Online Appendix to accompany Baliga, Hanany and Klibanoff, “Polarization and Ambiguity”, *American Economic Review*

This Appendix contains all proofs not included in the main text and some further results on the direction of updating.

A Proofs not in the Main Text

*Proof.* [Proof of Theorem 1] Bayesian updating is only well-defined following positive probability signals. Therefore, assume $\sum_i \hat{\eta}(\theta_i) \pi_{\theta_i}(x) > 0$ and $\sum_i \hat{\nu}(\theta_i) \pi_{\theta_i}(x) > 0$. We use proof by contradiction. Suppose two individuals use Bayesian updating and that $\hat{\eta}$ stochastically dominates $\hat{\nu}$ and $\hat{\nu}$ stochastically dominates $\hat{\eta}$ with at least one dominance strict (i.e., that polarization occurs). Observe that $\hat{\eta}$ stochastically dominates $\hat{\nu}$ implies $\hat{\eta}(\theta_1) \leq \hat{\nu}(\theta_1) = \frac{\hat{\eta}(\theta_1) \pi_{\theta_1}(x)}{\sum_i \hat{\eta}(\theta_i) \pi_{\theta_i}(x)}$ and $\hat{\eta}(\theta_{|\Theta}) \geq \hat{\nu}(\theta_{|\Theta}) = \frac{\hat{\eta}(\theta_{|\Theta}) \pi_{\theta_{|\Theta}}(x)}{\sum_i \hat{\eta}(\theta_i) \pi_{\theta_i}(x)}$. Simplifying, this implies

$$\pi_{\theta_1}(x) \geq \sum_i \hat{\eta}(\theta_i) \pi_{\theta_i}(x) \geq \pi_{\theta_{|\Theta}}(x). \quad (6)$$

Similarly, observe that $\hat{\nu}$ stochastically dominates $\hat{\eta}$ implies $\hat{\nu}(\theta_1) \geq \hat{\nu}(\theta_1) = \frac{\hat{\nu}(\theta_1) \pi_{\theta_1}(x)}{\sum_i \hat{\nu}(\theta_i) \pi_{\theta_i}(x)}$ and $\hat{\nu}(\theta_{|\Theta}) \leq \hat{\nu}(\theta_{|\Theta}) = \frac{\hat{\nu}(\theta_{|\Theta}) \pi_{\theta_{|\Theta}}(x)}{\sum_i \hat{\nu}(\theta_i) \pi_{\theta_i}(x)}$. Simplifying, this implies

$$\pi_{\theta_1}(x) \leq \sum_i \hat{\nu}(\theta_i) \pi_{\theta_i}(x) \leq \pi_{\theta_{|\Theta}}(x). \quad (7)$$

The only way for (6) and (7) to be satisfied simultaneously is when

$$\pi_{\theta_1}(x) = \sum_i \hat{\eta}(\theta_i) \pi_{\theta_i}(x) = \sum_i \hat{\nu}(\theta_i) \pi_{\theta_i}(x) = \pi_{\theta_{|\Theta}}(x). \quad (8)$$

Notice that under (8) $\hat{\eta}(\theta_1) = \hat{\nu}(\theta_1)$, $\hat{\eta}(\theta_{|\Theta}) = \hat{\nu}(\theta_{|\Theta})$, $\hat{\eta}(\theta_1) = \hat{\nu}(\theta_1)$ and $\hat{\eta}(\theta_{|\Theta}) = \hat{\nu}(\theta_{|\Theta})$. Given $\sum_i \hat{\eta}(\theta_i) \pi_{\theta_i}(x) = \sum_i \hat{\nu}(\theta_i) \pi_{\theta_i}(x)$, consider the
induction hypothesis that, for some $1 \leq n < |\Theta|$, 

$$\hat{\eta}(\theta_i) = \hat{\nu}(\theta_i) \text{ and } \tilde{\eta}(\theta_i) = \tilde{\nu}(\theta_i) \text{ for } i = 1, ..., n.$$ 

Under this hypothesis, $\tilde{\eta}$ stochastically dominates $\tilde{\nu}$ implies $\tilde{\eta}(\theta_{n+1}) \leq \tilde{\nu}(\theta_{n+1}) = \frac{\hat{\eta}(\theta_{n+1})\pi_1(x)}{\sum_i \hat{\eta}(\theta_i)\pi_\theta(x)}$ and $\hat{\nu}$ stochastically dominates $\hat{\eta}$ implies $\hat{\nu}(\theta_{n+1}) \geq \hat{\eta}(\theta_{n+1}) = \frac{\tilde{\eta}(\theta_{n+1})\pi_\theta(x)}{\sum_i \tilde{\eta}(\theta_i)\pi_\theta(x)}$. Therefore, 

$$\hat{\eta}(\theta_{n+1}) = \hat{\nu}(\theta_{n+1}) \text{ and } \tilde{\eta}(\theta_{n+1}) = \tilde{\nu}(\theta_{n+1}).$$

Since we showed above that the induction hypothesis holds for $n = 1$, we conclude that $\tilde{\eta}$ stochastically dominates $\tilde{\nu}$ and $\hat{\nu}$ stochastically dominates $\hat{\eta}$ implies $\tilde{\eta} = \tilde{\nu}$ and $\hat{\eta} = \hat{\nu}$. This contradicts our supposition of polarization. \(\square\)

**Proof.** [Proof of Proposition 1] It is immediate from (1) that $\alpha^*(x) \in (0,1)$ since $\mu \in (0,1)$ and $\phi' > 0$. To prove (i), fix any $x \in \mathcal{X}$ and, from (2), observe that for any $y \in \mathcal{X}$, $\alpha^*(y)$ is a strictly increasing function of $\alpha^*(x)$ in any solution of the system of first-order conditions. This and the fact that $\phi$ is concave implies that the left-hand side of (1) is strictly increasing in $\alpha^*(x)$ and decreasing in $\frac{\pi_1(x)}{\pi_\theta(x)}$. The right-hand side of (1) is strictly increasing in $\mu$ and constant in $\alpha^*(x)$. Therefore, $\alpha^*(x)$ is well-defined and strictly increasing in $\mu$ and $\frac{\pi_1(x)}{\pi_\theta(x)}$. The first-order condition describing the best constant prediction, which we denote here by $\bar{\alpha}$, is 

$$\frac{\bar{\alpha}}{1 - \bar{\alpha}} \frac{\phi'[-(\bar{\alpha})^2]}{\phi'[-(1 - \bar{\alpha})^2]} = \frac{\mu}{1 - \mu}. \quad (9)$$

Again, concavity of $\phi$ implies that the left-hand side is strictly increasing in $\bar{\alpha}$ and thus the best constant prediction is strictly increasing in $\mu$.

To prove (ii), let $\beta^*(x)$ denote the optimal prediction after observing $x$. By the first-order conditions for optimality, these predictions must satisfy 

$$\frac{\beta^*(x)}{1 - \beta^*(x)} \frac{\phi'[-(\beta^*(x))^2]}{\phi'[-(1 - \beta^*(x))^2]} = \frac{\nu_x}{1 - \nu_x}. \quad (10)$$


Therefore, by the same reasoning as in (i), \( \beta^*(x) \) is strictly increasing in the posterior \( \nu_x \). Comparing (9) and (10) and using concavity of \( \phi \) yields

\[
\nu_x \gtrless \mu
\]

if and only if

\[
\beta^*(x) \gtrless \bar{\alpha}.
\]

Finally, under dynamically consistent updating, from (3), the posteriors must satisfy

\[
\frac{\alpha^*(x)}{1 - \alpha^*(x)} \frac{\phi'[-(\alpha^*(x))^2]}{\phi'[-(1 - \alpha^*(x))^2]} = \frac{\nu_x}{1 - \nu_x}.
\]

Therefore, \( \beta^*(x) = \alpha^*(x) \).

\( \square \)

Proof. [Proof of Proposition 2] Dynamically consistent updating implies that (3) is satisfied in addition to (1). Combining the two equalities yields,

\[
\nu_x \gtrless \mu
\]

if and only if

\[
\frac{\phi'[-(\alpha^*(x))^2]}{\phi'[-(1 - \alpha^*(x))^2]} \gtrless \frac{\phi'[E_{\pi_0}(-(\alpha^*(X))^2)]}{\phi'[E_{\pi_1}(-(1 - \alpha^*(X))^2)]} \pi_0(x) \pi_1(x).
\]

\( \square \)

Proof. [Proof of Theorem 2] By Proposition 2,

\[
\nu_{x,H} \geq \mu
\]

if and only if

\[
\frac{\phi'[-(\alpha^*(x)^H)^2]}{\phi'[-(1 - \alpha^*(x)^H)^2]} \geq \frac{\phi'[E_{\pi_0}(-(\alpha^*(X)^H))^2]}{\phi'[E_{\pi_1}(-(1 - \alpha^*(X)^H))^2]} \pi_0(x^H) \pi_1(x^H),
\]

(11)
For all \( y \in \mathcal{X} \), since \( \frac{\pi_1(x_H)}{\pi_0(x_H)} \geq \frac{\pi_1(y)}{\pi_0(y)} \), it follows from part (i) of Proposition 1 that

\[
\alpha^*(x^H) \geq \alpha^*(y).
\]

Therefore \( (\alpha^*(x^H))^2 \geq E_{\pi_0}(\alpha^*(X))^2 \) and \( (1 - \alpha^*(X))^2 \leq E_{\pi_1}(1 - \alpha^*(X))^2 \).

As \( \phi \) is concave, this implies

\[
\frac{\phi'[-(\alpha^*(x^H))^2]}{\phi'[-(1 - \alpha^*(x^H))^2]} \geq \frac{\phi'[E_{\pi_0}(-(\alpha^*(X))^2)]}{\phi'[E_{\pi_1}(-(1 - \alpha^*(X))^2)]},
\]

(12)

Since \( \frac{\pi_1(x_H)}{\pi_0(x_H)} \geq 1 \), (11) follows. Thus \( \nu_{xH} \geq \mu \).

Furthermore, (3), (12) and (1) imply

\[
\frac{\nu_{xH}}{1 - \nu_{xH}} = \frac{\alpha^*(x^H)}{1 - \alpha^*(x^H)} \frac{\phi'[-(\alpha^*(x^H))^2]}{\phi'[-(1 - \alpha^*(x^H))^2]} \geq \frac{\alpha^*(x^H)}{1 - \alpha^*(x^H)} \frac{\phi'[E_{\pi_0}(-(\alpha^*(X))^2)]}{\phi'[E_{\pi_1}(-(1 - \alpha^*(X))^2)]} = \frac{\mu}{1 - \mu} \frac{\pi_1(x^H)}{\pi_0(x^H)},
\]

where the last expression is the posterior ratio generated by Bayesian updating of \( \mu \) after observing \( x^H \).

An analogous argument shows \( \mu \geq \nu_{xL} \) and

\[
\frac{\nu_{xL}}{1 - \nu_{xL}} \leq \frac{\mu}{1 - \mu} \frac{\pi_1(x^L)}{\pi_0(x^L)}.
\]

\[\square\]

Proof. [Proof of Theorem 3] Recall that the optimal prediction \( \alpha^*(x) \) is continuous and increasing in the prior probability of \( \theta = 1 \). Denote this probability by \( \eta \). As the optimal prediction is 0 if \( \eta = 0 \) and 1 if \( \eta = 1 \), considering \( \eta \) close enough to 0 or \( \eta \) close enough to 1 is equivalent to considering \( \alpha^*(x) \) close enough to 0 or 1 respectively. The proof strategy for determining updating for sufficiently extreme beliefs will be to consider updating for sufficiently extreme predictions.
Observe, by applying (1) and (3), that dynamically consistent updating of $\eta$ after seeing $x \in \mathcal{X}$ will be shaded upward/equal to/shaded downward compared to Bayesian updating if and only if

$$\phi'[-(\alpha^*(x))^2]\phi'[\sum_{y \in \mathcal{X}} \pi_1(y)(1 - \alpha^*(y))^2] \geq \phi'[-(1 - \alpha^*(x))^2]\phi'[\sum_{y \in \mathcal{X}} \pi_0(y)(\alpha^*(y))^2].$$

From (2), $\alpha^*(y) = \beta_{\pi_1,\pi_0}(\alpha^*(x); y)$ where $\beta_{\pi_1,\pi_0} : [0, 1] \times \mathcal{X} \to [0, 1]$ is defined by $\beta_{\pi_1,\pi_0}(z; y) = \frac{z \pi_1(y) + (1-z) \pi_0(y)}{z \pi_1(y) + (1-z) \pi_0(y)}$ for all $z \in [0, 1]$ and $y \in \mathcal{X}$. Define the function $f : [0, 1] \to \mathbb{R}$ such that

$$f(z) = \frac{\phi'[\sum_{y \in \mathcal{X}} \pi_1(y)(1 - \beta_{\pi_1,\pi_0}(z; y))^2] - \phi'[-\sum_{y \in \mathcal{X}} \pi_0(y)(\beta_{\pi_1,\pi_0}(z; y))^2]}{\phi'[-(1 - z)^2]}.$$

Under our assumptions, $f$ is continuous and differentiable. By comparing $f$ with (13), observe that when $z = \alpha^*(x) \in (0, 1)$, the direction in which updating is shaded relative to Bayesian updating is determined by the sign of $f$. Therefore we want to determine the sign of $f(z)$ when $z$ is close 0 and when it is close to 1. By the assumptions in the statement of the theorem, $0 < \phi'(0) < \phi'(-1) < \infty$ where the last inequality comes from the fact that $\phi'$ is continuous on $[-1, 0]$ and thus bounded. Then $f(0) = f(1) = 0$. Therefore, the sign of $f(z)$ when $z$ is close 0 and when it is close to 1 is determined by the sign of $f'(z)$ at 0 and 1 respectively. Differentiating $f$ (and denoting the
derivative of $\beta_{\pi_1,\pi_0}$ with respect to $z$ evaluated at $(z; y)$ by $\beta'_{\pi_1,\pi_0}(z; y)$ yields,

$$f'(z) = \frac{2\phi''[\sum_{y \in \mathcal{X}} \pi_1(y) (1 - \beta_{\pi_1,\pi_0}(z; y))^2 \sum_{y \in \mathcal{X}} \pi_1(y) (1 - \beta_{\pi_1,\pi_0}(z; y)) \beta'_{\pi_1,\pi_0}(z; y)}{\phi'[-(1 - z)^2]}$$

$$- \frac{2\phi'[\sum_{y \in \mathcal{X}} \pi_1(y) (1 - \beta_{\pi_1,\pi_0}(z; y))^2 \sum_{y \in \mathcal{X}} \pi_1(y) (1 - \beta_{\pi_1,\pi_0}(z; y)) \beta'_{\pi_1,\pi_0}(z; y)}{\phi'[-(1 - z)^2](1 - z)}$$

$$+ \frac{2\phi''[-\sum_{y \in \mathcal{X}} \pi_0(y) (1 - \beta_{\pi_1,\pi_0}(z; y))^2 \sum_{y \in \mathcal{X}} \pi_0(y) (1 - \beta_{\pi_1,\pi_0}(z; y)) \beta'_{\pi_1,\pi_0}(z; y)}{(\phi'[-(1 - z)^2])^2}$$

$$- \frac{2\phi''[-(1 - z)^2] \phi'[\sum_{y \in \mathcal{X}} \pi_0(y) (1 - \beta_{\pi_1,\pi_0}(z; y))^2]{\phi'[-(1 - z)^2]}{\phi'[-(1 - z)^2]}.$$

Thus,

$$f'(0) = 2 \left( -\frac{\phi''(-1)}{\phi'(-1)} \right) \left[ 1 - \sum_{y \in \mathcal{X}} \pi_1(y) \beta'_{\pi_1,\pi_0}(0; y) \right] + (0) \left( -\frac{\phi''(0)}{\phi'(0)} \right) \left[ 1 - \sum_{y \in \mathcal{X}} \pi_0(y) \beta'_{\pi_1,\pi_0}(0; y) \right]$$

and

$$f'(1) = (0) \left( -\frac{\phi''(0)}{\phi'(0)} \right) \left[ 1 - \sum_{y \in \mathcal{X}} \pi_1(y) \beta'_{\pi_1,\pi_0}(1; y) \right] + 2 \left( -\frac{\phi''(-1)}{\phi'(-1)} \right) \left[ 1 - \sum_{y \in \mathcal{X}} \pi_0(y) \beta'_{\pi_1,\pi_0}(1; y) \right].$$

Since $\phi''$ is negative and finite (since $\phi''$ is continuous on a bounded interval), the coefficient of ambiguity aversion, $-\frac{\phi''}{\phi'}$, is everywhere positive and finite. This allows us to conclude that the sign of $f'(0)$ is the same as the sign of $1 - \sum_{y \in \mathcal{X}} \pi_1(y) \beta'_{\pi_1,\pi_0}(0; y)$, while the sign of $f'(1)$ is the sign of $1 - \sum_{y \in \mathcal{X}} \pi_0(y) \beta'_{\pi_1,\pi_0}(1; y)$. Differentiating $\beta_{\pi_1,\pi_0}(z; y)$ shows that $\beta'_{\pi_1,\pi_0}(0; y) = \frac{\pi_1(y)}{\pi_0(y)} \frac{\pi_1(x)}{\pi_0(x)}$ and $\beta'_{\pi_1,\pi_0}(1; y) = \frac{\pi_1(x)}{\pi_0(x)} \frac{\pi_1(y)}{\pi_0(y)}$. Thus $f'(0) < 0$ and $f'(1) < 0$ if and only if

$$\frac{1}{\sum_{y \in \mathcal{X}} \pi_0(y) \frac{\pi_0(y)}{\pi_1(y)}} < \frac{\pi_1(x)}{\pi_0(x)} < \sum_{y \in \mathcal{X}} \pi_1(y) \frac{\pi_1(y)}{\pi_0(y)}. \quad (14)$$

Summarizing, we have shown that $f$ is negative for values sufficiently close to 0 and positive for values sufficiently close to 1 if and only if (14) is satisfied. Therefore, it is exactly under these conditions that updating will be shaded downward compared to Bayesian updating for beliefs sufficiently close to 0 and
shaded upward compared to Bayesian updating for beliefs sufficiently close to 1.

We now show that a neutral signal necessarily satisfies (14). Note that

\[ \sum_{y \in \mathcal{X}} \pi_1(y) \frac{\pi_1(y)}{\pi_0(y)} \geq 1 \quad \text{and} \quad \sum_{y \in \mathcal{X}} \pi_0(y) \frac{\pi_0(y)}{\pi_1(y)} \geq 1 \]

because the strictly convex constrained minimization problem

\[
\min_{w_1, ..., w_{|\mathcal{X}|}} \sum_{i=1}^{|\mathcal{X}|} \frac{w_i^2}{v_i} \quad \text{subject to} \quad \sum_{i=1}^{|\mathcal{X}|} w_i = 1,
\]

assuming \( \sum_{i=1}^{|\mathcal{X}|} v_i = 1 \) and \( v_i > 0 \) for \( i = 1, ..., |\mathcal{X}| \), has first order conditions equivalent to \( \frac{w_i}{v_i} \) constant in \( i \), thus the minimum is achieved at \( \frac{1}{\sum_{i=1}^{|\mathcal{X}|} v_i} = 1 \) with \( w_i = \frac{v_i}{\sum_{i=1}^{|\mathcal{X}|} v_i} = v_i \). Moreover, since there exists at least one informative signal, i.e., \( y \in \mathcal{X} \) such that \( \frac{\pi_1(y)}{\pi_0(y)} \neq 1 \), the unique minimum is not attained and so \( \sum_{y \in \mathcal{X}} \pi_1(y) \frac{\pi_1(y)}{\pi_0(y)} > 1 \) and \( \sum_{y \in \mathcal{X}} \pi_0(y) \frac{\pi_0(y)}{\pi_1(y)} > 1 \). Thus, (14) is always satisfied if \( \frac{\pi_1(x)}{\pi_0(x)} = 1 \) (i.e., if \( x \) is a neutral signal).

Finally, observe that if \( x \) is a neutral signal, then, since Bayesian updating would be flat, updating shaded downward implies updating is downward and updating shaded upward implies updating is upward, generating polarization.

\[ \square \]

**Remark 3.** The theorem remains true if \( \phi'(0) = 0 \) and the requirements of the theorem are otherwise satisfied. This case requires an argument based on second-order comparisons. Intuitively, second-order differences that were previously masked may now become important in the limit because the zero creates unboundedly large ambiguity aversion (as measured by \( -\frac{\phi''}{\phi'} \)) near perfect predictions. Specifically, one can show that, for beliefs close to \( \theta \), a second-order comparison yields that the payoff following a neutral signal is larger than the expected payoff before seeing the signal. This drives the comparison of ex ante versus interim hedging effects and generates the polarization. Moreover, in this case, the polarization result may be extended beyond neutral signals to all signals having a likelihood ratio lying in an interval containing 1.

**Proof.** [Proof of Proposition 3] From Lemma 1, \( \nu_{x^M} \gtrless \mu \) if and only if

\[
\sum_{y \in \mathcal{X}} \pi_1(y) \frac{\left( \frac{\pi_1(x^M)}{\pi_0(x^M)} \right)^{1/2} - \frac{\pi_1(y)}{\pi_0(y)}}{\left[ \alpha^*(x^M) \frac{\pi_1(y)}{\pi_0(y)} + (1 - \alpha^*(x^M)) \frac{\pi_1(x^M)}{\pi_0(x^M)} \right]^2} \gtrless 0. \tag{15}
\]
We consider the following exhaustive list of possibilities:

(i) \( \left( \frac{\pi_1(x^M)}{\pi_0(x^M)} \right)^{\frac{1}{\gamma}+2} \geq \frac{\pi_1(x^H)}{\pi_0(x^H)} \). In this case, using \( \frac{\pi_1(x^L)}{\pi_0(x^L)} < \frac{\pi_1(x^M)}{\pi_0(x^M)} < \frac{\pi_1(x^H)}{\pi_0(x^H)} \), the left-hand side of (15) is strictly positive, and therefore updating is always upward, so set \( \tau(\gamma, \pi_0, \pi_1) = 0 \). Note that a necessary condition for this case is that \( \frac{\pi_1(x^M)}{\pi_0(x^M)} > 1 \).

(ii) \( \left( \frac{\pi_1(x^M)}{\pi_0(x^M)} \right)^{\frac{1}{\gamma}+2} < \frac{\pi_1(x^L)}{\pi_0(x^L)} \). In this case, using \( \frac{\pi_1(x^L)}{\pi_0(x^L)} < \frac{\pi_1(x^M)}{\pi_0(x^M)} < \frac{\pi_1(x^H)}{\pi_0(x^H)} \), the left-hand side of (15) is strictly negative, and therefore updating is always downward, so set \( \tau(\gamma, \pi_0, \pi_1) = 1 \). Note that a necessary condition for this case is that \( \frac{\pi_1(x^M)}{\pi_0(x^M)} < 1 \).

(iii) \( \frac{\pi_1(x^H)}{\pi_0(x^H)} > \left( \frac{\pi_1(x^M)}{\pi_0(x^M)} \right)^{\frac{1}{\gamma}+2} > \frac{\pi_1(x^L)}{\pi_0(x^L)} \). In this case, using \( \frac{\pi_1(x^L)}{\pi_0(x^L)} < \frac{\pi_1(x^M)}{\pi_0(x^M)} < \frac{\pi_1(x^H)}{\pi_0(x^H)} \), in the left-hand side of (15), the term for \( y = x^L \) is positive and has a denominator strictly decreasing in \( \alpha^*(x^M) \), the term for \( y = x^M \) is constant in \( \alpha^*(x^M) \), and the term for \( y = x^H \) is negative and has a denominator strictly increasing in \( \alpha^*(x^M) \). Therefore the whole sum is strictly increasing in \( \alpha^*(x^M) \) and thus can change signs at most once. Three sub-cases are relevant:

(iii)(a) the left-hand side of (15) is non-negative when 0 is plugged in for \( \alpha^*(x^M) \). In this case, updating is always upward, so set \( \tau(\gamma, \pi_0, \pi_1) = 0 \).

(iii)(b) the left-hand side of (15) is non-positive when 1 is plugged in for \( \alpha^*(x^M) \). In this case, updating is always downward, so set \( \tau(\gamma, \pi_0, \pi_1) = 1 \).

(iii)(c) otherwise. In this case, continuity and strict increasingness of the left-hand side of (15) in \( \alpha^*(x^M) \) implies there exists a unique solution for \( a \) in \( (0, 1) \) to

\[
\sum_{y \in X^\prime} \pi_1(y) \left( \frac{\pi_1(x^M)}{\pi_0(x^M)} \right)^{\frac{1}{\gamma}+2} - \frac{\pi_1(y)}{\pi_0(y)} = 0.
\]

Since (16) holds with equality when \( z = a \), using constant relative ambiguity aversion (\( \phi'(z) = (-z)^\gamma \)) and given the monotonicity of \( \alpha^*(x^M) \) in \( \mu \), the associated threshold for \( \mu \) may be found by substituting \( z = a \) into (16) with equality and solving for \( \mu = \tau(\gamma, \pi_0, \pi_1) \). Doing this yields

\[
\frac{\tau(\gamma, \pi_0, \pi_1)}{1 - \tau(\gamma, \pi_0, \pi_1)} = \left( \frac{a}{1 - a} \right)^{2\gamma+1}.
\]
Therefore
\[
\tau(\gamma, \pi_0, \pi_1) = \frac{a^{2\gamma+1}}{a^{2\gamma+1} + (1-a)^{2\gamma+1}}.
\]
Collecting these results into an overall expression, the threshold is defined by:
\[
\tau(\gamma, \pi_0, \pi_1) = \frac{b^{2\gamma+1}}{b^{2\gamma+1} + (1-b)^{2\gamma+1}},
\]
where
\[
b \equiv \begin{cases} 
0 & \text{if } S(0) \geq 0 \\
 a & \text{if } S(a) = 0 \text{ and } a \in (0, 1) \\
1 & \text{if } S(1) \leq 0
\end{cases}
\]
and
\[
S(\lambda) \equiv \sum_{y \in \{x_L, x_M, x_H\}} \pi_1(y) \frac{\left(\frac{\pi_1(x_M)}{\pi_0(x_M)}\right)^{\frac{1}{\gamma+2}} - \pi_1(y)}{(\lambda \frac{\pi_1(y)}{\pi_0(y)} + (1 - \lambda) \frac{\pi_1(x_M)}{\pi_0(x_M)})^2}.
\]

Proof. [Proof of Theorem 4] Polarization is equivalent to \(\hat{\nu} \geq \hat{\eta}\) and \(\tilde{\nu} \leq \tilde{\eta}\) with at least one inequality strict. If \(\gamma = 0\), updating is Bayesian and polarization is impossible by Theorem 1, so set \(\hat{\tau} = 1\) and \(\tilde{\tau} = 0\). By Proposition 3, if \(\gamma > 0\) then polarization occurs if and only if \(\hat{\eta} \geq \tau(\hat{\gamma}, \pi_0, \pi_1)\) and \(\tilde{\eta} \leq \tau(\tilde{\gamma}, \pi_0, \pi_1)\) with at least one inequality strict, where the \(\tau\) function is the one defined in that result.

Proof. [Proof of Corollary 1] From Proposition 3, \(\hat{\tau} = \tau(\hat{\gamma}, \pi_0, \pi_1)\) and \(\tilde{\tau} = \tau(\tilde{\gamma}, \pi_0, \pi_1)\). The rest is immediate from Theorem 4.

Proof. [Proof of Corollary 2] From Proposition 3, such a threshold exists. Since \(\pi_0(x^M) = \pi_1(x^M)\) implies \(\pi_0(x^L) - \pi_1(x^L) = \pi_1(x^H) - \pi_0(x^H) > 0\), calculation shows that the relevant case in the proof of Proposition 3 is case \((iii)(c)\). Thus \(\tau(\gamma, \pi_0, \pi_1) = \frac{a^{2\gamma+1}}{a^{2\gamma+1} + (1-a)^{2\gamma+1}} = \frac{1}{1 + (\frac{1-a}{a})^{2\gamma+1}}\) where \(a \in (0, 1)\) is the unique
solution of \( S(a) = 0 \). Simplifying yields

\[
\frac{1 - a}{a} = \sqrt{\frac{\pi_1(x^H) \pi_1(x^L)}{\pi_0(x^H) \pi_0(x^L)}}.
\]

\[ \square \]

## B Further Results on the Direction of Updating

The next result combines Proposition 2 and equations (2) and (3) to show a general form relating fundamentals to the direction of updating.

**Proposition 4.** The posterior \( \nu_x \) is above/equal to/below the prior \( \mu \) if and only if the fundamentals \((\mu, \phi, \pi_1, \pi_0)\) are such that

\[
\frac{z}{1 - z} \frac{\phi'[-z^2]}{\phi'[-(1 - z)^2]} \gtrless \frac{\mu}{1 - \mu},
\]

for the unique \( z \in (0, 1) \) solving

\[
\frac{z}{1 - z} \phi' \left[ -z^2 \sum_{y \in X} \pi_0(y) \left( \frac{\pi_1(y)}{\pi_0(y)} \right)^2 \right] \leq \frac{\mu}{1 - \mu}.
\]

**Proof.** Substituting (3) into (4) and rearranging yields

\[
\frac{\alpha^*(x)}{1 - \alpha^*(x)} \frac{\phi'[-\alpha^*(x)^2]}{\phi'[-(1 - \alpha^*(x))^2]} \gtrless \frac{\mu}{1 - \mu}.
\]

From (2), we obtain for all \( y \in X \),

\[
\alpha^*(y) = \frac{\alpha^*(x) \pi_1(y)}{\alpha^*(x) \pi_0(y) + (1 - \alpha^*(x)) \pi_1(y) \pi_0(x)}.
\]

10
Using this together with (3), $\alpha^*(x)$ is the unique solution to

$$
\frac{\alpha^*(x)}{1 - \alpha^*(x)} \phi' \left[ -\alpha^*(x) \sum_{y \in \mathcal{X}} \frac{\pi_0(y) \left( \frac{\pi_1(y)}{\pi_0(y)} \right)^2}{\pi_1(y) \left( \frac{\pi_1(y)}{\pi_0(y)} \right)^2 + (1 - \alpha^*(x)) \frac{\pi_0(y)}{\pi_0(x)}} \right] = \frac{\mu - \pi_1(x)}{1 - \mu \pi_0(x)}.
$$

In interpreting inequality (16), it is important to realize that $z$ is an increasing function of beliefs $\mu$ (as follows from the argument used in proving part (i) of Proposition 1 with $z$ playing the role of $\alpha^*(x)$). In fact, (17) combines (2) and (3). This implies that $z = \alpha^*(x)$, the optimal prediction given the observation $x$. From (17), in the case of ambiguity neutrality ($\phi$ affine) $\frac{z}{1-z}$ is simply a multiple of $\frac{\mu}{1-\mu}$ so that updating is either always upward (if $\frac{\pi_1(x)}{\pi_0(x)} \geq 1$) or always downward (if $\frac{\pi_1(x)}{\pi_0(x)} \leq 1$). Similarly, we see that under ambiguity aversion, $\frac{z}{1-z}$ is generally a non-linear function of $\frac{\mu}{1-\mu}$ (reflecting the balancing of the desire to hedge with the likelihood based motivation from the ambiguity neutral case) which creates the possibility that inequality (16) may change direction as beliefs $\mu$ change. In general, the regions where it goes one way and where it goes the other may be very complex. We now offer a characterization of when updating follows a threshold rule so that 16 changes direction at most once.

**Proposition 5.** There is a threshold rule for updating $\mu$ after observing $x$ if
and only if

\[
\frac{\phi'[-z^2]}{\phi'[-(1-z)^2]} \frac{\pi_1(x)}{\pi_0(x)} \pi_0(y) \left( \frac{\pi_1(y)}{\pi_0(y)} \right)^2 
= \frac{\phi'}{\phi'} \left[ -z^2 \sum_{y \in X} \pi_0(y) \left( \frac{\pi_1(y)}{\pi_0(y)} \right)^2 \right]
\]

as a function of \( z \) has at most one zero in \( (0, 1) \) and, if a zero exists, (18) is increasing at that zero.

**Proof.** The result follows by combining the definition of a threshold updating rule with the characterization of the direction of updating given by Proposition 4.

Finally, we present a lemma showing how inequality (4), which identifies the direction of updating after observing a signal, simplifies under the assumption of constant relative ambiguity aversion. In proving Theorem 4, we use this inequality to help establish and calculate the threshold rule.

**Lemma 1.** With constant relative ambiguity aversion \( \gamma > 0 \), the posterior \( \nu_x \) is above/equal to/below the prior \( \mu \) if and only if

\[
\sum_{y \in X} \pi_1(y) \left( \frac{\pi_1(x)}{\pi_0(x)} \right)^{1+\gamma} \left( \frac{\pi_1(y)}{\pi_0(y)} \right)^2 \left( \alpha^*(x) \frac{\pi_1(y)}{\pi_0(y)} + (1 - \alpha^*(x)) \frac{\pi_1(x)}{\pi_0(x)} \right)^2 \geq 0.
\]

**Proof.** From inequality (16) and equation (17), \( \nu_x \geq \mu \) if and only if

\[
\frac{\phi'[-(\alpha^*(x))^2]}{\phi'[-(1-\alpha^*(x))^2]} \frac{\pi_1(x)}{\pi_0(x)} \leq \frac{\phi'}{\phi'} \left[ -(\alpha^*(x))^2 \sum_{y \in X} \pi_0(y) \left( \frac{\pi_1(y)}{\pi_0(y)} \right)^2 \right]
\]

(20)
Under constant relative ambiguity aversion, $\phi'(z) = (-z)^\gamma$ and therefore (20) is equivalent to

$$
\left( \frac{\pi_1(x)}{\pi_0(x)} \right)^{1/2} \frac{\sum_{y \in X} \pi_0(y) \left( \frac{\pi_1(y)}{\pi_0(y)} \right)^2}{\sum_{y \in X} \pi_1(y) \left( \frac{\pi_1(y)}{\pi_0(y)} \right)^2} \leq \frac{\pi_0(y) \left( \frac{\pi_1(y)}{\pi_0(y)} \right)^2}{\sum_{y \in X} \pi_1(y) \left( \frac{\pi_1(y)}{\pi_0(y)} \right)^2}. 
$$

Simplifying yields inequality (19). \qed