha 1_{\pi A} + (1-\alpha)h \text{ for all } \alpha \in [0,1] \text{ and all acts } h \in \mathcal{F}. \tag{3.1}$$

This symmetry says that the decision maker is always indifferent between betting on an event and betting on its permutation. The use of the term "always" here means at least that this preference should hold no matter what other act the individual faces in combination with the bet. In an Anscombe-Aumann framework such as ours, this is expressed by (3.1). In the language of Ghirardato and Siniscalchi [24], note that, thinking of acts as state-contingent utility consequences of actions and h as a status quo, (3.1) says a move away from the status quo in the direction of 1_A is indifferent to the same size move away from the status quo in the direction of $1_{\pi A}$ no matter what the status quo h and no matter how far one moves away from it. The idea behind Event Symmetry is that such utility transfers are considered indifferent because the ordinates are viewed as (ex-ante) identical. Observe that for preferences satisfying the usual Anscombe-Aumann independence axiom, $1_A \sim 1_{\pi A}$ implies $\alpha 1_A + (1-\alpha)h \sim \alpha 1_{\pi A} + (1-\alpha)h$ for all $\alpha \in [0,1]$ and all acts h. For preferences that may violate independence (e.g., because of ambiguity concerns), this is not true, and thus we cannot substitute the former condition for the latter. Appendix B weakens Event Symmetry so as to accommodate preferences symmetric only conditional on some set of observables.

Remark 3.1. As written, Event Symmetry seems to depend on the choice of x^*, x_* in defining 1_A. In fact, in the presence of our other axioms, Event Symmetry implies that the analogous property holds for any choice of $x_*, x^* \in X$.

Combining all of these axioms defines symmetric preferences:

Definition 3.1. $\succsim$ satisfies *Symmetry* if it satisfies Weak Order, Monotonicity, Risk Independence, Non-triviality, and Event Symmetry.

When we say that $\succsim$ is Symmetric, we mean that it satisfies Symmetry.

In addition to Symmetry, we will often need some form of continuity of preference. We now state two forms of continuity that are used in the paper. The first, Mixture Continuity, is a standard axiom in the literature and serves partly to ensure a real-valued representation.

Axiom 6 (Mixture Continuity)**.** *For all $f, g, h \in \mathcal{F}$, the sets $\{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succsim h\}$ and $\{\lambda \in [0,1] : h \succsim \lambda f + (1-\lambda) g\}$ are closed in $[0,1]$.*

Our second continuity axiom will allow us to restrict attention to countably additive (as opposed to finitely additive) measures in the representations we obtain. To describe this axiom, it is notationally convenient to introduce the binary relation $\succsim^*$ (see e.g., Ghirardato, Maccheroni and Marinacci [22]) derived from $\succsim$:

$$f \succsim^* g \text{ if } \alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h \text{ for all } \alpha \in [0,1] \text{ and } h \in \mathcal{F}. \tag{3.2}$$

The axiom applies Arrow [5]'s monotone continuity to $\succsim^*$, as was done in Ghirardato, Maccheroni and Marinacci [22].

Axiom 7 (Monotone Continuity of $\succsim^*$)**.** *For all $x, x', x'' \in X$, if $A_n \searrow \emptyset$ and $x' \succ x''$, then $x' \succsim^* x A_n x''$ for some n.*

Definition 3.2. $\succsim$ satisfies *Continuous Symmetry* if it is Symmetric, Mixture Continuous and satisfies Monotone Continuity of $\succsim^*$.

When we say that $\succsim$ is Continuous Symmetric, we mean that it satisfies Continuous Symmetry. This is the class of preferences that is the focus of the paper. Our next result provides a representation for any Continuous Symmetric preference.

Theorem 3.1. *If $\succsim$ is Continuous Symmetric, then there is a non-constant vNM utility function u on X, a set $D \subseteq \Delta(S)$ and an isotonic, mixture continuous, and increasing functional G on $\tilde{F}$ such that*

$$U(f) \equiv G\left(\left(\int u(f)\, d\ell^{\infty}\right)_{\ell \in D}\right) \tag{3.3}$$

represents $\succsim$.

Additionally, if $\widehat{G}\left(\left(\int \widehat{u}(f)\, d\ell^{\infty}\right)_{\ell \in \widehat{D}}\right)$ is any other such representation of $\succsim$, $\widehat{u}$ is a positive affine transformation of u.

Proof. All proofs are contained in Appendix C. □

Remark 3.2. Given any functional of the form (3.3), the $\succsim$ derived from it must be Symmetric and satisfy Mixture Continuity, but need not satisfy Monotone Continuity of $\succsim^*$. If, additionally, D is finite then the $\succsim$ must be Continuous Symmetric. In Appendix A, we provide additional conditions on G such that, if the Locally Bounded Improvements axiom of Ghirardato and Siniscalchi [24] is added to the Continuous Symmetry axioms, the representation with general D is both necessary and sufficient.

The theorem shows that any Continuous Symmetric preference may be described by specifying (1) a set of marginals, D, (2) a vNM utility function, u, and (3) a function, G, aggregating the expected utilities with respect to the i.i.d. products of elements of D. Under slightly different assumptions, the fact that the set of expected utilities with respect to all i.i.d. measures can be monotonically aggregated to represent preferences was shown in Al-Najjar and De Castro [3]. Note that G and D are not generally unique. The remainder of this section identifies the unique essential elements of D and shows how they relate to perceived ambiguity and comparative ambiguity aversion.

3.2 Relevant measures

We define what it means for a marginal $\ell \in \Delta(S)$ to be relevant according to Continuous Symmetric preferences $\succsim$. We then show that the set of relevant measures is closed and that, a measure is not relevant if and only if the limiting frequency event generated by some open neighborhood of that measure is null according to the preferences. Finally, we relate relevant measures to the representation of $\succsim$ in (3.3).

For notational convenience, let $\mathcal{O}_\ell$ be the collection of open subsets of $\Delta(S)$ that contains ℓ. That is, for $\ell \in \Delta(S)$, $\mathcal{O}_\ell = \{L \subseteq \Delta(S) : L \text{ is open}, \ell \in L\}$.

Definition 3.3. A measure $\ell \in \Delta(S)$ is *relevant* (according to preferences $\succsim$) if, for any $L \in \mathcal{O}_\ell$, there are $f, g \in \mathcal{F}$ such that $f \nsim g$ and $\int f d\hat{\ell}^\infty = \int g d\hat{\ell}^\infty$ for all $\widehat{\ell} \in \Delta(S) \backslash L$.

In words, ℓ is relevant if it satisfies the following property: For each open set containing ℓ, there are acts that are not indifferent despite generating identical induced distributions over outcomes when any measure outside this set governs the independent realization of each ordinate S. The use of open neighborhoods is required only because $\Delta(S)$ is infinite. Why is it enough to consider equality of the lotteries generated by f and g for i.i.d. measures, $\hat{\ell}^\infty$? Under Continuous Symmetry, Theorem 3.1 shows that non-i.i.d. measures are not needed to represent preferences. Furthermore, as Continuous Symmetry implies expected utility on constant acts, one could replace $\int f d\hat{\ell}^\infty = \int g d\hat{\ell}^\infty$ by the analogous condition on expected utilities, $\int u(f) d\hat{\ell}^\infty = \int u(g) d\hat{\ell}^\infty$, without changing the meaning of the definition within our theory. Given any Continuous Symmetric $\succsim$, the relevant measures are uniquely determined.

This definition is in the spirit of "non-null" as the term is applied to events (e.g., Savage [39]).[5] An event is *non-null* if there are acts $f \nsim g$ such that $f = g$ on all states outside of that event. An event is *null* otherwise. We consider open sets of measures, $L \in \mathcal{O}_\ell$, instead of events, and $\int f d\hat{\ell}^\infty = \int g d\hat{\ell}^\infty$ for all other measures $\widehat{\ell}$ instead of $f = g$ on all other states. Our next result makes this connection explicit. In reading it, recall that, for $A \subseteq \Delta(S)$, $\Psi^{-1}(A)$ is the event that limiting frequencies over S lie in A. We use R to denote the set of relevant measures.

Theorem 3.2. *Assume $\succsim$ is Continuous Symmetric. For $\ell \in \Delta(S)$, $\ell \notin R$ if and only if, for some $L \in \mathcal{O}_\ell$, $\Psi^{-1}(L)$ is a null event. Moreover, R is closed.*

When R is finite, the same result holds without the use of neighborhoods, i.e., $\Psi^{-1}(\ell)$ is null if and only if $\ell \notin R$. The above result justifies thinking of R as the unique set of marginals subjectively viewed as possible, since the individual behaves as if only those outside of R are impossible. In this sense, relevant measures represent part of perceived ambiguity, as they are the measures an individual reveals that he treats as possible resolutions of his ambiguity. Specifically, given that perceived ambiguity is

[5] The definition is also reminiscent of the definition of relevant subjective state in Dekel, Lipman and Rustichini [10, Definition 1]. In the case of a finite subjective state space, a state is relevant if there are two menus $x \nsim y$, the valuations of which coincide on all other subjective states. The infinite case uses open neighborhoods just as we do.

subjective uncertainty about probability assignments, under Continuous Symmetry the relevant measures are the probability assignments in the support of that uncertainty.

Our next result connects the relevant measures to the representation of Continuous Symmetric preferences.

Theorem 3.3. *Any Continuous Symmetric $\succsim$ may be represented as in (3.3) setting $D = R$, and in any representation of $\succsim$ as in (3.3), $\overline{D} \supseteq R$. If every measure in D is relevant, then $\overline{D} = R$. Furthermore, in any representation (3.3) of $\succsim$, for $\ell \in D$, $\ell \notin R$ if and only if there exists an $L \in \mathcal{O}_\ell$ such that, $\tilde{f}, \tilde{g} \in \tilde{F}$* and *$\tilde{f}(\hat{\ell}) = \tilde{g}(\hat{\ell})$ for all $\hat{\ell} \notin L$ implies $G(\tilde{f}) = G(\tilde{g})$.*

The theorem says that in representing Continuous Symmetric preferences by (3.3), only relevant measures need appear, and, up to closure, all of them must do so. That is, R is the unique closed subset of marginals that are necessary and sufficient for D in such representations. Furthermore, any measures in D that are not relevant may be identified through the property that the aggregator G appearing in (3.3) is not responsive to the expected utilities generated by (some open neighborhood of) such measures.

3.3 Comparative ambiguity aversion and relevant measures

We now investigate the relationship between comparative ambiguity aversion and relevant measures. We adopt the following established notion for what it means for one preference to be more ambiguity averse than another. Definition 3.4 is essentially a restatement of the Epstein [14, (2.3)] and Ghirardato and Marinacci [23, Definition 4] definitions of comparative uncertainty/ambiguity aversion as applied to our Anscombe-Aumann setting. All these definitions, in turn, are natural adaptations to ambiguity aversion of Yaari [45]'s classic formulation of comparative (subjective) risk aversion.

Definition 3.4. $\succsim_A$ is *more ambiguity averse than* $\succsim_B$ if, for all $x \in X$ and $f \in F$, $f \succsim_A x \Rightarrow f \succsim_B x$ and $f \precsim_B x \Rightarrow f \precsim_A x$.

The difference from Yaari's definition is that he requires the implications only when x is an outcome (i.e., a degenerate lottery). Because Definition 3.4 applies to all lotteries x, it holds risk aversion fixed – if two preferences can be ordered in terms of ambiguity aversion then they must rank lotteries in the same way.

Yaari's definition, under sufficient conditions on the utility function (e.g., differentiability), implies that SEU preferences can be ranked in terms of risk aversion only if they share a common subjective probability measure. Thus changes in the subjective probability measure can neither increase nor decrease risk aversion. Analogously, our next result provides a sufficient condition so that, when Definition 3.4 is applied to Continuous Symmetric preferences, preferences may be ranked in terms of ambiguity aversion only if they share the same set of relevant measures. In this way, changes in relevant measures are shown to neither increase nor decrease ambiguity aversion.

Our sufficient condition is a form of strict monotonicity that is weak enough to fit many cases of interest, including the applications in the next section.

Definition 3.5. $\succsim$ is *strictly monotonic for bets on non-null limiting frequency events* if, for all $L \subseteq \Delta(S)$ such that $\Psi^{-1}(L)$ is a non-null event, and all $x, y, z \in X$ such that $x \succ y$, $x\Psi^{-1}(L)\,z \succ y\Psi^{-1}(L)\,z$.

Theorem 3.4. *Let $\succsim_A$ and $\succsim_B$ be Continuous Symmetric preferences with sets of relevant measures R_A and R_B and suppose $\succsim_A$ and $\succsim_B$ are strictly monotonic for bets on non-null limiting frequency events. Then $\succsim_A$ more ambiguity averse than $\succsim_B$ implies $R_A = R_B$.*

4 Relevant measures in specific decision models

We examine Continuous Symmetric versions of models from the ambiguity literature. We identify the relevant measures and describe the implications of comparative ambiguity aversion within these models. This section considers α-MEU, smooth ambiguity and Bewley-style representations. The online supplement (Klibanoff, Mukerji and Seo [32]) considers additional models.

4.1 The α-MEU model

Consider preferences having an α-MEU representation where the set of measures in the representation is a finite set of i.i.d. measures:

$$\alpha \min_{p\in\{\ell^{\infty}:\ell\in D\}} \int u(f)\,dp + (1-\alpha) \max_{p\in\{\ell^{\infty}:\ell\in D\}} \int u(f)\,dp \tag{4.1}$$

where $D \subseteq \Delta(S)$ is finite, u is a non-constant vNM utility function and $\alpha \in [0,1]$.[6] Call such preferences *i.i.d. α-MEU.*

We show that the relevant measures are the set D.

Theorem 4.1. *For any i.i.d. α-MEU preference, $R = D$.*

We next characterize comparative ambiguity aversion for these preferences.

Theorem 4.2. *Let $\succsim_A$ and $\succsim_B$ be two i.i.d. α-MEU preferences such that $\alpha_A, \alpha_B \in (0,1)$ and D_B is non-singleton. Then, $\succsim_A$ is more ambiguity averse than $\succsim_B$ if and only if $\alpha_A \geq \alpha_B$, $D_A = D_B$ and (up to normalization) $u_A = u_B$.*

The theorem tells us that, as long as D is non-singleton (so that α affects preferences), increases in ambiguity aversion correspond exactly to increases in α. Note that the "if" direction of the result holds for any $\alpha_A, \alpha_B \in [0,1]$, as $\alpha \in (0,1)$ is used only to ensure that preferences are strictly monotonic for bets on non-null limiting frequency

[6] Finiteness of D is necessary for these preferences to satisfy Monotone Continuity of $\succsim^*$.

events which is needed for the “only if” direction. Applying the most closely related result in the literature (Ghirardato, Maccheroni and Marinacci [22, Proposition 12]) to i.i.d. α-MEU preferences delivers only the following: If $D_A = D_B$, $u_A = u_B$ (up to normalization), and either $\alpha_A = 1$ and $\alpha_B = 0$ or $\alpha_A = \alpha_B$ then $\succsim_A$ is more ambiguity averse than $\succsim_B$. This limitation to α equals 0 or 1 is related to the critique in Eichberger et al. [12]. Part of Section 4.3 explains why applying Ghirardato, Maccheroni and Marinacci [22, Proposition 12] is informative only for α equals 0 or 1 and how Continuous Symmetry and the focus on relevant measures allows us to say more.

4.2 The Smooth Ambiguity model

Consider preferences having a smooth ambiguity representation where the support of the second-order measure consists of i.i.d. measures on the state space:

$$\int_{\Delta(S)} \phi\left(\int u(f) d\ell^{\infty}\right) d\mu\left(\ell\right) \tag{4.2}$$

where u is a non-constant vNM utility function, $\phi : u(X) \to \mathbb{R}$ is a strictly increasing continuous function, $\mu \in \Delta(\Delta(S))$ and either (i) there are $m, M > 0$ such that $m\left|a-b\right| \leq \left|\phi\left(a\right)-\phi\left(b\right)\right| \leq M\left|a-b\right|$ for all $a, b \in u\left(X\right)$ or, (ii) $\operatorname{supp}\mu$ is finite.[7] Call such preferences *i.i.d. smooth ambiguity.*

Theorem 4.3. *For any i.i.d. smooth ambiguity preference,* $R = \operatorname{supp}\mu$.

The result says that for i.i.d. smooth ambiguity preferences, the relevant measures are exactly the support of the second-order measure μ. Given this and the fact that these preferences satisfy strict monotonicity for bets on non-null events (since ϕ is strictly increasing), Theorem 3.4 implies that one preference is more ambiguity averse than another only if the support of the μ's for the two preferences coincide and (up to normalization) the associated u's are equal.

We next characterize comparative ambiguity aversion for these preferences. Since equality of the u's (up to normalization) is necessary for one preference to be more ambiguity averse than another, to ease the statement of the characterization, we simply assume this.

Theorem 4.4. *Let* $\succsim_A$ *and* $\succsim_B$ *be any two i.i.d. smooth ambiguity preferences such that* $u_A = u_B$*,* ϕ_A *and* ϕ_B *are continuously differentiable and* $\operatorname{supp}\mu_A$ *and* $\operatorname{supp}\mu_B$ *are non-singleton. Then,* $\succsim_A$ *is more ambiguity averse than* $\succsim_B$ *if and only if* ϕ_A *is more concave than* ϕ_B *and* $\mu_A = \mu_B$*.*

Continuous Symmetry allows our characterization to improve on that in Klibanoff, Marinacci and Mukerji [30, Theorem 2] in that equality of μ is part of the characterization rather than an assumption.

[7] The requirement that either (i) or (ii) is satisfied is necessary for these preferences to satisfy Monotone Continuity of $\succsim^*$.

4.3 A Bewley-style representation

Recall, from (3.2), the induced relation $\succsim^*$. This is the maximal sub-relation of $\succsim$ that satisfies the Anscombe-Aumann independence axiom. For Continuous Symmetric preferences violating independence, $\succsim^*$ will be incomplete. We provide a generalization of de Finetti's theorem based on a Bewley-style (Bewley [6]) representation result for $\succsim^*$ and show where the set of relevant measures appear in this representation. Compared to other Bewley-style representation results (e.g., Ghirardato, Maccheroni and Marinacci [22], Gilboa et. al. [26], Ghirardato and Siniscalchi [24], Nehring [35]) our key contribution is in showing that Continuous Symmetry allows a de Finetti-style decomposition of the representing set of measures, C, and in explaining the relationship between C and R. This relationship is of particular interest because C is the main alternative offered in the literature as possibly representing perceived ambiguity (see Ghirardato, Maccheroni and Marinacci [22], Nehring ([35],[36]), Ghirardato and Siniscalchi [24], Siniscalchi [41]).

In light of the possible incompleteness of $\succsim^*$, to state the representation result as an if and only if characterization, we need to weaken two of the Continuous Symmetry axioms. We replace Weak Order and Mixture Continuity with the following:

Axiom 8 (C-complete Preorder)**.** *$\succsim$ is reflexive, transitive and the restriction of $\succsim$ to X is complete.*

Axiom 9 (Mixture Continuity of $\succsim^*$)**.** *For all $f, g, h \in \mathcal{F}$, the sets $\{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succsim^* h\}$ and $\{\lambda \in [0,1] : h \succsim^* \lambda f + (1-\lambda) g\}$ are closed in $[0,1]$.*

Refer to a binary relation satisfying this weakened version of the Continuous Symmetry axioms as *Weak Continuous Symmetric.*

Theorem 4.5. *$\succsim$ is Weak Continuous Symmetric if and only if $\succsim$ is transitive and there exist a non-empty compact convex set $M \subseteq \Delta(\Delta(S))$ and a non-constant vNM utility function u such that*

$$f \succsim^* g \text{ if and only if } \int u(f)\, dp \geq \int u(g)\, dp \text{ for all } p \in C, \tag{4.3}$$

where $C = \left\{\int \ell^\infty dm(\ell) : m \in M\right\}$. Furthermore $R = \overline{\bigcup_{m \in M} \operatorname{supp} m}$ and M is unique.

De Finetti's theorem (see Hewitt and Savage [29]) says that for an expected utility preference displaying indifference under finite permutations (i.e., $f \sim \pi f$), the subjective probability measure is an exchangeable measure and thus can be described by a unique probability measure over marginals of i.i.d. measures. Theorem 4.5 expands on this in several respects. First, it applies to a much larger class of preferences. Second, instead of a single probability measure over marginals there is a unique compact convex set of such measures, M, used to represent $\succsim^*$. Finally, Theorem 4.5 identifies the relevant measures according to $\succsim$ from the supports of measures in M.

Theorem 4.5 features the set C and shows its exact relationship to R. For any Continuous Symmetric $\succsim$, just as $\succsim$ may be represented as in (3.3) with $D = R$, Cerreia-Vioglio et. al [7, Proposition 5] implies that $\succsim$ may be represented by a monotonic aggregation of expected utilities with respect to the measures in C. For preferences satisfying Definition 3.5, our results on comparative ambiguity aversion allow us to show that C, but not R, may be affected by increases or decreases in ambiguity aversion. Therefore, despite the fact that preferences can be represented using the exchangeable measures that make up C, C cannot be said to reflect only perceived ambiguity.

To understand this, consider two examples. First, consider i.i.d. α-MEU preferences. By Theorem 4.1, $R = D$. The allowed weights over D for m to be an element of M are determined completely by α.[8] For $\alpha \in (0, 1)$ and D non-singleton, Theorem 4.2 shows that α reflects comparative ambiguity aversion. Thus, for these preferences (which do not include SEU as a special case), the weights determined by the m (and thus C) are affected by ambiguity aversion, while the supports of the m (and thus R) are not. In particular, when $\alpha \in (0, 1)$, holding C fixed holds α fixed as well. This explains why, as mentioned in Section 4.1, the approach of Ghirardato, Maccheroni and Marinacci [22, Proposition 12] characterizing more ambiguity averse while holding C fixed is not informative when $\alpha \in (0, 1)$.

The second example comes from i.i.d. smooth ambiguity preferences with ϕ twice continuously differentiable. By Theorem 4.3, $R = \operatorname{supp} \mu$. The allowed weights over $\operatorname{supp} \mu$ for m to be an element of M are determined by a combination of ϕ' and μ.[9] To illustrate, suppose $\phi(x) = -\exp(-\theta x)$ for some $\theta > 0$, $\mu(\ell_1) = \frac{1}{2} = \mu(\ell_2)$ and $u(X) = [-1, 1]$. Then $M = \left\{ m \in \Delta(\Delta(S)) : m(\ell_1) = \lambda = 1 - m(\ell_2) \text{ and } \lambda \in \left[\frac{\exp(-\theta)}{\exp(-\theta)+\exp(\theta)}, \frac{\exp(\theta)}{\exp(-\theta)+\exp(\theta)} \right] \right\}$. Thus, as ϕ becomes more concave (i.e., θ increases), the set M (and thus C) gets strictly larger. Theorem 4.4 shows that ϕ more concave reflects more ambiguity aversion. Therefore, C is again affected by changes in ambiguity aversion, while R is not.[10] Contrast this with the i.i.d. smooth ambiguity preference with the same μ and u but with $\phi(x) = x$. This is an SEU preference, and C is a singleton. Specifically, $M = \{\mu\}$.

As Theorem 4.5 has shown, when Anscombe-Aumann independence is violated, C is determined, in part, by the multiple weightings defined by $m \in M$ being applied over R. These weightings, as in the examples, may depend on ambiguity aversion. This explains our focus on the relevant measures rather than C.

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[8] Specifically M in this case is the closed convex hull of the set $\{m \in \Delta(\Delta(S)) : m(\ell) = \alpha = 1 - m(\ell') \text{ with } \ell, \ell' \in D \text{ and } \ell \neq \ell'\}$.

[9] Specifically M in this case is the closed convex hull of the set $\left\{ m \in \Delta(\Delta(S)) : dm(\ell) = \frac{\phi'(\int e d\ell^\infty) d\mu(\ell)}{\int \phi'(\int e d\ell^\infty) d\mu(\ell)} \text{ and } e \in \operatorname{int} B(\Sigma, u(X)) \right\}$ where $\operatorname{int} B(\Sigma, u(X))$ is the interior of the set of all Σ-measurable functions $a : S^\infty \to \mathbb{R}$ for which there exist $\alpha, \beta \in u(X)$ satisfying $\alpha \leq a(\omega) \leq \beta$ for all $\omega \in S^\infty$ (informally, the interior of the set of bounded utility acts).

[10] Beyond this example, when $\operatorname{supp} \mu$ is non-singleton, it may be shown that ϕ restricts the measures over $\operatorname{supp} \mu$ that belong to M if and only if $\sup \bigcup_{r,t \in u(X)} \frac{\phi'(t)}{\phi'(r)} < +\infty$.

A Appendix A: An Equivalence Theorem

Consider the following axiom adopted from Ghirardato and Siniscalchi [24].

Axiom 10 (Locally Bounded Improvements)**.** *For every $h \in \mathcal{F}^{int}$, there are $y \in X$ and $g \in \mathcal{F}$ with $g(\omega) \succ h(\omega)$ for all ω such that for all $h_n \in \mathcal{F}$, $h_n \sim x_n \in X$ and $\lambda_n \subseteq [0,1]$ with $h_n \to h$ and $\lambda_n \searrow 0$*

$$\lambda^n g + (1-\lambda^n) h^n \prec \lambda^n y + (1-\lambda^n) x_n \textit{ eventually.}$$

In this Appendix, we provide an if and only if representation theorem for Continuous Symmetric preferences that also satisfy Locally Bounded Improvements. Note that all of the complete preferences used in Section 4 and the Online Supplement satisfy Locally Bounded Improvements.

Additional definitions: For locally Lipschitz $G : \tilde{F} \to \mathbb{R}$, the Clarke derivative of G at $\tilde{h}$ in the direction $\tilde{f}$ is $G^\circ\left(\tilde{h};\tilde{f}\right) = \limsup_{\tilde{g}\to\tilde{h}, t\searrow 0} \frac{G\left(\tilde{g}+t\tilde{f}\right)-G(\tilde{g})}{t}$. The Clarke differential of G at $\tilde{h}$ is $\partial G\left(\tilde{h}\right) = \left\{Q \in ba(D) : Q\left(\tilde{f}\right) \leq G^\circ\left(\tilde{h};\tilde{f}\right) \text{ for all } \tilde{f} \in \tilde{F}\right\}$. The normalized differential is $C\left(\tilde{h}\right) = \left\{Q/Q(D) : Q \in \partial G\left(\tilde{h}\right), Q(D) > 0\right\}$.

Theorem A.1. *The following are equivalent:*
(1) $\succsim$ is Continuous Symmetric and satisfies Locally Bounded Improvements, and
(2) There is a non-constant vNM utility function u on X, a set $D \subseteq \Delta(S)$ and an isotonic, mixture continuous and increasing functional G on $\tilde{F}$ such that i) G is locally Lipschitz in its interior, ii) $\overline{co}\left(\bigcup_{\tilde{h}\in int\tilde{F}} C\left(\tilde{h}\right)\right)$ is compact, iii)$G\left((c)_{\ell\in D}\right) = c$ for all $c \in u(X)$ and iv)

$$U(f) \equiv G\left(\left(\int u(f)\, d\ell^\infty\right)_{\ell\in D}\right) \tag{A.1}$$

represents $\succsim$.

Additionally, if $\widehat{G}\left(\left(\int \widehat{u}(f)\, d\ell^\infty\right)_{\ell\in\widehat{D}}\right)$ is any other such representation of $\succsim$, $\widehat{u}$ is a positive affine transformation of u.

B Appendix B: Relevance under Heterogeneous Environments

A decision maker may face a situation where non-identical experiments are repeated. For example, a doctor faces patients who may differ in ways important for the treatment problem at hand. Another example is an agent who wants to make a decision based on a regression model analysis where different data points may have different values of the regressors. We describe a variation of our model that allows these heterogeneous environments.

Let Ξ be a set of descriptions. We assume $\Xi = \left\{\xi^1, ..., \xi^K\right\}$ is a finite set for simplicity. Descriptions categorize the ordinates (of S^∞) so that it is only ordinates with the same description that are viewed as symmetric by the decision maker. Formally, we augment the state space S^∞ by attaching a description to each ordinate S. Thus, for a doctor facing many patients, each patient has a description $\xi \in \Xi$. A doctor faces a sequence of patients whose descriptions may be different from each other. Let $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2, \dots) \in \Xi^\infty$ be a sequence such that each element of Ξ appears infinitely often. Let $\succsim_{\tilde{\xi}}$ be a preference on $\mathcal{F}$ when faced with ordinates whose descriptions form the sequence $\tilde{\xi}$.

We assume the same axioms as in Section 3.1 on $\succsim_{\tilde{\xi}}$ with the exception of Event Symmetry. Instead we assume Partial Event Symmetry.

Axiom 11 (Partial Event Symmetry)**.** *For all finite cylinder events $A \in \Sigma$ and finite permutations $\pi \in \Pi$* such that $\tilde{\xi}_i = \tilde{\xi}_{\pi(i)}$ for all i,

$$\alpha 1_A + (1-\alpha)h \sim_{\tilde{\xi}} \alpha 1_{\pi A} + (1-\alpha)h \textit{ for all } \alpha \in [0,1] \textit{ and all acts } h \in \mathcal{F}.$$

Partial Event Symmetry says that an agent views ordinates with the same descriptions in the same way – as long as the descriptions are the same, the order does not matter. In contrast, no restrictions are placed on preferences towards ordinates that have different descriptions. For two ordinates with different descriptions, there is no reason to believe that the two are symmetric. Viewing our earlier framework as one in which there was only one possible description, Partial Event Symmetry is the natural generalization of Event Symmetry.

Formally, therefore, we replace the assumption of *Continuous Symmetry* with *Continuous Partial Symmetry*:

Definition B.1. $\succsim_{\tilde{\xi}}$ satisfies *Continuous Partial Symmetry* if it satisfies Weak Order, Monotonicity, Risk Independence, Non-triviality, Partial Event Symmetry, Mixture Continuity and $\succsim^*_{\tilde{\xi}}$ satisfies Monotone Continuity.

Now, we can define relevant measures under heterogeneous environments. Since beliefs may vary depending on descriptions, a relevant measure is a mapping $\boldsymbol{l}$ from Ξ into $\Delta(S)$. Let $\mathcal{O}_{\boldsymbol{l}}$ denote an open subset of $(\Delta(S))^\Xi$ containing $\boldsymbol{l}$ under the product topology. For $\boldsymbol{l} \in (\Delta(S))^\Xi$, denote by $\boldsymbol{l}\left(\tilde{\xi}\right)$ the product measure on S^∞ whose i-th coordinate marginal is $\boldsymbol{l}\left(\tilde{\xi}_i\right) \in \Delta(S)$. That is, $\boldsymbol{l}\left(\tilde{\xi}\right) = \boldsymbol{l}\left(\tilde{\xi}_1\right) \otimes \boldsymbol{l}\left(\tilde{\xi}_2\right) \otimes ....$

Definition B.2. A mapping $\boldsymbol{l} \in (\Delta(S))^\Xi$ is *relevant* (according to preferences $\succsim_{\tilde{\xi}}$) if, for any $L \in \mathcal{O}_{\boldsymbol{l}}$, there are $f, g \in \mathcal{F}$ such that $f \nsim_{\tilde{\xi}} g$ and $\int f d\hat{\boldsymbol{l}}\left(\tilde{\xi}\right) = \int g d\hat{\boldsymbol{l}}\left(\tilde{\xi}\right)$ for all $\hat{\boldsymbol{l}} \in (\Delta(S))^\Xi \backslash L$.

When $\Xi = \{\xi\}$ is a singleton, $\tilde{\xi} = (\xi, \xi, ...)$ and, therefore, it is as if $L \subseteq \Delta(S)$ and each $\hat{\boldsymbol{l}}\left(\tilde{\xi}\right)$ is i.i.d., and the above definition reduces to our earlier definition of relevant measures (Definition 3.3). As before we use R to denote the set of relevant mappings.

A standard linear regression is the case where the relevant measure is $\boldsymbol{l}$ and $\boldsymbol{l}(\xi_i)$ is normal with mean $\beta\xi_i$ and variance σ^2. Note that the description in this case is simply a vector giving the values of the regressors for a particular observation. An example of a set of relevant measures might be $\{\boldsymbol{l} \in (\Delta(S))^{\Xi} : \boldsymbol{l}(\xi_i)$ is normal with mean $\beta\xi_i$ and variance 1 for $\beta \in [\underline{b}, \overline{b}]^2\}$. This reflects knowledge of normality and the variance, and bounds on the coefficients within which any coefficients are seen as possible.

Relative to the homogeneous case, this framework: (1) allows for ordinates to differ according to Ξ, and (2) allows relevant measures to reflect beliefs about how the marginals for one $\xi \in \Xi$ relate to the marginals for another $\xi' \in \Xi$. This last point is useful, for example, in capturing the case, mentioned above, where Ξ is related to S according to a linear regression model.

We provide results analogous to those in the homogeneous case:

Theorem B.1. *If $\succsim_{\tilde{\xi}}$ is Continuous Partial Symmetric, then there is a non-constant vNM utility function u on X, a set $D \subseteq (\Delta(S))^{\Xi}$ and an isotonic, mixture continuous, and increasing functional G on $\tilde{F}$ such that*

$$U(f) \equiv G\left(\left(\int u(f)\, d\boldsymbol{l}\left(\tilde{\xi}\right)\right)_{\boldsymbol{l}\in D}\right) \tag{B.1}$$

represents $\succsim_{\tilde{\xi}}$.

Additionally, if $\widehat{G}\left(\left(\int \widehat{u}(f)\, d\boldsymbol{l}\left(\tilde{\xi}\right)\right)_{\boldsymbol{l}\in\widehat{D}}\right)$ is any other such representation of $\succsim_{\tilde{\xi}}$, $\widehat{u}$ is a positive affine transformation of u.

Define $\Psi(\omega) \in (\Delta(S))^{\Xi}$ by $\Psi(\omega)\left(\xi^k\right)(A) = \lim_n \left(\sum_{t=1}^n I\left(\tilde{\xi}_t = \xi^k\right)\right)^{-1} \sum_{t=1}^n I\left(\tilde{\xi}_t = \xi^k, \omega_t \in A\right)$ for each event A in S. For a given k, $\Psi(\omega)\left(\xi^k\right)$ gives an empirical frequency limit when considering the experiments of description ξ^k, that is, all coordinates t such that $\tilde{\xi}_t = \xi^k$. If the limit does not exist let $\Psi(\omega)\left(\xi^k\right)(A) = 0$. The inverses are $\Psi^{-1}(\boldsymbol{l}) = \{\omega : \Psi(\omega) = \boldsymbol{l}\}$ and $\Psi^{-1}(L) = \{\omega : \Psi(\omega) \in L\}$ for $\boldsymbol{l} \in (\Delta(S))^{\Xi}$ and $L \subseteq (\Delta(S))^{\Xi}$.

Theorem B.2. *Assume $\succsim_{\tilde{\xi}}$ is Continuous Partial Symmetric. For $\boldsymbol{l} \in (\Delta(S))^{\Xi}$, $\boldsymbol{l} \notin R$ if and only if, for some $L \in \mathcal{O}_{\boldsymbol{l}}$, $\Psi^{-1}(L)$ is a null event. Moreover, R is closed.*

Theorem B.3. *Any Continuous Partial Symmetric $\succsim_{\tilde{\xi}}$ may be represented as in (B.1) setting $D = R$, and in any representation of $\succsim_{\tilde{\xi}}$ as in (B.1), $\overline{D} \supseteq R$. If every mapping in D is relevant, then $\overline{D} = R$. Furthermore, in any representation (B.1) of $\succsim_{\tilde{\xi}}$, for $\boldsymbol{l} \in D$, $\boldsymbol{l} \notin R$ if and only if there exists an $L \in \mathcal{O}_{\boldsymbol{l}}$ such that, $\tilde{f}, \tilde{g} \in \tilde{F}$* and $\tilde{f}(\hat{\boldsymbol{l}}) = \tilde{g}(\hat{\boldsymbol{l}})$ for all $\hat{\boldsymbol{l}} \notin L$ implies $G(\tilde{f}) = G(\tilde{g})$.

Definition B.3. $\succsim_{\tilde{\xi}}$ is *strictly monotonic for bets on non-null limiting frequency events* if, for all $L \subseteq (\Delta(S))^{\Xi}$ such that $\Psi^{-1}(L)$ is a non-null event, and all $x, y, z \in X$ such that $x \succ_{\tilde{\xi}} y$, $x\Psi^{-1}(L)\, z \succ_{\tilde{\xi}} y\Psi^{-1}(L)\, z$.

Theorem B.4. *Let $\succsim_{\tilde{\xi}A}$ and $\succsim_{\tilde{\xi}B}$ be Continuous Partial Symmetric preferences with sets of relevant mappings R_A and R_B and suppose $\succsim_{\tilde{\xi}A}$ and $\succsim_{\tilde{\xi}B}$ are strictly monotonic for bets on non-null limiting frequency events. Then $\succsim_{\tilde{\xi}A}$ is more ambiguity averse than $\succsim_{\tilde{\xi}B}$ implies $R_A = R_B$.*

Theorem B.5. *$\succsim_{\tilde{\xi}}$ satisfies Continuous Partial Symmetry with Mixture Continuity relaxed to Mixture Continuity of $\succsim^*$ and Weak Order relaxed to C-complete Preorder if and only if $\succsim_{\tilde{\xi}}$ is transitive and there exist a non-empty compact convex set $M \subseteq \Delta\left((\Delta S)^{\Xi}\right)$ and a non-constant vNM utility function u such that ,*

$$f \succsim^*_{\tilde{\xi}} g \text{ if and only if } \int u(f)\,dp \geq \int u(g)\,dp \text{ for all } p \in C, \tag{B.2}$$

where $C = \left\{\int \boldsymbol{l}\left(\tilde{\xi}\right) dm(\boldsymbol{l}) : m \in M\right\}$. Furthermore $R = \overline{\bigcup_{m\in M} \operatorname{supp} m} \subseteq (\Delta S)^{\Xi}$ and M is unique.

C Appendix C: Proofs

Denote by $B(S)$ the set of bounded measurable functions on S. Similarly for $B(\Delta(S))$ and $B(S^{\infty})$.

C.1 Proofs of Theorems 3.1 and B.1

The two proofs are essentially the same and we prove the first only.

First, note that Mixture Continuity is stronger than the Archimedean axiom. Given this, it is routine to show that there are a non-constant vNM utility function u on X and a utility function $I : B_0(\Omega) \to \mathbb{R}$ such that I is sup-norm continuous and $U(f) = I(u(f))$ represents $\succsim$ (Cerreia-Vioglio et. al [7, Proposition 1], for example).[11] Theorem 4.5 guarantees the existence of sets C and M derived there from $\succsim$. Let $D = \bigcup_{m\in M} \operatorname{supp} m \subseteq \Delta S$, and define $\tilde{F}$ accordingly. Define G on $\tilde{F}$ by $G(\tilde{f}) = U(f)$, which is well-defined because $\int u(f)\,d\ell^{\infty} = \int u(g)\,d\ell^{\infty}$ for all $\ell \in D$ implies $\int u(f)\,dp = \int u(g)\,dp$ for all $p \in C$, which, by Theorem 4.5, implies $f \sim g$. Thus $f \longmapsto G(\tilde{f})$ represents $\succsim$.

It is straightforward that Mixture Continuity of $\succsim$ implies Mixture Continuity of G.

We next show that G is increasing. For $\hat{f}, \hat{g} \in \tilde{F}$, assume $\hat{f}(\ell) \geq \hat{g}(\ell)$ for each $\ell \in D$. Fix $g \in \mathcal{F}$ so that $\tilde{g} = \hat{g}$. We can take $f \in \mathcal{F}$ such that $u(f(\omega)) = u(g(\omega)) + \hat{f}(\ell) - \hat{g}(\ell)$ for $\omega \in \Psi^{-1}(\ell)$ and $u(f(\omega)) = u(g(\omega)) + \varepsilon$ otherwise, where $\varepsilon > 0$. Then, $\tilde{f} = \hat{f}$ on D. Monotonicity implies that $f \succsim g$ and thus $G(\hat{f}) = U(f) \geq U(g) = G(\hat{g})$. That G is isotonic can be shown similarly.

Uniqueness of u up to positive affine transformations is standard, as $\succsim$ restricted to constant acts is expected utility.

[11] $B_0(\Omega)$ is the set of simple measurable functions on Ω.

C.2 Proof of Remark 3.2

We show that when D is finite in (3.3), Monotone Continuity of $\succsim^*$ is satisfied. Suppose that D is finite. Assume $A_n \searrow \emptyset$ and $x' \succ x''$, and let $u(x') = 1 > u(x') = t > u(x'') = 0$, without loss of generality. Then, because of countable additivity, there is N such that $t \geq \ell^\infty(A_N)$ for all $\ell \in D$. Hence, $\alpha t + (1-\alpha)\int u(h)\, d\ell^\infty \geq \alpha \ell^\infty(A_N) + (1-\alpha)\int h d\ell^\infty$ for all $\alpha \in [0,1]$, $h \in \mathcal{F}$ and $\ell \in D$. Monotonicity of G implies

$$\begin{aligned} U(\alpha x' + (1-\alpha)h) &= G\left(\left(\alpha t + (1-\alpha)\int u(h)\, d\ell^\infty\right)_{\ell \in D}\right) \\ &\geq G\left(\left(\alpha \ell^\infty(A_N) + (1-\alpha)\int u(h)\, d\ell^\infty\right)_{\ell \in D}\right) = G(\alpha x A_n x'' + (1-\alpha)h). \end{aligned}$$

Thus, $x' \succsim^* x A_n x''$ and Monotone Continuity of $\succsim^*$ is satisfied.

C.3 Proofs of Theorems 3.2 and B.2

We prove the first only since the proof of the second is essentially the same.

Assume $\ell \notin R$. By definition, there is $L \in \mathcal{O}_\ell$ such that $\int f' d\hat{\ell}^\infty = \int g' d\hat{\ell}^\infty$ for all $\hat{\ell} \in \Delta(S) \backslash L$ implies $f' \sim g'$. For any f and g, if $f' = f\Psi^{-1}(L)\, g$ and $g' = g$, we have $\int f' d\hat{\ell}^\infty = \int g' d\hat{\ell}^\infty$ for all $\hat{\ell} \in \Delta(S) \backslash L$. Thus $f' \sim g'$, which implies $\Psi^{-1}(L)$ is null.

Assume that, for some $L \in \mathcal{O}_\ell$, $\Psi^{-1}(L)$ is null. Take any $f, g \in \mathcal{F}$ such that $\int f d\hat{\ell}^\infty = \int g d\hat{\ell}^\infty$ for all $\hat{\ell} \in \Delta(S) \backslash L$. Define $f', g' \in \mathcal{F}$ by

$$f'(\omega) = \begin{cases} \int f d\hat{\ell}^\infty & \text{if } \omega \in \Psi^{-1}\left(\hat{\ell}\right) \text{ for some } \hat{\ell} \in \Delta(S) \\ x_* & \text{otherwise} \end{cases}$$

and

$$g'(\omega) = \begin{cases} \int g d\hat{\ell}^\infty & \text{if } \omega \in \Psi^{-1}\left(\hat{\ell}\right) \text{ for some } \hat{\ell} \in \Delta(S) \\ x_* & \text{otherwise} \end{cases}.$$

Then, f' and g' differ only on $\Psi^{-1}(L)$ which is null, and thus $f' \sim g'$. Moreover, $f \sim f'$ and $g \sim g'$ by Theorem 3.1 and $\Psi^{-1}(L)$ being null. By transitivity, $f \sim g$, which implies $\ell \notin R$.

To show that R is closed, take any $\ell \notin R$. Then, there is $L \in \mathcal{O}_\ell$ such that $\Psi^{-1}(L)$ is null. It suffices to show that $L \subseteq R^c$. Take any $\widehat{\ell} \in L$ and observe that $L \in \mathcal{O}_{\widehat{\ell}}$. Since $\Psi^{-1}(L)$ is null, $\widehat{\ell} \notin R$. Thus, $L \subseteq R^c$ and R is closed.

C.4 Proofs of Theorems 3.3 and B.3

Again, we prove the first only.

We invoke Theorem 4.5 to define $C \subseteq \Delta(\Omega)$ and $M \subseteq \Delta(\Delta(S))$ as in that theorem.

Step 1. Let $R' = \bigcup_{m \in M} \operatorname{supp} m$ and note that $R' \subseteq R$, since Theorem 4.5 says $R = \overline{R'}$.

Step 2. D may be R: The proof of Theorem 3.1 sets $D = R'$ when showing that a representation of the form in (3.3) exists. By Step 1, this $D \subseteq R$. Since D may always be enlarged while representing the same preferences (for example by making G unresponsive to the extra expectations), D may be set to R.

Step 3. $\overline{D} \supseteq R$: Assume $\ell \notin \overline{D}$. Noting that $\overline{D}$ is closed, we can take $L \in \mathcal{O}_\ell$ such that $L \subseteq \left(\overline{D}\right)^c$. Since no measure in L appears in the utility function (3.3), $\Psi^{-1}(L)$ is null. Thus, by Theorem 3.2, $\ell \notin R$.

Step 4. If every measure in D is relevant, $\overline{D} = R$: By the assumption, $D \subseteq R$. Since R is closed, $\overline{D} \subseteq R$. Combining with Step 3, we obtain $\overline{D} = R$.

Step 5. (last sentence of the theorem): Since $\ell \notin R$, by Theorem 3.2 there is some $L \in \mathcal{O}_\ell$ such that $\Psi^{-1}(L)$ is null. Suppose that $\tilde{f}(\hat{\ell}) = \tilde{g}(\hat{\ell})$ for all $\hat{\ell} \notin L$. For each $\hat{\ell} \notin L$, take $x_{\hat{\ell}} \in X$ such that $u(x_{\hat{\ell}}) = \tilde{f}(\hat{\ell})$. For each $\hat{\ell} \in L$, take $y_{\hat{\ell}}, z_{\hat{\ell}} \in X$ such that $u(y_{\hat{\ell}}) = \tilde{f}(\hat{\ell})$ and $u(z_{\hat{\ell}}) = \tilde{g}(\hat{\ell})$. Now define $f', g' \in \mathcal{F}$ by

$$f'(\omega) = \begin{cases} x_{\hat{\ell}} & \text{if } \omega \in \Psi^{-1}(\hat{\ell}) \text{ and } \hat{\ell} \notin L \\ y_{\hat{\ell}} & \text{if } \omega \in \Psi^{-1}(\hat{\ell}) \text{ and } \hat{\ell} \in L \\ x_* & \text{otherwise} \end{cases}$$

and

$$g'(\omega) = \begin{cases} x_{\hat{\ell}} & \text{if } \omega \in \Psi^{-1}(\hat{\ell}) \text{ and } \hat{\ell} \notin L \\ z_{\hat{\ell}} & \text{if } \omega \in \Psi^{-1}(\hat{\ell}) \text{ and } \hat{\ell} \in L \,. \\ x_* & \text{otherwise} \end{cases}$$

Then, f' and g' differ only on $\Psi^{-1}(L)$, and thus $f' \sim g'$. Moreover, since $\tilde{f}' = \tilde{f}$ and $\tilde{g}' = \tilde{g}$, we have $G(\tilde{f}) = G(\tilde{f}') = G(\tilde{g}') = G(\tilde{g})$.

For the other direction, fix ℓ and suppose there exists an $L \in \mathcal{O}_\ell$ such that, $\tilde{f}, \tilde{g} \in \tilde{F}$ and $\tilde{f}(\hat{\ell}) = \tilde{g}(\hat{\ell})$ for all $\hat{\ell} \notin L$ implies $G(\tilde{f}) = G(\tilde{g})$. By the representation (3.3) and the definition of relevant measure (Definition 3.3), $\ell \notin R$.

C.5 Proof of Theorems 3.4 and B.4

Suppose $\succsim_A$ is more ambiguity averse than $\succsim_B$ and each preference is strictly monotonic for bets on non-null limiting frequency events. Since $\succsim_A$ is more ambiguity averse than $\succsim_B$ implies the two preferences agree on lotteries, fix lotteries $x, y \in X$ such that $x \succ_A y$ and $x \succ_B y$.

Suppose $R_A \not\supseteq R_B$ and fix $\ell' \in R_B \backslash R_A$. Since $\ell' \notin R_A$, by Theorem 3.2 there exists an $L' \in \mathcal{O}_{\ell'}$ such that $\Psi^{-1}(L')$ is null for $\succsim_A$, and so $x \sim_A y \Psi^{-1}(L')\, x$. Observe that $\succsim_B$ strictly monotonic for bets on non-null limiting frequency events and $\ell' \in R_B$ implies $x \succ_B y \Psi^{-1}(L')\, x$. Thus, $y \Psi^{-1}(L')\, x \succsim_A x$ but $y \Psi^{-1}(L')\, x \prec_B x$, contradicting $\succsim_A$ more ambiguity averse than $\succsim_B$.

Suppose $R_A \nsubseteq R_B$, and fix $\ell'' \in R_A \backslash R_B$. Since $\ell'' \notin R_B$, by Theorem 3.2 there exists an $L'' \in \mathcal{O}_{\ell''}$ such that $\Psi^{-1}(L'')$ is null for $\succsim_B$, and so $y \sim_B x\Psi^{-1}(L'')\,y$. Observe that $\succsim_A$ strictly monotonic for bets on non-null limiting frequency events and $\ell'' \in R_A$ implies $y \prec_A x\Psi^{-1}(L'')\,y$. Thus, $x\Psi^{-1}(L'')\,y \precsim_B y$ but $x\Psi^{-1}(L'')\,y \succ_A y$, contradicting $\succsim_A$ more ambiguity averse than $\succsim_B$.

Thus $R_A = R_B$.

C.6 Proof of Theorem 4.1

Suppose $\succsim$ is an i.i.d. α-MEU preference. We first show that $D \subseteq R$. Suppose $\hat{\ell} \in D$ and fix any $K \in \mathcal{O}_{\hat{\ell}}$. Consider $f = 1_{\Psi^{-1}(K)}$ and $g = 1_{\emptyset}$ and observe that $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S) \backslash K$. Note that $\int u(f)\, d\ell^\infty > \int u(g)\, d\ell^\infty$ for all $\ell \in K$ while $\int u(f)\, d\ell^\infty \geq \int u(g)\, d\ell^\infty$ for all $\ell \in D$. Thus, if $\alpha \in [0,1)$, $f \succ g$ and $\hat{\ell}$ is relevant. If $\alpha = 1$, consider instead $f = \frac{1}{2}1_{\Psi^{-1}(K)} + \frac{1}{2}1_{\Psi^{-1}(\Delta(S)\backslash K)}$ and $g = \frac{1}{2}1_{\emptyset} + \frac{1}{2}1_{\Psi^{-1}(\Delta(S)\backslash K)}$ and observe that $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S) \backslash K$ while $\min_{\ell \in D} \int u(f)\, d\ell^\infty = \frac{1}{2}u(x^*) + \frac{1}{2}u(x_*) > u(x_*) = \min_{\ell \in D} \int u(g)\, d\ell^\infty$ so that $f \succ g$ and again $\hat{\ell}$ is relevant.

We show that $\succsim$ is Continuous Symmetric. All axioms except Monotone Continuity of $\succsim^*$ are straightforward. To check the latter, consider $V_1(f) \equiv \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp$ first. The set C (as in Theorem 4.5) in the representation of $\succsim^*$ associated with V_1 is $co(\{\ell^\infty : \ell \in D\}) \subseteq \Delta(\Omega)$ and it is weak*-compact because D is finite (Dunford and Schwartz [11, Theorems IV.9.1 and V.6.1]). Thus, the preference represented by V_1 satisfies Monotone Continuity of $\succsim^*$ by Ghirardato, Maccheroni and Marinacci [22, Remark 1]. Similarly, the preference represented by $V_0(f) \equiv \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp$ also satisfies Monotone Continuity of $\succsim^*$. Take $A_n \searrow \emptyset$ and $x, x', x'' \in X$ such that $u(x') > u(x'')$. Then, there is $\bar{n}_1$ and $\bar{n}_0$ such that

$$V_1(\lambda x' + (1-\lambda)h) \geq V_1(\lambda x A_n x'' + (1-\lambda)h)$$

for all $\lambda \in [0,1]$, $h \in \mathcal{F}$ and $n \geq \bar{n}_1$, and

$$V_0(\lambda x' + (1-\lambda)h) \geq V_0(\lambda x A_n x'' + (1-\lambda)h)$$

for all $\lambda \in [0,1]$, $h \in \mathcal{F}$ and $n \geq \bar{n}_2$. Defining $V \equiv \alpha V_1 + (1-\alpha)V_0$,

$$V(\lambda x' + (1-\lambda)h) \geq V(\lambda x A_n x'' + (1-\lambda)h) \text{ for } n = \max(\bar{n}_1, \bar{n}_2).$$

Thus, for any i.i.d. α-MEU preference, Monotone Continuity of $\succsim^*$ is satisfied.

Since $\succsim$ is Continuous Symmetric and every measure in D is relevant, Theorem 3.3 implies $R = \overline{D}$. Since D is finite, $\overline{D} = D$.

C.7 Proof of Theorem 4.2

We prove the "only if" direction first. Suppose $\succsim_A$ is more ambiguity averse than $\succsim_B$. This implies the two preferences agree on constant acts, which is equivalent to $u_A = u_B$

up to normalization. Theorem 4.1 shows that $D = R$ for such preferences. Theorem 3.4 implies $D_A = D_B$, since $\alpha_A, \alpha_B \in (0,1)$ implies $\succsim_A$ and $\succsim_B$ are strictly monotonic for bets on non-null limiting frequency events. (To see this last implication, observe that $\Psi^{-1}(L)$ non-null implies $L \cap D \neq \emptyset$. Assume $u(x) > u(y)$. If $L \cap D = D$, the desired strict monotonicity is direct. Assume non-null $\Psi^{-1}(L)$ with $L \cap D \subset D$. If $x \succ y \succsim z$, the α-MEU functional evaluates the bets $x\Psi^{-1}(L)\,z$ and $y\Psi^{-1}(L)\,z$ by $\alpha u(z) + (1-\alpha)u(x)$ and $\alpha u(y) + (1-\alpha)u(z)$, and thus $x\Psi^{-1}(L)\,z \succ y\Psi^{-1}(L)\,z$. When $z \succsim x \succ y$, $x\Psi^{-1}(L)\,z \succ y\Psi^{-1}(L)\,z$ can be shown similarly. Finally, if $x \succ z \succ y$, $x\Psi^{-1}(L)\,z \succ z\Psi^{-1}(L)\,z \succ y\Psi^{-1}(L)\,z$.) Since D_B is non-singleton and $D_A = D_B$, for $\ell \in D_A$, $1_{\Psi^{-1}(\ell)} \sim_A \alpha_A x_* + (1-\alpha_A)x^*$ and $1_{\Psi^{-1}(\ell)} \sim_B \alpha_B x_* + (1-\alpha_B)x^*$. Thus $\succsim_A$ is more ambiguity averse than $\succsim_B$ requires $1_{\Psi^{-1}(\ell)} \succsim_A \alpha_A x_* + (1-\alpha_A)x^*$implies $1_{\Psi^{-1}(\ell)} \succsim_B \alpha_A x_* + (1-\alpha_A)x^*$, which, in turn, implies $\alpha_A \geq \alpha_B$.

Turn to the "if" direction. Suppose $\alpha_A \geq \alpha_B$, $D_A = D_B$ and (up to normalization) $u_A = u_B$. Let u be a common normalization of u_A and u_B and $D \equiv D_A = D_B$. Then $f \succsim_A x$ if and only if $\alpha_A \min_{p\in\{\ell^\infty:\ell\in D\}} \int u(f)\,dp + (1-\alpha_A) \max_{p\in\{\ell^\infty:\ell\in D\}} \int u(f)\,dp \geq u(x)$, which implies $\alpha_B \min_{p\in\{\ell^\infty:\ell\in D\}} \int u(f)\,dp + (1-\alpha_B) \max_{p\in\{\ell^\infty:\ell\in D\}} \int u(f)\,dp \geq u(x)$ and thus $f \succsim_B x$. This proves $\succsim_A$ is more ambiguity averse than $\succsim_B$. Note that this direction did not use the restriction of α_A, α_B to $(0,1)$.

C.8 Proof of Theorem 4.3

Suppose $\succsim$ is an i.i.d. smooth ambiguity preference. We first show that $\operatorname{supp}\mu \subseteq R$. Suppose $\hat{\ell} \in \operatorname{supp}\mu$ and fix any $L \in \mathcal{O}_{\hat{\ell}}$. Consider $f = 1_{\Psi^{-1}(L)}$ and $g = 1_\emptyset$ and observe that $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S)\backslash L$. Since ϕ is strictly increasing, $\phi\left(\int u(f)\,d\ell^\infty\right) > \phi\left(\int u(g)\,d\ell^\infty\right)$ for all $\ell \in L$ and $\phi\left(\int u(f)\,d\ell^\infty\right) \geq \phi\left(\int u(g)\,d\ell^\infty\right)$ for all $\ell \in \operatorname{supp}\mu$. By the definition of $\operatorname{supp}\mu$, $\mu(L) > 0$. Thus, $f \succ g$ and $\hat{\ell}$ is relevant.

We next show that $\succsim$ satisfies Continuous Symmetry. We directly verify only the following axioms: Monotone Continuity of $\succsim^*$ and Mixture Continuity. That the remaining axioms are satisfied is straightforward.

Monotone Continuity of $\succsim^*$: Suppose that condition (i) holds in the definition of i.i.d. smooth ambiguity preferences, so there are $m, M > 0$ such that $m\,|a-b| \leq |\phi(a) - \phi(b)| \leq M\,|a-b|$ for all $a, b \in u(X)$. Fix any $x, x', x'' \in X$ with $x' \succ x''$. If $x' \succsim x$ then Monotone Continuity of $\succsim^*$ follows from Monotonicity. Therefore, consider $x \succ x'$. Without loss of generality, normalize u so that $u(x) = 1 > u(x') = t' > u(x'') = 0$ and $[0,1] \subseteq u(X)$. Suppose $A_n \searrow \emptyset$. Take $\varepsilon', \varepsilon > 0$ so that

$$\varepsilon' < t' \text{ and } m\,(t' - \varepsilon')\,(1-\varepsilon) \geq M\,(1-t')\,\varepsilon.$$

Define $\zeta_n : \Delta(S) \to \mathbb{R}$ by $\zeta_n(\ell) = \ell^\infty(A_n)$, and temporarily equip $\Delta(S)$ with the weak convergence (wc) topology (i.e., the weakest topology that makes the integrals of continuous bounded functions continuous). Since wc open sets are weak* open, μ is well-defined on the Borel σ-algebra generated by wc open sets. Then, by Lusin's theorem (Aliprantis and Border [2, Theorem 12.8]), there is a wc compact set $L \subseteq \Delta(S)$ such that $\mu(L) > 1-\varepsilon$ and all ζ_n are wc continuous. Note that ζ_n converges monotonically to 0 pointwise. Then by Dini's Theorem (Aliprantis and Border [2, Theorem 2.66]), ζ_n on L converges uniformly to 0. Hence there is $N > 0$ such that $\zeta_N = \ell^\infty(A_N) < \varepsilon'$ for all $\ell \in L$. To see $x' \succsim^* xA_N x''$, and thus Monotone Continuity of $\succsim^*$, compute, for any $\alpha \in [0,1]$ and $h \in \mathcal{F}$,

$$
\begin{aligned}
&\int_{\Delta(S)} \phi\left(\int u(\alpha x' + (1-\alpha)h) d\ell^\infty\right) d\mu(\ell) - \int_{\Delta(S)} \phi\left(\int u(\alpha x A_N x'' + (1-\alpha)h) d\ell^\infty\right) d\mu(\ell) \\
&= \int_L \left(\phi\left(\alpha t' + (1-\alpha)\int u(h) d\ell^\infty\right) - \phi\left(\alpha \ell^\infty(A_N) + (1-\alpha)\int u(h) d\ell^\infty\right)\right) d\mu(\ell) \\
&+ \int_{\Delta(S)\setminus L} \left(\phi\left(\alpha t' + (1-\alpha)\int u(h) d\ell^\infty\right) - \phi\left(\alpha \ell^\infty(A_N) + (1-\alpha)\int u(h) d\ell^\infty\right)\right) d\mu(\ell) \\
&> \int_L \left(\phi\left(\alpha t' + (1-\alpha)\int u(h) d\ell^\infty\right) - \phi\left(\alpha \varepsilon' + (1-\alpha)\int u(h) d\ell^\infty\right)\right) d\mu(\ell) \\
&+ \int_{\Delta(S)\setminus L} \left(\phi\left(\alpha t' + (1-\alpha)\int u(h) d\ell^\infty\right) - \phi\left(\alpha + (1-\alpha)\int u(h) d\ell^\infty\right)\right) d\mu(\ell) \\
&\geq \int_L \alpha m(t' - \varepsilon') d\mu(\ell) + \int_{\Delta(S)\setminus L} \alpha M(t'-1) d\mu(\ell) \\
&= \alpha\left[m(t'-\varepsilon')\mu(L) - M(1-t')(1-\mu(L))\right] \\
&\geq \alpha\left[m(t'-\varepsilon')(1-\varepsilon) - M(1-t')\varepsilon\right] \geq 0.
\end{aligned}
$$

Turn to the case where (ii) holds in the definition of i.i.d. smooth ambiguity preferences, so that $\operatorname{supp}\mu$ is finite. Again suppose $A_n \searrow \emptyset$ and $x \succ x' \succ x''$. Since $\operatorname{supp}\mu$ is finite, $\sup_{\ell \in \operatorname{supp}\mu} \ell^\infty(A_n) \to 0$. Thus, for $\varepsilon > 0$ satisfying $u(x') > \varepsilon u(x) + (1-\varepsilon) u(x'')$, there is $n > 0$ such that $\ell^\infty(A_n) < \varepsilon$ for all $\ell \in \operatorname{supp}\mu$. This implies

$$
\begin{aligned}
&\int_{\Delta(S)} \phi\left(\int u(\alpha x' + (1-\alpha)h) d\ell^\infty\right) d\mu(\ell) - \int_{\Delta(S)} \phi\left(\int u(\alpha x A_N x'' + (1-\alpha)h) d\ell^\infty\right) d\mu(\ell) \\
&= \int (\phi\left(\alpha u(x') + (1-\alpha)\int u(h) d\ell^\infty\right) \\
&- \phi\left(\alpha\left(\ell^\infty(A_n) u(x) + (1-\ell^\infty(A_n)) u(x'')\right) + (1-\alpha)\int u(h) d\ell^\infty\right)) d\mu(\ell) \\
&\geq 0
\end{aligned}
$$

for all $\alpha \in [0,1]$ and $h \in \mathcal{F}$. Therefore, $x' \succsim^* xA_n x''$ and Monotone Continuity of $\succsim^*$ holds.

Mixture Continuity: Fix acts $f, g, h \in \mathcal{F}$ and consider a sequence $\lambda_n \in [0,1]$ such that $\lambda_n \to \lambda$ and $\lambda_n f + (1-\lambda_n) g \succsim h$ for all n. Therefore, for all n,

$$\int_{\Delta(S)} \phi\left(\lambda_n \int u(f) d\ell^\infty + (1-\lambda_n)\int u(g) d\ell^\infty\right) d\mu(\ell)$$
$$\geq \int_{\Delta(S)} \phi\left(\int u(h) d\ell^\infty\right) d\mu(\ell).$$

Since ϕ is continuous, by the Dominated Convergence Theorem (e.g., Aliprantis and Border [2, Theorem 11.21])

$$\int_{\Delta(S)} \phi\left(\lambda_n \int u(f) d\ell^\infty + (1-\lambda_n)\int u(g) d\ell^\infty\right) d\mu(\ell)$$
$$\to \int_{\Delta(S)} \phi\left(\lambda \int u(f) d\ell^\infty + (1-\lambda)\int u(g) d\ell^\infty\right) d\mu(\ell)$$

so that $\lambda f + (1-\lambda) g \succsim h$ and thus the upper set is closed. The same argument using a sequence such that $\lambda_n f + (1-\lambda_n) g \precsim h$ may be used to show the lower set is closed.

Since $\int_{\Delta(S)} \phi\left(\int u(f) d\ell^\infty\right) d\mu(\ell) = \int_{\operatorname{supp}\mu} \phi\left(\int u(f) d\ell^\infty\right) d\mu(\ell)$ is a representation of the form in Theorem 3.1 with $D = \operatorname{supp}\mu$, $\succsim$ is Continuous Symmetric and $\operatorname{supp}\mu \subseteq R$, Theorem 3.3 implies $R = \overline{\operatorname{supp}\mu}$. Since, by definition, $\operatorname{supp}\mu$ is relative weak* closed, $R = \operatorname{supp}\mu$.

C.9 Proof of Theorem 4.4

We begin with two lemmas that will be used in proving the theorem. The first lemma may be viewed as generalizing a remark in Yaari [45] so that it applies to not-necessarily-convex preferences.

We need the following definitions and notation to state the first lemma. We say a function is *differentiable* if it is Fréchet differentiable. Denote the Fréchet differential of a functional I by DI. The interior of a set A is denoted by $\operatorname{int} A$.

Lemma C.1. *Let $\succsim_A$ and $\succsim_B$ be two preferences represented by $U_A(f) = I_A(u(f))$ and $U_B(f) = I_B(u(f))$ respectively. Suppose I_A and I_B are differentiable at $c \in \operatorname{int} u(X)$ and $U_A(x) = U_B(x)$ for all $x \in X$. If $\succsim_A$ is more ambiguity averse than $\succsim_B$, $DI_A(c) = DI_B(c)$.*

Proof. Since $\succsim_A$ is more ambiguity averse than $\succsim_B$, for any $x \in X$, $f \in \mathcal{F}$ such that $f \sim_A x$,

$$U_A(f) = U_A(x) = U_B(x) \leq U_B(f).$$

Thus,

$$\frac{I_A(c+\lambda\varphi) - I_A(c)}{\lambda} \leq \frac{I_B(c+\lambda\varphi) - I_B(c)}{\lambda}$$

for any bounded measurable $\varphi : \Omega \to \mathbb{R}$ and $\lambda > 0$ small enough so that $c + \lambda\varphi \in u(X)^{\Omega}$. This implies $DI_A(c)(\varphi) \leq DI_B(c)(\varphi)$. Since the same is true for $-\varphi$ and $DI_A(c)(\varphi)$ is linear in φ, we have

$$DI_A(c)(\varphi) = -DI_A(c)(-\varphi) \geq -DI_B(c)(-\varphi) = DI_B(c)(\varphi).$$

Thus, $DI_A(c) = DI_B(c)$. □

To state the second lemma, let $S_i(f) \equiv \int_{\Delta(S)} \phi_i \left(\int u_i(f) d\ell^{\infty}\right) d\mu_i(\ell)$ for $i = A, B$.

Lemma C.2. *$\succsim_A$ is more ambiguity averse than $\succsim_B$ if and only if $\phi_B^{-1}(S_B(f)) \geq \phi_A^{-1}(S_A(f))$ for all acts $f \in \mathcal{F}$.*

Proof. To see the "only if" direction, observe that $\phi_B^{-1}(S_B(f)) < \phi_A^{-1}(S_A(f))$ means that, for any $y \in X$ such that $\phi_B^{-1}(S_B(f)) < u(y) \leq \phi_A^{-1}(S_A(f))$, $f \succsim_A y$ and $f \prec_B y$, contradicting $\succsim_A$ more ambiguity averse than $\succsim_B$. To see the "if" direction, note that $f \succsim_A x \Rightarrow \phi_A^{-1}(S_A(f)) \geq u(x) \Rightarrow \phi_B^{-1}(S_B(f)) \geq u(x) \Rightarrow f \succsim_B x$, thus $\succsim_A$ is more ambiguity averse than $\succsim_B$. □

Consider the "only if" direction of the theorem. Suppose $\succsim_A$ is more ambiguity averse than $\succsim_B$.

We first show $\mu_A = \mu_B$. We claim that $\phi_A'(c) > 0$ and $\phi_B'(c) > 0$ for some $c \in \operatorname{int} u(X)$. Since ϕ_A is strictly increasing, $\phi_A'(\bar{c}) > 0$ for some $\bar{c} \in \operatorname{int} u(X)$. Continuity of ϕ_A' implies that there is $a > 0$ such that $\phi_A'(d) > 0$ for all $d \in (\bar{c} - a, \bar{c} + a)$. The Mean Value Theorem implies that there is $c \in (\bar{c} - a, \bar{c} + a)$ such that $\phi_B'(c) = \frac{\phi_B(\bar{c}+a) - \phi_B(\bar{c}-a)}{2a} > 0$.

Now, compute

$$D\phi_A^{-1}\left(\int_{\Delta(S)} \phi_A\left(\int \cdot d\ell^{\infty}\right) d\mu_A(\ell)\right)(c) = \frac{\phi_A'(c)\int_{\Delta(S)} \ell^{\infty} d\mu_A(\ell)}{\phi_A'(c)} = \int_{\Delta(S)} \ell^{\infty} d\mu_A(\ell).$$

The analogous equality holds for B. By Lemma C.1, $\succsim_A$ more ambiguity averse than $\succsim_B$ implies $\int_{\Delta(S)} \ell^{\infty} d\mu_A(\ell) = \int_{\Delta(S)} \ell^{\infty} d\mu_B(\ell)$. Thus, for any $L \subseteq \Delta(S)$, $\mu_A(L) = \int_{\Delta(S)} \ell^{\infty}(\Psi^{-1}(L)) d\mu_A(\ell) = \int_{\Delta(S)} \ell^{\infty}(\Psi^{-1}(L)) d\mu_B(\ell) = \mu_B(L)$. Therefore $\mu_A = \mu_B$.

We now show ϕ_A is more concave than ϕ_B. Define $h : \phi_B(u(X)) \to \mathbb{R}$ by $h = \phi_A \circ \phi_B^{-1}$. Note that $\phi_A = h \circ \phi_B$. Since ϕ_A and ϕ_B are continuous and strictly increasing, so is h. It remains to show that h is concave. By Lemma C.2, $\phi_B^{-1}(S_B(f)) \geq \phi_A^{-1}(S_A(f))$ for all acts $f \in \mathcal{F}$. Observe that $\phi_B^{-1}(S_B(f)) \geq \phi_A^{-1}(S_A(f))$ if and only if $h(S_B(f)) \geq S_A(f)$. Letting $\mu \equiv \mu_A = \mu_B$, since $\operatorname{supp} \mu$ is non-singleton, there exists a set $L \subseteq \Delta(S)$ such that $0 < \mu(L) \equiv t < 1$. For any $x', x'' \in X$, let $f_{x',x''}$ denote the act defined as follows:

$$f_{x',x''} = \begin{cases} x' & \text{if } \omega \in \Psi^{-1}(L) \\ x'' & \text{otherwise} \end{cases}.$$

Thus, $h(S_B(f_{x',x''})) \geq S_A(f_{x',x''})$ if and only if $h(t\phi_B(u(x')) + (1-t)\phi_B(u(x''))) \geq t\phi_A(u(x')) + (1-t)\phi_A(u(x'')) = th(\phi_B(u(x'))) + (1-t)h(\phi_B(u(x'')))$. Since this holds for all $x', x'' \in X$, h is concave (by e.g., Klibanoff, Marinacci and Mukerji [30, Lemma 6]).

Now turn to the "if" direction of the theorem. Let $\mu \equiv \mu_A = \mu_B$. Since $\phi_A = h \circ \phi_B$ for a strictly increasing and concave h, the Jensen inequality implies

$$h\left(S_B(f)\right) \geq \int_{\Delta(S)} h\left(\phi_B\left(\int u(f) d\ell^\infty\right)\right) d\mu\left(\ell\right)$$

for all acts $f \in \mathcal{F}$. But this is the same as $h(S_B(f)) \geq S_A(f)$ and thus $\phi_B^{-1}(S_B(f)) \geq \phi_A^{-1}(S_A(f))$ for all $f \in \mathcal{F}$ which is equivalent to $\succsim_A$ more ambiguity averse than $\succsim_B$ by Lemma C.2. Note that this direction did not use the differentiability or non-singleton assumptions.

Remark C.1. From the proof, it is apparent that the assumption of continuous differentiability of ϕ_A and ϕ_B can be weakened to differentiability of both functions at a common point, c, in the interior of $u(X)$ with $\phi_A'(c) > 0$ and $\phi_B'(c) > 0$.

C.10 Proofs of Theorems 4.5 and B.5

The first is a special case of the latter and we prove the latter here.

We first prove sufficiency of the stated axioms. We start by showing that $\succsim_{\tilde{\xi}}^*$ satisfies the properties assumed in Gilboa et. al. [26, Theorem 1]. Preorder, Monotonicity, Mixture Continuity, Non-triviality, C-Completeness and Independence of $\succsim_{\tilde{\xi}}^*$ follow directly from the axioms we assume and the definition of $\succsim_{\tilde{\xi}}^*$. Therefore, by Gilboa et. al. [26, Theorem 1], there exists a unique non-empty weak* closed and convex set $C \subseteq ba_1^+\left(S^\infty\right)$ and a non-constant vNM utility function, $u : X \to \mathbb{R}$, such that

$$f \succsim_{\tilde{\xi}}^* g \text{ if and only if } \int u\left(f\right) dp \geq \int u\left(g\right) dp \text{ for all } p \in C.$$

By Alaoglu's Theorem, C is weak* compact. Monotone Continuity of $\succsim_{\tilde{\xi}}^*$ implies $C \subseteq \Delta\left(S^\infty\right)$ by Ghirardato, Maccheroni and Marinacci [22, Remark 1]. Moreover, Partial Event Symmetry implies every $p \in C$ is partially symmetric on finite cylinder events.

Next, we prove the claim that every $p \in C$ is of the form $\int \boldsymbol{l}\left(\tilde{\xi}\right) dm\left(\boldsymbol{l}\right)$ for some $m \in \Delta\left(\left(\Delta S\right)^{\Xi}\right)$. (We prove this claim here because we did not find a proof in the literature.) The proof is based on the idea of Hewitt and Savage [29]. Let $\mathcal{P}_{\tilde{\xi}}$ be the set of partially symmetric measures. $\mathcal{P}_{\tilde{\xi}}$ is convex and also weak-convergence compact as $\Delta\left(S^\infty\right)$ is. Then, the Choquet Theorem (Phelps [38, p.14]) implies that any element in $\mathcal{P}_{\tilde{\xi}}$ is a mixture of its extreme points. We need to show that each extreme point is of the form $\boldsymbol{l}\left(\tilde{\xi}\right)$. We prove this for the case where $\tilde{\xi} = (\xi^1, \xi^2, \xi^1, \xi^2, ...)$. The general case follows from the same arguments. Take any extreme point p, $n \geq 1$ and event $A \subseteq S^n$. For each finite cylinder B,

$$p\left(B\right) = p\left(\pi B\right) = p\left(A\right) p\left(\pi B | A\right) + p\left(A^c\right) p\left(\pi B | A^c\right),$$

where $\pi \in \Pi$ is defined as follows: If n is even,

$$\pi(i) = i + n.$$

If n is odd,

$$\pi(i) = n + i - (-1)^i.$$

(Since B is a finite cylinder, π can be made a finite permutation.) For example, if $B \subset S^2$ and $n = 1$, then $\pi(1) = 3, \pi(2) = 2, \pi(3) = 1$, and $\pi(k) = k$ for $k \geq 4$, and hence $\pi B = \{\omega : (\omega_3, \omega_2) \in B\}$. Note that A and πB depend on different coordinates. Define $q_1, q_2 \in \Delta(S^\infty)$ by

$$\begin{aligned} q_1(B) &= p(\pi B|A) \text{ and} \\ q_2(B) &= p(\pi B|A^c) \end{aligned}$$

for each finite cylinder B. Noting that $\tilde{\xi}_i = \tilde{\xi}_{\pi(i)}$ for all $i = 1, 2, ...$, one can verify that $q_1, q_2 \in \mathcal{P}_{\tilde{\xi}}$. We have just shown that p is a mixture of q_1 and q_2 that lie in $\mathcal{P}_{\tilde{\xi}}$. Since p is an extreme point, $p = q_1 = q_2$. Therefore we have $p(B) = p(A \times \pi B)/p(A)$ where π is defined as above. By the fact that $p(B) = p(\pi B)$, $p(A)p(\pi B) = p(A \times \pi B)$ which proves that p is a product measure. By partial symmetry w.r.t. $\tilde{\xi} = (\xi^1, \xi^2, \xi^1, \xi^2, ...)$, $p = \ell_1 \otimes \ell_2 \otimes \ell_1 \otimes \ell_2 \otimes ...$ and is of the form $\boldsymbol{l}\left(\tilde{\xi}\right)$. Therefore, any element in $\mathcal{P}_{\tilde{\xi}}$ is a mixture of product measures of the form $\boldsymbol{l}\left(\tilde{\xi}\right)$.

Thus, $C = \left\{\int \boldsymbol{l}\left(\tilde{\xi}\right) dm(\boldsymbol{l}) : m \in M\right\}$ for some non-empty $M \subseteq \Delta\left((\Delta S)^\Xi\right)$. M is convex since C is.

To see that M is weak* compact, take any net $m_\alpha \in M$. Since C is weak* compact, there is $m' \in M$ and a subnet m'_λ of m_α such that

$$\int \left(\int \varphi d\boldsymbol{l}\left(\tilde{\xi}\right)\right) dm'_\lambda(\boldsymbol{l}) \to \int \left(\int \varphi d\boldsymbol{l}\left(\tilde{\xi}\right)\right) dm'(\boldsymbol{l}) \text{ for each } \varphi \in B(S^\infty).$$

It suffices to show that each $\phi \in B\left((\Delta S)^\Xi\right)$ can be written as $\boldsymbol{l} \mapsto \int \varphi d\boldsymbol{l}\left(\tilde{\xi}\right)$ for some $\varphi \in B(S^\infty)$. In fact,

$$\phi(\boldsymbol{l}) = \int_{S^\infty} \phi\left(\Psi(\omega)\left(\xi^1\right), ..., \Psi(\omega)\left(\xi^K\right)\right) d\boldsymbol{l}\left(\tilde{\xi}\right)(\omega).$$

Conclude that m'_λ converges to m'.

To show necessity, assume such a set M. $\succsim^*_{\tilde{\xi}}$ satisfies C-complete Preorder, Monotonicity and Risk Independence and thus $\succsim_{\tilde{\xi}}$ inherits these properties, with the exception of the directly assumed transitivity, as well. Partial Event Symmetry follows since each element of C is of the form $\int \boldsymbol{l}\left(\tilde{\xi}\right) dm(\boldsymbol{l})$ for some $m \in M$. Non-triviality of

$\succsim_{\tilde{\xi}}$ follows from non-constancy of u. Monotone Continuity of $\succsim^*_{\tilde{\xi}}$ follows from weak* compactness of C, which is implied by that of M. Mixture Continuity of $\succsim^*_{\tilde{\xi}}$ follows from Mixture Continuity of expected utility and the fact that intersections of closed sets are closed.

Uniqueness of M follows from uniqueness of C.

Finally, we show $R = \overline{\bigcup_{m \in M} \operatorname{supp} m}$. The first step is to show $R \subseteq \overline{\bigcup_{m \in M} \operatorname{supp} m}$: Take any $\boldsymbol{l} \notin \overline{\bigcup_{m \in M} \operatorname{supp} m}$. Then, there is $L \in \mathcal{O}_{\boldsymbol{l}}$ such that $L \subseteq \left(\overline{\bigcup_{m \in M} \operatorname{supp} m}\right)^c$. Take any $f, g \in \mathcal{F}$. Note that $\int f \Psi^{-1}_{\tilde{\xi}}(L)\, g dp = \int g dp$ for any $p \in C$. Thus $f \Psi^{-1}_{\tilde{\xi}}(L)\, g \sim^*_{\tilde{\xi}} g$ and so $f \Psi^{-1}_{\tilde{\xi}}(L)\, g \sim_{\tilde{\xi}} g$. Therefore, $\Psi^{-1}_{\tilde{\xi}}(L)$ is null, and $\boldsymbol{l} \notin R$.

The second step is to show $\bigcup_{m \in M} \operatorname{supp} m \subseteq R$: Take any $\boldsymbol{l} \in \bigcup_{m \in M} \operatorname{supp} m$ and $L \in \mathcal{O}_{\boldsymbol{l}}$. By the representation, $1_{\Psi^{-1}_{\tilde{\xi}}(L)} \succsim^*_{\tilde{\xi}} 1_{\emptyset}$ since $\int 1_{\Psi^{-1}_{\tilde{\xi}}(L)} dp \geq \int 1_{\emptyset} dp$ for all $p \in \Delta(S^\infty)$. Now show that $1_{\Psi^{-1}_{\tilde{\xi}}(L)} \not\precsim^*_{\tilde{\xi}} 1_{\emptyset}$. Note that, by definition of $\bigcup_{m \in M} \operatorname{supp} m$, there is $\hat{m} \in M$ such that $L \cap \operatorname{supp} \hat{m} \neq \emptyset$. Let $p = \int \hat{\boldsymbol{l}}\left(\tilde{\xi}\right) d\hat{m}\left(\hat{\boldsymbol{l}}\right)$ and compute

$$\int 1_{\Psi^{-1}_{\tilde{\xi}}(L)} dp = \hat{m}(L) > 0 = \int 1_{\emptyset} dp.$$

By the representation, $1_{\Psi^{-1}_{\tilde{\xi}}(L)} \not\precsim^*_{\tilde{\xi}} 1_{\emptyset}$. Therefore we have $1_{\Psi^{-1}_{\tilde{\xi}}(L)} \not\sim^*_{\tilde{\xi}} 1_{\emptyset}$, which implies that

$$\alpha 1_{\Psi^{-1}_{\tilde{\xi}}(L)} + (1-\alpha) h \succ_{\tilde{\xi}} \alpha 1_{\emptyset} + (1-\alpha) h$$

for some $\alpha \in [0,1]$ and $h \in \mathcal{F}$. Note that both sides coincide outside of $\Psi^{-1}_{\tilde{\xi}}(L)$ and hence $\boldsymbol{l} \in R$.

Finally, since by Theorem B.2 R is closed, $\overline{\bigcup_{m \in M} \operatorname{supp} m} \subseteq \overline{R} = R$, conclude that $R = \overline{\bigcup_{m \in M} \operatorname{supp} m}$.

C.11 Proof of Theorem A.1

It is convenient to define $\tilde{\succsim}$ on $\tilde{F}$ by

$$\tilde{f} \tilde{\succsim} \tilde{g} \text{ if } f \succsim g,$$

and similarly define $\tilde{\succsim}^*$.

Prove necessity of the axioms. Symmetry and Mixture Continuity are immediate from the properties of the representation. Ghirardato and Siniscalchi [25, Proposition S.1] show that a representing functional on utility acts being locally Lipschitz in its interior implies the represented preferences satisfy Locally Bounded Improvements. From this, G locally Lipschitz in its interior, the fact that U represents $\succsim$ and the definition of $\tilde{\succsim}$, it follows that $\tilde{\succsim}$ satisfies Locally Bounded Improvements. By considering acts and lotteries that generate the corresponding elements of $\tilde{F}$ and $u(X)$, and using the fact that the generated elements of $\tilde{F}$ and $u(X)$ are all that matter for $\succsim$, Locally Bounded Improvements for $\tilde{\succsim}$ implies Locally Bounded Improvements for $\succsim$.

Show that Monotone Continuity of $\succsim^*$ is satisfied. Ghirardato and Siniscalchi [24, Theorem 2] show that

$$\tilde{f}\tilde{\succsim}^*\tilde{g} \text{ if and only if } \int \tilde{f}dm \geq \int \tilde{g}dm \text{ for all } m \in \overline{co}\left(\bigcup_{\tilde{h}\in int\tilde{F}} C\left(\tilde{h}\right)\right).$$

Fix any $x, x', x'' \in X$ with $x' \succ x''$. The only non-immediate case has $x \succ x'$. Without loss of generality, normalize u so that $u(x) = 1 > u(x') = t' > u(x'') = 0$ and $[0,1] \subseteq u(X)$. Suppose $A_n \searrow \emptyset$. Let $L_n = \{\ell \in D : A_n \cap \Psi^{-1}(\ell) \neq \emptyset\}$. Then, $A_n \cap \Psi^{-1}(D) \subseteq \Psi^{-1}(L_n)$ and $L_n \searrow \emptyset$. Ghirardato, Maccheroni and Marinacci [22, Remark 1] implies that

$$t'\tilde{\succsim}^* 1L_n0$$

for some n. This implies $x' \succsim^* x\Psi^{-1}(L_n)x''$ for that n. Fix n. Since $A_n \cap \Psi^{-1}(D) \subseteq \Psi^{-1}(L_n)$ and only $\ell \in D$ appear in U, $x\Psi^{-1}(L_n)x'' \succsim^* xA_n \cap \Psi^{-1}(D)x'' \sim^* xA_nx''$. Thus $x' \succsim^* xA_nx''$.

Turn to sufficiency. Theorem 3.1 implies the existence of the required representation except i), ii) and iii) in the statement of Theorem A.1. For iii), given any $U(f) \equiv G\left(\left(\int u(f)\,d\ell^\infty\right)_{\ell\in D}\right)$ where G violates iii), replace G by $\chi^{-1} \circ G$ where $\chi : u(X) \to \mathbb{R}$ is defined by $\chi(c) = G\left((c)_{\ell\in D}\right)$ for all $c \in u(X)$. For i), Locally Bounded Improvements of $\succsim$ implies Locally Bounded Improvements of $\tilde{\succsim}$. This and Ghirardato and Siniscalchi [25, Proposition S.1] applied to $\tilde{\succsim}$ imply G is locally Lipschitz in its interior. For ii), $\overline{co}\left(\bigcup_{\tilde{h}\in int\tilde{F}} C\left(\tilde{h}\right)\right)$ is compact by applying Ghirardato and Siniscalchi [24, Theorem 2] and Ghirardato, Maccheroni and Marinacci [22, Remark 1] to $\tilde{\succsim}^*$.

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D Online Supplement to "Perceived Ambiguity and Relevant Measures" by Klibanoff, Mukerji and Seo

This supplement contains results identifying the relevant measures and implications of comparative ambiguity aversion for Continuous Symmetric versions of several additional models from the ambiguity literature: the extended MEU with contraction model (see e.g., Gajdos et. al. [6], Gajdos, Tallon and Vergnaud [7], Kopylov [12], Tapking [43]), the vector expected utility model (see Siniscalchi [15]) and the second-order Choquet representation (see Amarante [1]) of invariant biseparable preferences (defined by Ghirardato, Maccheroni and Marinacci [8]). It concludes by describing a technique for identifying relevant measures in additional models.

D.1 The Extended MEU with contraction model

Motivated by the contraction representation of Gajdos et al. [6, Theorem 6], this model has a functional form[12] that is a convex combination of MEU and expected utility.

Consider preferences having a representation of the form:

$$\beta \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp + (1-\beta) \int u(f)\, dq$$

where $D \subseteq \Delta(S)$ is finite, $q = \int \ell^\infty dm(\ell)$ for some $m \in \Delta(\Delta(S))$ such that $\operatorname{supp} m \subseteq D$, $0 < \beta < 1$ and u is a non-constant vNM utility function. Call such preferences *i.i.d. extended MEU with contraction.*

Theorem D.1. *For any i.i.d. extended MEU with contraction preference, $R = D$.*

The role of the finiteness restriction on D is to ensure Monotone Continuity of $\succsim^*$. Notice that any such preference also has a representation of the form $\min_{p \in \{\beta\ell^\infty + (1-\beta)q : \ell \in D\}} \int u(f)\, dp$, and therefore these preferences are a subset of the Continuous Symmetric MEU preferences.

In the representation of Gajdos et al. [6, Theorem 6], q is fully determined by the set over which the minimum is taken. It is a particular convex combination, known as the Steiner point, of the extreme points of that set. A property of the Steiner point of the set $\{\ell^\infty : \ell \in D\}$ is that $\operatorname{supp} m = D$. We do not require q to be the Steiner point, but do impose this support restriction in our comparative ambiguity aversion result below.

Theorem D.2. *Let $\succsim_A$ and $\succsim_B$ be any two i.i.d. extended MEU with contraction preferences such that D_B is non-singleton,* $\operatorname{supp} m_A = D_A$ *and* $\operatorname{supp} m_B = D_B$. *Then, $\succsim_A$ is more ambiguity averse than $\succsim_B$ if and only if $\beta_A \geq \beta_B$, $D_A = D_B$, $\ell \in D_B$ implies $m_B(\ell) \geq \frac{1-\beta_A}{1-\beta_B} m_A(\ell)$ and (up to normalization) $u_A = u_B$.*

[12] This functional form is much older than the derivation of it in Gajdos et al [6]. It appears in Ellsberg [5], and is acknowledged there as based upon a concept of Hodges and Lehmann [11].

Under the conditions in the theorem, β reflects comparative ambiguity aversion and D is the set of relevant measures and is therefore related to perceived ambiguity. This offers an additional perspective on the model compared to Gajdos et. al. [6]. In their setting, the sets D are objectively given and larger β reflects more imprecision aversion in the sense of stronger preference for having singleton sets.

Remark D.1. Observe that the inequality relating m_A and m_B is automatically satisfied if $q_A = q_B$. Equality of the q's is, for example, implied if the q's were Steiner points as in Gajdos et al. [6]. A step in the proof of Theorem D.2 is to show that $\succsim_A$ and $\succsim_B$ are strictly monotonic for bets on non-null limiting frequency events. This provides a set of Continuous Symmetric MEU preferences for which the conclusions of Theorem 3.4 hold.[13]

D.2 The Vector Expected Utility (VEU) model

Next we turn to a version of the VEU model of Siniscalchi [15]. Consider preferences having a representation of the form:

$$\int u(f)\,dp + A\left(\left(\int \zeta_i u(f)\,dp\right)_{1\leq i\leq n}\right)$$

where (i) u is a non-constant vNM utility function, (ii) $p \in \Delta(\Omega)$, (iii) $\zeta = (\zeta_1, ..., \zeta_n)$ is a bounded, measurable vector-valued function on Ω into $\mathbb{R}^n$ such that for each i, $\int \zeta_i dp = 0$, (iv) $A(0) = 0$, and $A(a) = A(-a)$ for all $a \in \mathbb{R}^n$, and (v) the whole functional is weakly monotonic. Call such preferences *VEU*.

Consider also preferences that are VEU and have a VEU representation that additionally satisfies (vi) n is finite, (vii) $p = \int \ell^\infty dm(\ell)$ for some $m \in \Delta(\Delta(S))$, (viii) for each i, for all $\pi \in \Pi$, $\zeta_i(\omega) = \zeta_i(\pi\omega)$, p-almost-everywhere and (ix) A is Lipschitz continuous. Call such preferences *i.i.d. VEU*.

Theorem D.3. *For any i.i.d. VEU preference, $R = \operatorname{supp} m$.*

Thus, for such preferences, the relevant measures, R, are those $\ell \in \Delta(S)$ given weight by p. It is interesting to observe that these are the same relevant measures as for the expected utility preference represented by $\int u(f)\,dp$. Note that the symmetry conditions on p and the ζ_i are imposed to ensure Event Symmetry, while n finite and Lipschitz continuity of A are imposed to ensure Monotone Continuity of $\succsim^*$. The remaining conditions are standard for the VEU model.

In characterizing comparative ambiguity aversion it turns out that Continuous Symmetry is not required and our result applies to VEU preferences with differentiable A:

[13] More generally, any MEU preference with a set of exchangeable measures such that each measure has the same set of i.i.d. measures in its support will be strictly monotonic for bets on non-null limiting frequency events.

Theorem D.4. *Let $\succsim_A$ and $\succsim_B$ be any two VEU preferences such that A_A and A_B are Fréchet differentiable. Then, $\succsim_A$ is more ambiguity averse than $\succsim_B$ if and only if $p_A = p_B$, $A_A\left(\left(\int \zeta_i^A u(f)\, dp_A\right)_{1\leq i\leq n}\right) \leq A_B\left(\left(\int \zeta_i^B u(f)\, dp_B\right)_{1\leq i\leq n}\right)$ for all $f \in \mathcal{F}$ and (up to normalization) $u_A = u_B$.*

Compared to Siniscalchi's result on comparative ambiguity aversion in VEU (Siniscalchi [15, Proposition 4]), this theorem derives equality of p as an implication rather than assuming it. The differentiability assumption is what allows this.

D.3 The Second-order Choquet model of Invariant Biseparable preferences

As shown by Amarante [1], the Invariant Biseparable preferences defined in Ghirardato, Maccheroni and Marinacci [8] may be represented by a Choquet integral of expected utilities. These preferences generalize both the MEU model of Gilboa and Schmeidler [10] and the Choquet Expected Utility model of Schmeidler [14]. Here we consider a Continuous Symmetric version, where the expected utilities are calculated with respect to i.i.d. measures.

Some notation and definitions are necessary in order to formally describe the representation of such preferences. Let υ be a capacity mapping subsets of $\Delta(S)$ to $[0,1]$. An event E is υ-non-null if there is an event E' such that $\upsilon(E \cup E') > \upsilon(E')$. The support of υ, denoted $\operatorname{supp}\upsilon$, is defined to be the set of elements $\ell \in \Delta(S)$ such that any open set containing ℓ is υ-non-null.

Consider preferences having a representation of the form:

$$\int \int u(f) d\ell^{\infty} d\upsilon(\ell)$$

where u is a non-constant vNM utility function, υ is a capacity on $\Delta(S)$ with finite support and the outer integral is taken in the sense of Choquet. Call such preferences *i.i.d. second-order Choquet.*

Theorem D.5. *For any i.i.d. second-order Choquet preference, $R = \operatorname{supp}\upsilon$.*

Therefore the relevant measures are exactly the measures in the support of the representing capacity υ. Our next result shows that an everywhere lower υ characterizes more ambiguity aversion.

Theorem D.6. *Let $\succsim_A$ and $\succsim_B$ be any two i.i.d. second-order Choquet preferences. $\succsim_A$ is more ambiguity averse than $\succsim_B$ if and only if $\upsilon_B \geq \upsilon_A$ and (up to normalization) $u_A = u_B$.*

Observe that if $\upsilon_A(\{\ell\}) > 0$ for all $\ell \in \operatorname{supp}\upsilon_A$ and $\upsilon_B(\Delta(S) \backslash \{\ell\}) < 1$ for all $\ell \in \operatorname{supp}\upsilon_B$, then $\upsilon_B \geq \upsilon_A$ implies $\operatorname{supp}\upsilon_A = \operatorname{supp}\upsilon_B$. Thus, these conditions are sufficient for R to be unaffected by increases and decreases in ambiguity aversion.

D.4 A Technique for Further Applications

Theorems 4.5 and B.5 provide a way to leverage the fact that there are extant characterizations of the set C for a variety of models as a step towards identifying R in Continuous Symmetric instances of such models. Given an explicit characterization of the set C for a continuous symmetric preference, Theorem 4.5 shows how to determine the relevant measures. For example, Cerreia-Vioglio et al. [2] characterize the set C for the variational preferences of Maccheroni, Marinacci and Rustichini [13] as the closure of the set of probability measures assigned finite values by the cost function c in the variational representation. Therefore, for Continuous Symmetric variational preferences, by Theorem 4.5, all of the measures in this closure are exchangeable measures and R is determined by looking at the marginals of i.i.d. measures appearing in the supports. See Cerreia-Vioglio et al. [2] and Ghirardato and Siniscalchi [9] for characterizations of C for a variety of models.

D.5 Proofs of Results in the Online Supplement

D.5.1 Proof of Theorem D.1

Suppose $\succsim$ is an i.i.d. extended MEU with contraction preference. We first show that $D \subseteq R$. Suppose $\hat{\ell} \in D$ and and fix any $L \in \mathcal{O}_{\hat{\ell}}$. Consider $f = 1_{\Psi^{-1}(L)}$ and $g = 1_{\emptyset}$ and observe that $\int f d\ell^{\infty} = \int g d\ell^{\infty}$ for all $\ell \in \Delta(S) \setminus L$. Observe that $\int u(f)\, d\ell^{\infty} > \int u(g)\, d\ell^{\infty}$ for all $\ell \in L$ while $\int u(f)\, d\ell^{\infty} \geq \int u(g)\, d\ell^{\infty}$ for all $\ell \in D$ and thus also $\int u(f)\, dq \geq \int u(g)\, dq$. Therefore, if $q(\Psi^{-1}(L)) > 0$, $f \succ g$ and $\hat{\ell}$ is relevant. If $q(\Psi^{-1}(L)) = 0$, consider instead $f = \frac{1}{2}1_{\Psi^{-1}(L)} + \frac{1}{2}1_{\Psi^{-1}(\Delta(S)\setminus L)}$ and $g = \frac{1}{2}1_{\emptyset} + \frac{1}{2}1_{\Psi^{-1}(\Delta(S)\setminus L)}$ and observe that $\int f d\ell^{\infty} = \int g d\ell^{\infty}$ for all $\ell \in \Delta(S) \setminus L$ while $\min_{\ell \in D} \int u(f)\, d\ell^{\infty} = \frac{1}{2}u(x^*) + \frac{1}{2}u(x_*) > u(x_*) = \min_{\ell \in D} \int u(g)\, d\ell^{\infty}$ so that $f \succ g$ and again $\hat{\ell}$ is relevant.

We now show that $\succsim$ satisfies Continuous Symmetry. Since W is a real-valued representation, $\succsim$ satisfies Weak Order. Since W is sup norm continuous, $\succsim$ satisfies Mixture Continuity. All the remaining axioms will be shown by way of Theorem 4.5, as we now demonstrate that $\succsim^*$ may be represented as in (4.3). Suppose $\int u(f)\, dp \geq \int u(g)\, dp$ for all $p \in co\{\beta\ell^{\infty} + (1-\beta)q : \ell \in D\}$. Fix any $\lambda \in [0,1]$ and acts $f, g, h \in \mathcal{F}$, and let $\hat{\ell}^{\infty} \in \arg\min_{p \in \{\ell^{\infty} : \ell \in D\}} \int u(\lambda f + (1-\lambda)h)\, dp$. Then

$$\begin{aligned} W(\lambda f + (1-\lambda)h) &= \int u(\lambda f + (1-\lambda)h)\, d\left(\beta\hat{\ell}^{\infty} + (1-\beta)q\right) \\ &\geq \int u(\lambda g + (1-\lambda)h)\, d\left(\beta\hat{\ell}^{\infty} + (1-\beta)q\right) \\ &\geq W(\lambda g + (1-\lambda)h) \end{aligned}$$

so that $f \succsim^* g$. Going the other direction, suppose $f \succsim^* g$ and that there exists a $\hat{p} \in co\{\beta\ell^{\infty} + (1-\beta)q : \ell \in D\}$ such that $\int u(f)\, d\hat{p} < \int u(g)\, d\hat{p}$. This implies that there exists an $\hat{\ell} \in D$ such that $\int u(f)\, d\left(\beta\hat{\ell}^{\infty} + (1-\beta)q\right) < \int u(g)\, d\left(\beta\hat{\ell}^{\infty} + (1-\beta)q\right)$.

Let $\hat{h} = 1_{\Psi^{-1}(D \setminus \hat{\ell})}$. Choose $\hat{\lambda} \in (0,1)$ small enough to satisfy

$$\begin{aligned}&(1-\hat{\lambda})(u(x^*) - u(x_*))\\ &> \hat{\lambda} \max[\int u(f)\, d\hat{\ell}^\infty - \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp, \int u(g)\, d\hat{\ell}^\infty - \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(g)\, dp].\end{aligned}$$

Then,

$$\begin{aligned}\min_{p \in \left\{\ell^\infty : \ell \in D, \ell \neq \hat{\ell}\right\}} \int u\left(\hat{\lambda} f + (1-\hat{\lambda})\hat{h}\right) dp &\geq \hat{\lambda} \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp + (1-\hat{\lambda}) u(x^*)\\ &> \hat{\lambda} \int u(f)\, d\hat{\ell}^\infty + (1-\hat{\lambda}) u(x_*) = \int u\left(\hat{\lambda} f + (1-\hat{\lambda})\hat{h}\right) d\hat{\ell}^\infty\end{aligned}$$

which implies $\hat{\ell}^\infty \in \arg\min_{p \in \{\ell^\infty : \ell \in D\}} \int u\left(\hat{\lambda} f + (1-\hat{\lambda})\hat{h}\right) dp$.
Similarly, $\hat{\ell}^\infty \in \arg\min_{p \in \{\ell^\infty : \ell \in D\}} \int u\left(\hat{\lambda} g + (1-\hat{\lambda})\hat{h}\right) dp$. Thus,

$$\begin{aligned}\min_{p \in \{\ell^\infty : \ell \in D\}} \int u\left(\hat{\lambda} f + (1-\hat{\lambda})\hat{h}\right) dp &= \int u\left(\hat{\lambda} f + (1-\hat{\lambda})\hat{h}\right) d\hat{\ell}^\infty\\ &< \int u\left(\hat{\lambda} g + (1-\hat{\lambda})\hat{h}\right) d\hat{\ell}^\infty\\ &= \min_{p \in \{\ell^\infty : \ell \in D\}} \int u\left(\hat{\lambda} g + (1-\hat{\lambda})\hat{h}\right) dp.\end{aligned}$$

Therefore, as $\beta > 0$,

$$\begin{aligned}W\left(\hat{\lambda} f + (1-\hat{\lambda})\hat{h}\right) &= \int u\left(\hat{\lambda} f + (1-\hat{\lambda})\hat{h}\right) d\left(\beta\hat{\ell}^\infty + (1-\beta)q\right)\\ &< \int u\left(\hat{\lambda} g + (1-\hat{\lambda})\hat{h}\right) d\left(\beta\hat{\ell}^\infty + (1-\beta)q\right)\\ &= W(\lambda g + (1-\lambda)h)\end{aligned}$$

contradicting $f \succsim^* g$. Summarizing, we have shown that

$$f \succsim^* g \text{ if and only if } \int u(f)\, dp \geq \int u(g)\, dp \text{ for all } p \in co\left\{\beta\ell^\infty + (1-\beta)q : \ell \in D\right\}.$$

Therefore, applying Theorem 4.5 and noting that $co\left\{\beta\ell^\infty + (1-\beta)q : \ell \in D\right\}$ is weak* compact because D is finite, $\succsim$ represented by $W(f)$ satisfies Continuous Symmetry.

Since $\succsim$ is Continuous Symmetric and every measure in D is relevant, Theorem 3.3 implies $R = \overline{D}$. Since D is finite, $\overline{D} = D$.

D.5.2 Proof of Theorem D.2

Consider the "only if" direction. That $u_A = u_B$ up to normalization is equivalent to the two preferences agreeing on constant acts, a necessary condition for $\succsim_A$ is more ambiguity averse than $\succsim_B$.

Let $W_i(f) = \beta_i \min_{p \in \{\ell^\infty : \ell \in D_i\}} \int u_i(f)\, dp + (1-\beta_i) \int u_i(f)\, dq_i$ with $q_i = \int \ell^\infty dm_i(\ell)$ for $i = A, B$.

By Theorem D.1, $D = R$ for such preferences. Applying Theorem 3.4, we obtain $D_B = D_A$ since $\beta_A, \beta_B < 1$ implies $\succsim_A$ and $\succsim_B$ are strictly monotonic for bets on non-null limiting frequency events. (To see the latter, let $\Psi^{-1}(L)$ be a non-null event, $x, y, z \in X$ and $x \succ y$. Then, $L \cap D \neq \emptyset$ and $\int u(x\Psi^{-1}(L)z)\, dq > \int u(y\Psi^{-1}(L)z)\, dq$. Since $\beta < 1$, $x\Psi^{-1}(L)z \succ y\Psi^{-1}(L)z$.)

Consider the act $1_{\Psi^{-1}(\ell)}$ for some $\ell \in D_A$. Since $D_A = D_B$ is non-singleton, this act is evaluated as $W_A(1_{\Psi^{-1}(\ell)}) = (1-\beta_A)m_A(\ell)$ and $W_B(1_{\Psi^{-1}(\ell)}) = (1-\beta_B)m_B(\ell)$ respectively. Since $W_A = W_B$ on constant acts, $[f \succsim_A x \Rightarrow f \succsim_B x$ for all $x \in X$ and $f \in F]$ implies $[W_B(f) \geq W_A(f)$ for all $f \in F]$. Therefore

$$(1-\beta_B)m_B(\ell) \geq (1-\beta_A)m_A(\ell) \text{ for all } \ell \in D_A. \tag{D.1}$$

Summing (D.1) over $\ell \in D_A$ yields $\beta_A \geq \beta_B$. For all $\ell \in D_B$, (D.1) yields $m_B(\ell) \geq \frac{1-\beta_A}{1-\beta_B} m_A(\ell)$.

Turn to the "if" direction. For all acts f and measures $q = \int \ell^\infty dm(\ell)$ for some $m \in \Delta(\Delta(S))$ such that $\operatorname{supp} m \subseteq D$, $\min_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp \leq \int u(f)\, dq$, so that increasing β can only lower the valuation of an act. Observe that W evaluates f by taking a convex combination of the $\int u(f)\, d\ell^\infty$ values. The weights in this convex combination are $(1-\beta)m(\ell)$ for all but some ℓ yielding the lowest of the values, which is assigned weight $(1-\beta)m(\ell) + \beta$. The conditions on m_A and m_B ensure that $(1-\beta_B)m_B(\ell) \geq (1-\beta_A)m_A(\ell)$ so that each ℓ yielding a value other than the minimum is assigned weakly more weight by $W_B(f)$ than by $W_A(f)$. Therefore, $W_B(f) \geq W_A(f)$ with equality for constant acts, so that $f \succsim_A x \Rightarrow f \succsim_B x$ for all $x \in X$ and $f \in F$.

D.5.3 Proof of Theorem D.3

First we show $\operatorname{supp} m \subseteq R$. Suppose $\hat{\ell} \in \operatorname{supp} m$ and fix any $L \in \mathcal{O}_{\hat{\ell}}$. Take $x_1, x_2, x_3 \in X$ such that $x_2 \sim \frac{1}{2}x_1 + \frac{1}{2}x_3$ and $x_1 \succ x_3$. Define two acts f and g by

$$f(\omega) = \begin{cases} x_1 & \text{if } \Psi(\omega) \in L \\ x_2 & \text{otherwise} \end{cases} \text{ and } g(\omega) = \begin{cases} x_3 & \text{if } \Psi(\omega) \in L \\ x_2 & \text{otherwise} \end{cases}.$$

Since $\int f d\ell^{\infty} = \int g d\ell^{\infty}$ for all $\ell \in \Delta(S) \backslash L$, it suffices to show that $f \nsim g$. For each $i = 1, ..., n$,

$$\begin{aligned}\int \zeta_i u(f) \, dp &= \int_{\Psi^{-1}(L)} \zeta_i u(x_1) \, dp + \int_{\Omega \backslash \Psi^{-1}(L)} \zeta_i u(x_2) \, dp \\ &= \int_{\Psi^{-1}(L)} \zeta_i [u(x_1) - u(x_2)] \, dp + \int_{\Omega} \zeta_i u(x_2) \, dp \\ &= \int_{\Psi^{-1}(L)} \zeta_i [u(x_1) - u(x_2)] \, dp \\ &= \int_{\Psi^{-1}(L)} \zeta_i [u(x_2) - u(x_3)] \, dp = -\int \zeta_i u(g) \, dp.\end{aligned}$$

The third equality follows because $\int \zeta_i dp = 0$, and the fourth comes from $x_2 \sim \frac{1}{2}x_1 + \frac{1}{2}x_3$. Because $A(a) = A(-a)$,

$$A\left(\left(\int \zeta_i u(f) \, dp\right)_{1 \leq i \leq n}\right) = A\left(\left(\int \zeta_i u(g) \, dp\right)_{1 \leq i \leq n}\right).$$

Then,

$$\int u(f) \, dp + A\left(\left(\int \zeta_i u(f) \, dp\right)_{1 \leq i \leq n}\right) \neq \int u(g) \, dp + A\left(\left(\int \zeta_i u(g) \, dp\right)_{1 \leq i \leq n}\right)$$

because $\int u(f) \, dp > \int u(g) \, dp$ by the fact that $m(L) > 0$ and $x_1 \succ x_3$. Thus, $f \nsim g$ and each measure in $\operatorname{supp} m$ is relevant.

Next, we show that $\succsim$ satisfies Continuous Symmetry. The form assumed for p and the symmetry property assumed for each ζ_i ensure that Event Symmetry is satisfied. The other properties in Symmetry along with Mixture Continuity follow directly from the properties of VEU (see Siniscalchi [15]). To see Monotone Continuity of $\succsim^*$, observe that $x' \succsim^* x A_k x''$ if and only if, for all $\alpha \in [0, 1]$ and $h \in \mathcal{F}$,

$$\begin{aligned}&\alpha u(x') + A\left(\left((1-\alpha) \int u(h) \zeta_i dp\right)_{1 \leq i \leq n}\right) \\ &\geq \alpha \left(p(A_k) u(x) + (1 - p(A_k) u(x''))\right) \\ &+ A\left(\left(\alpha \left[u(x) \int_{A_k} \zeta_i dp + u(x'') \int_{A_k^c} \zeta_i dp\right] + (1-\alpha) \int u(h) \zeta_i dp\right)_{1 \leq i \leq n}\right).\end{aligned}$$

Since p is countably additive and ζ_i is bounded and measurable, $A_k \searrow \emptyset$ implies $p(A_k) \to 0$ and $\int_{A_k} \zeta_i dp \to 0$ and $\int_{A_k^c} \zeta_i dp \to \int_{S^{\infty}} \zeta_i dp = 0$. Therefore, since n is finite and A is Lipschitz continuous, there exists a k such that A_k is small enough so that $x' \succsim^* x A_k x''$. This proves Monotone Continuity of $\succsim^*$.

Because $\succsim$ is Continuous Symmetric and every measure in $\operatorname{supp} m$ is relevant, we can apply Theorem 3.3 to conclude $R = \operatorname{supp} m$.

D.5.4 Proof of Theorem D.4

Consider the "only if" direction. Suppose $\succsim_A$ is more ambiguity averse than $\succsim_B$.

Since the two preferences agree on constant acts and thus $u_A = u_B$ up to normalization. Let $u = u_A = u_B$.

Note that if A is differentiable and $A(a) = A(-a)$ for all $a \in \mathbb{R}^n$, $DA(0) = 0$. This can be seen from $DA(a) = -DA(-a)$ for all $a \in \mathbb{R}^n$ and setting $a = 0$. Let $I_k(u(f)) \equiv \int u(f)\, dp_k + A_k\left(\left(\int \zeta_i^k u(f)\, dp_k\right)_{1\le i\le n}\right)$ for $k = A, B$. Then,

$$DI_A(c)(\varphi) = \int \varphi dp_A + DA_A\left((0)_{1\le i\le n}\right)\left(\left(\int \zeta_i^A \varphi dp_A\right)_{1\le i\le n}\right) = \int \varphi dp_A.$$

Similarly, $DI_B(c)(\varphi) = \int \varphi dp_B$. Then, Lemma C.1 implies $p_A = p_B$. The rest of the result including the "if" direction follows directly from Proposition 4 of Siniscalchi [15].

D.5.5 Proof of Theorem D.5

Suppose $\succsim$ is an i.i.d. second-order Choquet preference. We first show that $\operatorname{supp} \upsilon \subseteq R$. Suppose $\hat{\ell} \in \operatorname{supp} \upsilon$ and take $L \in \mathcal{O}_{\hat{\ell}}$. By definition, there is $L' \subseteq \Delta(S)$ such that $\upsilon(L \cup L') > \upsilon(L')$. Consider $f = 1_{\Psi^{-1}(L\cup L')}$ and $g = 1_{\Psi^{-1}(L')}$. Then, $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S) \backslash L$. Compute $\int\int u(f) d\ell^\infty d\upsilon(\ell) = \upsilon(L \cup L') > \upsilon(L') = \int\int u(g) d\ell^\infty d\upsilon(\ell)$. Thus, each $\hat{\ell} \in \operatorname{supp} \upsilon$ is relevant.

We now show that $\succsim$ satisfies Continuous Symmetry. Monotone Continuity of $\succsim^*$ and Mixture Continuity follow by Cerreia-Vioglio et al. [3, Theorem 7] and the fact that a capacity with finite support is continuous. That the other axioms are satisfied is straightforward.

We have shown that every measure in $\operatorname{supp} \upsilon$ is relevant and $\succsim$ satisfies Continuous Symmetry. Thus we can apply Theorem 3.3 to conclude $R = \operatorname{supp} \upsilon$.

D.5.6 Proof of Theorem D.6

Consider the "if" direction first. Let $u = u_A = u_B$ without loss of generality. It is a property of the Choquet integral that $\upsilon_B \geq \upsilon_A$ implies $\int\int u(f) d\ell^\infty d\upsilon_B(\ell) \geq \int\int u(f) d\ell^\infty d\upsilon_A(\ell)$ for all $f \in F$ with equality for constant acts (see e.g., Denneberg [4, Propositions 5.1 and 5.2]). This implies $\succsim_A$ is more ambiguity averse than $\succsim_B$.

Now turn to the "only if" direction. That $u_A = u_B$ up to normalization is equivalent to the two preferences agreeing on constant acts, a necessary condition for $\succsim_A$ is more ambiguity averse than $\succsim_B$. Suppose, to the contrary, that $\succsim_A$ is more ambiguity averse than $\succsim_B$ but $\upsilon_B(L') < \upsilon_A(L')$ for some $L' \subseteq \Delta(S)$. Then $1_{\Psi^{-1}(L')} \sim_A \upsilon_A(L')x^* + (1 - \upsilon_A(L'))x_* \succ_B 1_{\Psi^{-1}(L')}$, contradicting $\succsim_A$ is more ambiguity averse than $\succsim_B$.

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