Relevance and Symmetry*

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Abstract

This paper provides a method to identify components of preference reflecting information and those reflecting only tastes. Important to this method is the identification of a unique set of revealed probability assignments (called *relevant measures*) from preferences over acts. We characterize these relevant measures and show where they appear in representations of preferences. This method works for a large set of preference models provided that the state space is treated as if it had a symmetric, "i.i.d. with unknown parameters," structure. Relevant measures are shown to characterize revealed information and to help in identifying taste components of preference representations. We apply our findings to four well-known representations of ambiguity-sensitive preferences: the α-MEU model, the smooth ambiguity model, the extended MEU with contraction model and the vector expected utility model. For each representation, the theory identifies both the set of relevant measures and components of the representation that reflect only tastes.

Keywords: Symmetry, tastes, beliefs, ambiguity, ambiguity attitude, comparative statics of information

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1 Introduction

In Savage's subjective expected utility (SEU) theory [40], an individual's preference over acts (maps from states of the world to outcomes) can be described using two ar-

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guments, a subjective probability over states that enables her to identify each act with a distribution over outcomes, and a von Neumann-Morgenstern (vNM) utility function that describes her risk attitude (i.e., preference over distributions over outcomes). As is well known, the assumption of state independence implicit in Savage's postulates P3 and P4 makes the taste aspect of an SEU preference independent of the subjective probability over states, hence separating it from beliefs. This separation has enabled theorists to examine non-expected utility models of risk attitudes while retaining the same subjective probability as in SEU, thereby facilitating the analysis of systematic violations of the vNM independence axiom in a subjective setting. From a more applied perspective, this separation provides foundation for the useful and common practice in economic modeling of thinking of beliefs as the component of a preference (representation) that may vary as information varies while taste components such as risk attitude do not.

Other violations of Savage's assumptions have motivated models that require a richer description of uncertainty about states and attitudes toward this uncertainty. In particular, this richness is useful for describing the individual's perception of and attitude towards ambiguity.[1] Just as it proved useful in SEU to identify components of preference reflecting information and those reflecting only tastes, it is useful to have a method for doing so that can be applied more generally. Our main contribution is providing such a method. Our method applies to a large set of preference models provided that the state space is treated as if it had a symmetric, "i.i.d. with unknown parameters," structure. Thus, think of the state space as an infinite product, with the unknown parameter being the distribution, ℓ, on a single ordinate. If ℓ were given, the distribution on the whole state space would be the i.i.d. product ℓ^∞. We describe below how our method exploits this symmetry. In Section 1.2, we discuss how the restrictions imposed by symmetry leave ample scope for application of the theory.

Central to our theory is defining a unique set of probability assignments that an individual's preferences reveal. We characterize such sets and show where they appear in representations of preferences. We also show that these probabilities characterize revealed information. Standard economic theory formally identifies information with the realization of an event in the state space. A preference reveals the information corresponding to a particular event if the complement of that event is Savage null according to the preference and no closed strict subset of that event has a Savage null complement. Considering symmetric preferences makes it possible for sets of probabilities over a single ordinate of the state space to be identified with events in the whole state space, and thus, formally, with revealed information. Given state independent preferences, it is natural for us to define components of preference as reflecting only tastes if they are independent of revealed information. Thus, tastes are components of preference that are fixed and not affected by changing information. In this way, our

[1]We use ambiguity to mean subjective uncertainty about probabilities, in the sense of the decision theory literature following Ellsberg [14]. See e.g., Ghirardato [22] who states ". . . 'ambiguity' corresponds to situations in which some events do not have an obvious, unanimously agreeable, probability assignment."

characterization of revealed probability assignments aids in identifying components of preference representations reflecting information and those reflecting only tastes.

We illustrate our findings by applying them to four well-known representations of ambiguity-sensitive preferences: the α-MEU model[2] (see e.g., Ghirardato, Maccheroni and Marinacci [23]), the smooth ambiguity model (see e.g., Klibanoff, Marinacci and Mukerji [31], Nau [34], Seo [41]), the extended MEU with contraction model[3] (see e.g., Gajdos et. al. [21], Gajdos, Tallon and Vergnaud [20], Kopylov [33], Tapking [44]) and the vector expected utility model (see Siniscalchi [43]). For each representation, we identify both the set of revealed probability assignments and components of the representation that reflect only tastes.

Next, we describe our notion of revealed probability assignments. We want to model an individual behaving as if only certain distributions on states matter for his preferences over acts. To illustrate, consider the literature on model uncertainty in macroeconomics and finance (see e.g., Hansen and Sargent [27]). In this literature, a dynamic stochastic general equilibrium (DSGE) model gives as output a probability distribution on observables of the economy (e.g., GDP, inflation, interest rates, asset prices, etc.), and these observables make up the states on which acts are defined. Different DSGE models or different values of parameters within a given class of models give rise to different distributions on these states. Consider an investor who is uncertain about the DSGE model and is choosing a portfolio of assets whose payoffs are determined by future realizations of the observables. Assume the universe of models generates i.i.d. distributions over states. We say that a given set of distributions L matters for the investor's preference if there are two portfolios, f and g, that yield the same distribution of payoffs as each other under the remaining distributions and yet the individual strictly prefers f over g. Given an investor who ultimately cares only about the distribution over payoffs, this preference reveals that L matters because under the other distributions there is no reason to choose f over g. We call a distribution, ℓ, on a single ordinate a *relevant measure* if the set of i.i.d. distributions generated by each open set containing ℓ matters in the sense just described. These relevant measures are the revealed probability assignments.

Consider such an investor who believes a particular set of DSGE models/i.i.d. distributions are the only ones that matter. Our theory says that one should model this by making exactly these distributions the relevant measures in the investor's objective function. A typical comparative statics exercise under model uncertainty or ambiguity is to vary the set of distributions appearing in the investor's objective function. We provide foundations for interpreting when such a manipulation corresponds to changing only the set of distributions that the investor thinks matter. Moreover, our theory provides foundations for an additional comparative static – varying the class of preferences

[2] The α-MEU terminology comes from the fact that the representing functional is a convex combination of the maxmin expected utility (MEU) model of Gilboa and Schmeidler [26] and the corresponding maxmax expected utility, with weights $\alpha \in [0, 1]$ and $1 - \alpha$ respectively.

[3] This model has a functional form that is a convex combination of MEU and expected utility with coefficients β and $1 - \beta$ respectively.

(e.g., moving from α-MEU to smooth ambiguity preferences) while holding the relevant measures fixed.

These foundations are not obvious from standard preference representations or axiomatizations as commonly found in the literature. To illustrate this, consider two MEU preferences over acts f mapping from a state space S to an outcome space X, where ℓ_1, ℓ_2 are distinct probability distributions over S and u is a vNM utility function:

$$\min_{p\in\{\ell_1,\ell_2\}} \int u(f)\, dp, \tag{1.1}$$

and

$$\min_{p\in\left\{\frac{3}{4}\ell_1+\frac{1}{4}\ell_2, \frac{1}{4}\ell_1+\frac{3}{4}\ell_2\right\}} \int u(f)\, dp. \tag{1.2}$$

One might claim that it is "obvious" from these representations that these two individuals have different distributions in mind since the sets of distributions in the preference representation differ.[4] However, (1.2) can be equivalently written as an α-MEU preference with $\alpha = 3/4$:

$$\frac{3}{4} \min_{p\in\{\ell_1,\ell_2\}} \int u(f)\, dp + \frac{1}{4} \max_{p\in\{\ell_1,\ell_2\}} \int u(f)\, dp.$$

From this perspective, it seems just as "obvious" that these two individuals have the same distributions in mind, and differ only in that (1.1) is more ambiguity averse than (1.2) considering α as an index of ambiguity aversion.

A key element of our strategy for distinguishing between these possibilities is to consider "symmetric" preferences over acts defined on the product space, S^∞. Consider preferences on this larger space that agree with the preferences over acts on a single "slice" (i.e., an ordinate S) written above, and for which all the distributions appearing in the representations are convex combinations of i.i.d. distributions (i.e., are symmetric or exchangeable). One such extension of the above preferences is:

$$\min_{p\in\{(\lambda\ell_1+(1-\lambda)\ell_2)^\infty : \lambda\in[0,1]\}} \int u(f)\, dp, \tag{1.3}$$

and

$$\min_{p\in\left\{(\lambda\ell_1+(1-\lambda)\ell_2)^\infty : \lambda\in[\frac{1}{4},\frac{3}{4}]\right\}} \int u(f)\, dp. \tag{1.4}$$

A different extension is:

$$\min_{p\in\left\{\ell_1^\infty,\ell_2^\infty\right\}} \int u(f)\, dp, \tag{1.5}$$

[4]Note that the convex hull of $\left\{\frac{3}{4}\ell_1 + \frac{1}{4}\ell_2, \frac{1}{4}\ell_1 + \frac{3}{4}\ell_2\right\}$ is a strict subset of the convex hull of $\{\ell_1, \ell_2\}$ so that whether we write the preferences as above or, equivalently, replace the sets above with their respective convex hulls, the sets differ.

and

$$\min_{p \in \left\{ \frac{3}{4}\ell_1^\infty + \frac{1}{4}\ell_2^\infty, \frac{1}{4}\ell_1^\infty + \frac{3}{4}\ell_2^\infty \right\}} \int u(f)\, dp. \tag{1.6}$$

Notice that under the first extension, (1.4) can no longer be re-written as an α-MEU representation with $\alpha = 3/4$, while under the second extension, (1.6) is equivalent to:

$$\frac{3}{4} \min_{p \in \left\{ \ell_1^\infty, \ell_2^\infty \right\}} \int u(f)\, dp + \frac{1}{4} \max_{p \in \left\{ \ell_1^\infty, \ell_2^\infty \right\}} \int u(f)\, dp.$$

Our notion of relevant measures picks up this distinction – (1.3) and (1.4) have different sets of relevant measures ($\{\lambda\ell_1 + (1-\lambda)\ell_2 : \lambda \in [0,1]\}$ and $\left\{\lambda\ell_1 + (1-\lambda)\ell_2 : \lambda \in [\frac{1}{4}, \frac{3}{4}]\right\}$ respectively), while (1.5) and (1.6) share the same set of relevant measures, $\{\ell_1, \ell_2\}$. Thus by moving to the symmetric product state space environment, we see how examining the relevant measures lets one say that individuals with preferences as in (1.3) and (1.4) reveal that they have distinct distributions on a slice S in mind (as in the first "obvious" interpretation of (1.1) and (1.2)), while individuals with preferences as in (1.5) and (1.6) reveal that they have the same distributions on a slice S in mind and differ only in an aspect of preference that we show (see section 4) can formally be identified as taste (as in the second "obvious" interpretation of (1.1) and (1.2)). With symmetric preferences, the individual's revealed information is completely characterized by the relevant measures.

The rest of the paper is organized as follows. In the remainder of this section, we discuss related literature (section 1.1) and the extent to which the symmetric setting of our theory is of broad interest and applicability (section 1.2). Section 2 describes the formal setting and notation. Section 3 defines Continuous Symmetric preferences and the notion of relevant measure and provides the fundamental results relating relevant measures to representations of Continuous Symmetric preferences. It also contains our definition of revealed information and of tastes and relates them to relevant measures. Section 4 applies these results to identify relevant measures and components representing tastes in four specific decision models. Appendix A shows how our theory may be extended to preferences that violate symmetry due to observable differences across slices/ordinates of the state space. All proofs and related material are contained in Appendix B.

1.1 Related literature

There is an alternative approach in the literature to identifying sets of distributions over states from an individual's preferences (see Ghirardato, Maccheroni and Marinacci [23], Nehring ([35],[36]), Ghirardato and Siniscalchi [24], Siniscalchi [42]). Loosely, this approach identifies distributions from marginal rates of substitution in utility space. A brief comparison with our approach is in order. An advantage of the alternative approach is that it does not require a product state space or symmetry conditions on preferences. As Ghirardato and Siniscalchi [24] emphasize, the distributions identified in their approach are those that "identify candidate solutions to optimization problems." (p. 3) This is a different purpose than ours, and for the goals of this paper, a disadvantage of the alternative is that any tastes not captured by the vNM utility function u, such as ambiguity attitudes, will be incorporated into and affect the set of distributions identified. For example, for the preference represented by (1.6) the set of distributions identified by this alternative approach is the convex hull of $\left\{\frac{3}{4}\ell_1^\infty + \frac{1}{4}\ell_2^\infty, \frac{1}{4}\ell_1^\infty + \frac{3}{4}\ell_2^\infty\right\}$ which incorporates not only the relevant measures that our approach identifies but also the parameters $\frac{3}{4}, \frac{1}{4}$ that our approach identifies as tastes. That this occurs is not surprising given the connection between the distributions and optimization problems identified by Ghirardato and Siniscalchi – in general, one would expect the solution to an optimization problem to depend on all tastes other than the vNM risk attitude that they filter out, in addition to beliefs.

Another approach simply takes sets of probability distributions over the state space as an objective primitive. Such models include those in Gajdos et. al. [21], Gajdos, Tallon and Vergnaud [20], Kopylov [33], Wang [45], and Cerreia-Vioglio et. al. [9]. Our theory provides a useful linkage with the objective approach. One illustration of this is our Theorem 4.3 which shows that when the objectively given set in the extended MEU with contraction model of Gajdos et. al. [21] consists of i.i.d. measures, these are exactly the i.i.d. measures generated by the relevant measures. This confirms that in this case the objective set of measures is indeed what the individual is behaving as if he has in mind. In this sense, our approach is complementary to an objective approach.[5]

Our paper imposes a symmetry property on preferences. In doing so, we are following the work of de Finetti [11] and Hewitt and Savage [29] in the context of expected utility and recent extensions of this work to larger classes of preferences and various notions of symmetry by Epstein and Seo ([15],[16],[17],[18]), Al-Najjar and de Castro ([7],[8]) and Cerreia-Vioglio et. al. [9]. None of these papers use any of these "symmetries" to explore the concept of which i.i.d. measures (or generalizations thereof) are relevant nor the implications of this relevance for identifying tastes and information. Our particular formalization of symmetry is a preference axiom we call Event Symmetry (see Section 3). The relationship between this axiom and similar preference based notions in the literature is detailed in Klibanoff, Mukerji and Seo [32].

1.2 Illustrating the ubiquity of symmetric (and partially symmetric) environments

We would like to convince the reader that restricting attention to symmetric preferences leaves ample scope for application of the theory. Start by thinking of the preferences as those of a doctor who sees and treats a sequence of patients each of whose condition is determined by an associated disease state in S. A sequence of disease states, one

[5] Less related are models of preferences over sets of lotteries as in Olszewski [38] and Ahn [1]. As these models lack acts and a state space, the question of which probabilities are relevant in evaluating acts doesn't arise.

for each patient, is thus an element of S^∞. We do not mean to suggest that the theory is particularly tailored or appropriate for medical decisions, but it is useful when thinking about abstract concepts to be able to bring intuition derived from more familiar, concrete settings to bear. Symmetry is meant to reflect that the doctor thinks the uncertainty about the disease state of each patient is the same.

If the doctor thinks that data on other patients can tell him anything about the disease state of the patient at hand, by far the simplest and most common way of modeling this (as is done, for example, in any setting where an i.i.d. data sample is used to say something about a population that contains many members not in the sample) is to assume symmetry of the doctor's uncertainty across patients where the specific i.i.d. distribution is unknown. Thus, even when modeling a decision concerning only one patient, whenever such a decision may rely on information from treating other patients, in the background is naturally a product state space of many patients with a symmetric preference structure as in this paper.

What if not all ordinates are naturally considered symmetric for the decision at hand? For a doctor, it is usual to think of categorizing patients according to observable symptoms and history, the results of diagnostic tests, and demographic information. Patients in different categories might react to treatment according to different distributions. Replacing the overall symmetry assumption with symmetry conditional on descriptions (where a description is a vector of observable characteristics) allows our analysis and findings to be extended to such situations. This is done formally in Appendix A by replacing our Event Symmetry axiom with an assumption of Partial Symmetry. In our corresponding results, i.i.d. measures are replaced by functions mapping descriptions to i.i.d. measures. A standard linear regression model is an example of such a function; given a description, ξ, the i.i.d. measure is normal with mean $\beta\xi$ and variance σ^2. The description in this case is simply a vector giving the values of the regressors for a particular patient. The analogue of a set of relevant measures is a set of pairs (β, σ) denoting a corresponding set of regression models. For example, a doctor might act as if he views as relevant all regression models having β within certain bounds (e.g., within a confidence interval) and σ fixed. Thus, although each patient is different, a doctor who classifies them for treatment purposes based on a set of observables naturally falls within the scope of our theory.

Next we move on to more economic contexts and discuss how the symmetric framework fits in with three major strands of the economics literature applying ambiguity models: experiments, macro-finance and game theory. Bossaerts et. al. [6] analyze portfolio choices in an experiment involving a market for Arrow securities based on a draw from an Ellsberg urn. They model the portfolio as chosen using an α-MEU model with the set of probabilities reflecting the information provided to the subjects about the composition of the urn. Our theory gives a foundation for treating the set of probabilities in the α-MEU model in this way under the assumption that this α-MEU model is representing preferences on one slice of a larger, symmetric problem. Given that the bets are about a single draw that is, in principle, repeatable and that each draw would be informative about the distribution of the other draws, just as in the example of the doctor, it is natural to assume symmetry of a subject's uncertainty across draws where the specific composition of the Ellsberg urn is unknown.

In dynamic models of asset pricing with model uncertainty/ambiguity (e.g., Ju and Miao [30], Hansen and Sargent [27][28], Collard et. al. [10], and Epstein and Schneider [19]), the state space is an infinite product, S^∞, where a single ordinate represents the uncertainty about per capita consumption growth and/or dividend growth in a given period. When the unknown growth process is assumed to be i.i.d. by the representative agent, our symmetry assumption is satisfied. The modeler typically chooses which i.i.d. processes to write down in the agent's objective function based on calibration to realized real-world data (e.g., all distributions approximately matching certain moments in the data). The relevant measures identified by our theory will be exactly the i.i.d. processes the modeler has included. This provides a foundation for interpreting the agent as viewing these as the possible i.i.d. processes, and thus the agent is informed by real-world data to the same extent that the modeler's choice of these distributions was. For more complex processes, such as Markov, symmetry would need to be weakened, but a similar exercise could be carried out. For Markov, one would impose invariance with respect to *some* finite permutations instead of *all* finite permutations as in Event Symmetry.[6] In this case, the unknown process is Markov and the analogue of relevant measures are relevant transition matrices of Markov processes with each matrix specifying the marginal distribution on an ordinate as a function of the realization of the previous ordinate. Thus it is as if there is a different set of relevant measures for each possible previous period's growth.

Another context to which our theory readily applies is that of large-population models of games. In such models, there is a large society of individuals from which agents are drawn at random and matched to play a game G repeatedly. After each play, agents are separated and re-matched with (almost certainly) different co-players to eliminate strategic repeated game effects – thus, at each play, agents myopically maximize their current preferences. The approach is used to capture the idea that players form their current beliefs about the action choices of their opponents by extrapolating from past play they have encountered. Thus, in such models, the opponents' actions are viewed as if generated according to some unknown population distribution that is common across all plays of the game. Viewing an ordinate of the state space as representing uncertainty over opponents' actions in a particular play of the game, the product state space and symmetry are natural parts of such a framework. This literature typically aims to restrict the player's view of which unknown population distributions are possible to be those distributions consistent with information the player has observed from past plays. Our theory of relevant measures again provides a foundation for the practice of incorporating these restrictions through the measures put in the representations of the player's preferences. For an example explicitly referencing ambiguity in this context see Battigalli et. al. [4].

[6] Diaconis and Freedman [13] study the group of permutations characterizing a mixture of Markov processes.

2 Setting and Notation

Let S be a compact metric space and $\Omega = S^\infty$ the state space with generic element $\omega = (\omega_1, \omega_2, ...)$. The state space Ω is also compact metric (Aliprantis and Border [2, Theorems 2.61 and 3.36]). Denote by Σ_i the Borel σ-algebra on the i-th copy of S, and by Σ the product σ-algebra on S^∞. An act is a simple Anscombe-Aumann act, a measurable $f : S^\infty \to X$ having finite range (i.e., $f(S^\infty)$ is finite) where X is the set of lotteries (i.e., finite support probability measures on an outcome space Z). The set of acts is denoted by $\mathcal{F}$, and $\succsim$ is a binary relation on $\mathcal{F} \times \mathcal{F}$. As usual, we identify a constant act (an act yielding the same element of X on all of S^∞) with the element of X it yields.

Denote by Π the set of all finite permutations on $\{1, 2, ...\}$ i.e., all one-to-one and onto functions $\pi : \{1, 2, ...\} \to \{1, 2, ...\}$ such that $\pi(i) = i$ for all but finitely many $i \in \{1, 2, ...\}$. For $\pi \in \Pi$, let $\pi\omega = \left(\omega_{\pi(1)}, \omega_{\pi(2)}, ...\right)$ and $(\pi f)(\omega) = f(\pi\omega)$.

For any topological space Y, $\Delta(Y)$ denotes the set of (countably additive) Borel probability measures on Y. Unless stated otherwise, a measure is understood as a countably additive Borel measure. For later use, $ba(Y)$ is the set of finitely additive bounded real-valued set functions on Y, and $ba^1_+(Y)$ the set of nonnegative probability charges in $ba(Y)$. A measure $p \in \Delta(S^\infty)$ is called symmetric if the order doesn't matter, i.e., $p(A) = p(\pi A)$ for all $\pi \in \Pi$, where $\pi A = \{\pi\omega : \omega \in A\}$. Denote by ℓ^∞ the i.i.d. measure with the marginal $\ell \in \Delta(S)$. Define $\int_{S^\infty} f dp \in X$ by $\left(\int_{S^\infty} f dp\right)(B) = \left(\int_{S^\infty} f(\omega)(B)\, dp(\omega)\right)$. (Since f is simple, this is well-defined.)

Fix $x_*, x^* \in X$ such that $x^* \succ x_*$. For any event $A \in \Sigma$, 1_A denotes the act giving x^* on A and x_* otherwise. Informally, this is a bet on A. A finite cylinder event $A \in \Sigma$ is any event of the form $\{\omega : \omega_i \in A_i \text{ for } i = 1, ..., n\}$ for $A_i \in \Sigma_i$ and some finite n.

Endow $\Delta(S)$, $\Delta(\Delta(S))$ and $\Delta(S^\infty)$ with the relative weak* topology. To see what this is, consider, for example, $\Delta(S)$. The relative weak* topology on $\Delta(S)$ is the collection of sets $V \cap \Delta(S)$ for weak* open $V \subseteq ba(S)$, where the weak* topology on $ba(S)$ is the weakest topology for which all functions $\ell \longmapsto \int \psi d\ell$ are continuous for all bounded measurable ψ on S. Also note that a net $\ell_\alpha \in ba(S)$ converges to $\ell \in ba(S)$ under the weak* topology if and only if $\int \psi d\ell_\alpha \to \int \psi d\ell$ for all bounded measurable ψ on S. For a set $D \subseteq \Delta(S)$, denote the closure of D in the relative weak* topology by $\overline{D}$.

The support of a probability measure $m \in \Delta(\Delta(S))$, denoted $\operatorname{supp} m$, is a relative weak* closed set such that $m((\operatorname{supp} m)^c) = 0$ and if $G \cap \operatorname{supp} m \neq \emptyset$ for relative weak* open G, $m(G \cap \operatorname{supp} m) > 0$. (See e.g., Aliprantis and Border [2, p.441].)

Let $\Psi_n(\omega) \in \Delta(S)$ denote the empirical frequency operator $\Psi_n(\omega)(A) = \frac{1}{n}\sum_{t=1}^n I(\omega_t \in A)$ for each event A in S. Define the limiting frequency operator Ψ by $\Psi(\omega)(A) = \lim_n \Psi_n(\omega)(A)$ if the limit exists and 0 otherwise. Also, to map given limiting frequencies or sets of limiting frequencies to events in S^∞, we consider the natural inverses $\Psi^{-1}(\ell) = \{\omega : \Psi(\omega) = \ell\}$ and $\Psi^{-1}(L) = \{\omega : \Psi(\omega) \in L\}$ for $\ell \in \Delta(S)$ and $L \subseteq \Delta(S)$.

3 Symmetry and Relevance

3.1 Symmetric Preferences

We start by stating the conditions on preferences over acts $\mathcal{F}$ that delineate the scope of our theory of relevance. The theory will apply to preferences satisfying the following axioms.

Axiom 1 (C-complete Preorder)**.** *$\succsim$ is reflexive, transitive and the restriction of $\succsim$ to X is complete.*

Notice that we allow $\succsim$ to be incomplete. Some of our results will later invoke completeness.

Axiom 2 (Monotonicity)**.** *If $f(\omega) \succsim g(\omega)$ for all $\omega \in S^\infty$, $f \succsim g$.*

Monotonicity rules out state-dependence of preferences over X. This allows us to focus on states purely as specifying the resolution of acts.

Axiom 3 (Risk Independence)**.** *For all $x, x', x'' \in X$ and $\alpha \in (0,1)$, $x \succsim x'$ if and only if $\alpha x + (1-\alpha) x'' \succsim \alpha x' + (1-\alpha) x''$.*

This is the standard vNM Independence axiom on lotteries. This rules out non-expected utility preferences over lotteries. It allows us to separate attitudes toward risk from other aspects of preferences in a simple way, using a familiar vNM utility function.

Axiom 4 (Non-triviality)**.** *There exist $x, y \in X$ such that $x \succ y$.*

To describe our remaining axioms, it is notationally convenient to introduce the binary relation $\succsim^*$ derived from $\succsim$:

$$f \succsim^* g \text{ if } \alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h \text{ for all } \alpha \in [0,1] \text{ and } h \in \mathcal{F}.$$

Ghirardato, Maccheroni and Marinacci [23] refer to $\succsim^*$ as an unambiguous preference. We will not use this terminology here for reasons that will become clear later. As they state, Klaus Nehring is the first one to suggest using this maximal independent restriction $\succsim^*$ of a given $\succsim$, in a 1996 talk. See also Nehring ([35], [36], [37]). Observe that, given Monotonicity and Risk Independence, $\succsim^*$ and $\succsim$ are identical when restricted to constant acts, while, for more general acts, $f \succsim^* g$ implies $f \succsim g$ but the converse may be false.

The key axiom delineating the domain of our theory is Event Symmetry which says that the ordinates of S^∞ are viewed as interchangeable.

Axiom 5 (Event Symmetry)**.** *For all finite cylinder events $A \in \Sigma$ and finite permutations $\pi \in \Pi$, $1_A \sim^* 1_{\pi A}$.*

A natural notion of symmetry, as expressed through preferences, is that the decision maker is always indifferent between betting on an event and betting on its permutation. The use of the term "always" here means at least that this preference should hold no matter what other act the individual faces in combination with the bet. In an Anscombe-Aumann framework such as ours, this may be expressed by the statement that

$$\alpha 1_A + (1-\alpha)h \sim \alpha 1_{\pi A} + (1-\alpha)h \text{ for all } \alpha \in [0,1] \text{ and all acts } h, \tag{3.1}$$

which is exactly $1_A \sim^* 1_{\pi A}$. In the language of Ghirardato and Siniscalchi [24], note that, thinking of acts as state-contingent utility consequences of actions and h as a status-quo, (3.1) says a move away from the status quo in the direction of 1_A is indifferent to the same size move away from the status quo in the direction of $1_{\pi A}$ no matter what the status quo h and no matter how far one moves away from it. The idea behind Event Symmetry is that such utility transfers are considered indifferent because the ordinates are viewed as (ex-ante) identical. For preferences satisfying the usual Anscombe-Aumann independence axiom, $1_A \sim 1_{\pi A}$ implies $1_A \sim^* 1_{\pi A}$. For preferences that may violate independence (e.g., because of ambiguity concerns), this is not true, and thus we cannot substitute the former condition for the latter.

Remark 3.1. As written, Event Symmetry seems to depend on the choice of x^*, x_* in defining 1_A. In fact, in the presence of our other axioms, Event Symmetry implies that the analogous property holds for any choice of $x_*, x^* \in X$.

Combining all of these conditions defines the class of preferences we will work with:

Definition 3.1. $\succsim$ satisfies *Symmetry* if it satisfies C-complete Preorder, Monotonicity, Risk Independence, Non-triviality, and Event Symmetry.

When we say that $\succsim$ is Symmetric, we mean that it satisfies Symmetry.

In addition to Symmetry, we will often need some form of continuity of preference. Different forms of continuity will be more or less convenient for subsequent results. We now state three forms of continuity that are used in the paper. The first and second are standard mixture continuity requirements.

Axiom 6 (Mixture Continuity of $\succsim$)**.** *For all $f, g, h \in \mathcal{F}$, the sets $\{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succsim h\}$ and $\{\lambda \in [0,1] : h \succsim \lambda f + (1-\lambda) g\}$ are closed in* $[0,1]$.

Mixture continuity of $\succsim$ appears many places in the literature. A weakening of this requirement is the Mixture Continuity of $\succsim^*$:[7]

[7] To see that this is a weakening, observe that

$$\{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succsim^* h\} \\ = \bigcap_{\alpha \in [0,1], f' \in \mathcal{F}} \{\lambda \in [0,1] : \lambda(\alpha f + (1-\alpha) f') + (1-\lambda)(\alpha g + (1-\alpha) f') \succsim \alpha h + (1-\alpha) f'\}.$$

Mixture Continuity of $\succsim$ implies this set is closed since it is the intersection of closed sets. The same reasoning applies for the set $\{\lambda \in [0,1] : h \succsim^* \lambda f + (1-\lambda) g\}$.

Axiom 7 (Mixture Continuity of $\succsim^*$)**.** *For all $f, g, h \in \mathcal{F}$, the sets $\{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succsim^* h\}$ and $\{\lambda \in [0,1] : h \succsim^* \lambda f + (1-\lambda) g\}$ are closed in $[0,1]$.*

We will want additional continuity in order to restrict attention to countably additive measures. The standard approach to this in the literature is based on the application to $\succsim^*$ of the monotone continuity of Arrow [3], as in Ghirardato, Maccheroni and Marinacci [23].

Axiom 8 (Monotone Continuity of $\succsim^*$)**.** *For all $x, x', x'' \in X$, if $A_n \searrow \emptyset$ and $x' \succ x''$, then $x' \succsim^* x A_n x''$ for some n.*

Definition 3.2. $\succsim$ satisfies *Continuous Symmetry* if it is Symmetric and satisfies Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$.

When we say that $\succsim$ is Continuous Symmetric, we mean that it satisfies Continuous Symmetry.

3.2 Relevance

We now formalize what it means for a measure $\ell \in \Delta(S)$ to be relevant according to preferences $\succsim$. For notational convenience, let $\mathcal{O}_\ell$ be the collection of open subsets of $\Delta(S)$ that contains ℓ. That is, for $\ell \in \Delta(S)$, $\mathcal{O}_\ell = \{L \subseteq \Delta(S) : L \text{ is open}, \ell \in L\}$.

Definition 3.3. A measure $\ell \in \Delta(S)$ is *relevant* (according to preferences $\succsim$) if, for any $L \in \mathcal{O}_\ell$, there are $f, g \in \mathcal{F}$ such that $f \nsim g$ and $\int f d\hat{\ell}^\infty = \int g d\hat{\ell}^\infty$ for all $\widehat{\ell} \in \Delta(S) \backslash L$.

In words, ℓ is relevant if it satisfies the following property: For each open ball around ℓ, there are acts that are not indifferent despite generating identical induced distributions over outcomes when any measure outside this ball governs the independent realization of each ordinate S. The use of open balls is required only because $\Delta(S)$ is infinite. This definition is in the spirit of the notion of non-null as traditionally used in decision theory (e.g., Savage [40]).[8] To see the connection, recall that an event is non-null if there are acts $f \nsim g$ such that $f = g$ on all states outside of that event. We consider open sets of measures, $L \in \mathcal{O}_\ell$, instead of events, and $\int f d\hat{\ell}^\infty = \int g d\hat{\ell}^\infty$ for all other measures $\widehat{\ell}$ instead of $f = g$ on all other states.

Why is it enough to consider equality of the lotteries generated by f and g for i.i.d. measures, $\hat{\ell}^\infty$ (and by linearity of the integral, therefore, for any mixtures over these i.i.d. measures)? When $\succsim$ is Continuous Symmetric, we will show there is a natural sense in which mixtures over i.i.d. measures (i.e., exchangeable measures) will be the only ones that matter for preference. Furthermore, as Continuous Symmetry implies expected utility on constant acts, one could replace $\int f d\hat{\ell}^\infty = \int g d\hat{\ell}^\infty$ by the analogous

[8] The definition is also reminiscent of the definition of relevant subjective state in Dekel, Lipman and Rustichini [12, Definition 1]. In the case of a finite subjective state space, a state is relevant if there are two menus $x \nsim y$, the valuations of which coincide on all other subjective states. The infinite case uses open neighborhoods just as we do.

condition on expected utilities, $\int u(f)d\hat{\ell}^{\infty} = \int u(g)d\hat{\ell}^{\infty}$, without changing the meaning of the definition within our theory.

Next we introduce an alternative notion of relevance (or, more precisely, irrelevance) based on bets on events generated by limiting frequencies. In reading the definition recall that, for $A \subseteq \Delta(S)$, $\Psi^{-1}(A)$ is the event that limiting frequencies over S lie in A.

Definition 3.4. A measure $\ell \in \Delta(S)$ is *irrelevant* (according to preferences $\succsim$) if, for some $L \in \mathcal{O}_{\ell}$, $\Psi^{-1}(L)$ is Savage null i.e., $f_{\Psi^{-1}(L)}g \sim g$ for all $f, g \in \mathcal{F}$.

That is, in an i.i.d. environment, $\ell \in \Delta(S)$ is irrelevant when what an act yields on the limiting frequency event generated by an open neighborhood containing ℓ never affects preference. It is as if the individual knows this limiting frequency event will not happen.

3.3 Relevance and Continuous Symmetric Preferences

Assuming Continuous Symmetry, we show that the two notions of relevance offered above agree, and we provide a representation of the set of relevant measures in $\Delta(S)$. We also show that any such preferences may be represented by an increasing functional on the expected utilities generated by the relevant measures. Furthermore, up to closure, all relevant measures are needed for such a representation.

We first provide a Bewley-style (Bewley [5]) representation result for the induced relation $\succsim^*$. Compared to similar results in the literature (e.g., Ghirardato, Maccheroni and Marinacci [23], Gilboa et. al. [25], Ghirardato and Siniscalchi [24], Nehring [35]) the key difference is that Symmetry (and in particular, Event Symmetry) allows a de Finetti-style decomposition of the representing set of measures, C, the Bewley set.

Lemma 3.1. *Suppose $\succsim$ is reflexive and transitive. Then $\succsim$ is Continuous Symmetric if and only if there exist a non-empty compact convex set $M \subseteq \Delta(\Delta(S))$ and a non-constant vNM utility function u such that*

$$f \succsim^* g \text{ if and only if } \int u(f)\, dp \geq \int u(g)\, dp \text{ for all } p \in C, \tag{3.2}$$

where $C = \left\{\int \ell^{\infty} dm(\ell) : m \in M\right\}$. Furthermore M is unique.

Given this representation, define the set $R \equiv \bigcup_{m \in M} \operatorname{supp} m \subseteq \Delta(S)$. The set R is our candidate for the set of relevant measures in $\Delta(S)$. De Finetti's theorem (see Hewitt and Savage [29]) says that if we (or an agent) have a SEU preference, and if we are indifferent among the orderings of experiments, then the agent's subjective probability measure can be decomposed into parameters, corresponding to i.i.d. measures, and a unique probability measure over them. Our result goes beyond expected utility, and even beyond probabilistic sophistication, and says that Continuous Symmetry playing the role of indifference among the ordering of experiments, is equivalent to existence of a similar decomposition. Instead of a unique probability measure, when $\succsim$ is incomplete and/or violates the Anscombe-Aumann Independence axiom, our result delivers a compact convex set of probability measures, M, over parameters corresponding to i.i.d. measures. In this sense, R, the union of the supports of measures in M, is the set of parameters given weight under $\succsim$. Indeed, we now show that R is the set of relevant measures according to preferences $\succsim$.

Theorem 3.1. *Assume $\succsim$ is Continuous Symmetric, and take R accordingly. Then, R is the set of all relevant measures and is closed. Moreover, R^c is the set of all irrelevant measures.*

The theorem also says that R is the set of measures that are *not* irrelevant, and therefore our two notions of relevance agree.

When R is finite, the same result holds without the use of neighborhoods in defining irrelevant, i.e., $\Psi^{-1}(\ell)$ is Savage null if and only if $\ell \notin R$.

The above results justify thinking of R as the unique set of parameters viewed as subjectively possible since any other set of measures in $\Delta(S)$ will either leave out some relevant measures or include some irrelevant ones.

For complete preferences satisfying Continuous Symmetry, our next result shows that (up to closure) all relevant measures are needed to represent preferences and thus the i.i.d. measures generated from R, the set of all relevant measures, is the unique minimal closed set of i.i.d. measures to do so.

Theorem 3.2. *Suppose $\succsim$ is Continuous Symmetric and admits a real-valued representation. Then, there is a non-constant vNM utility function u on X and a weakly increasing functional G on*

$$\left\{\tilde{f} \in [u(X)]^R : \tilde{f}(\ell) = \int u(f)\, d\ell^\infty \text{ for some } f \in \mathcal{F}\right\}$$

such that

$$f \longmapsto G\left(\left(\int u(f)\, d\ell^\infty\right)_{\ell \in R}\right)$$

represents $\succsim$. Furthermore, the measures in the representation are essentially unique – if $D \subseteq \Delta(S)$ and every element in D is relevant, $\tilde{u}$ is a non-constant vNM utility function, H is a functional on

$$\left\{\tilde{f} \in [\tilde{u}(X)]^D : \tilde{f}(\ell) = \int (\tilde{u}(f)) d\ell^\infty \text{ for some } f \in \mathcal{F}\right\}$$

and

$$f \longmapsto H\left(\left(\int (\tilde{u}(f)) d\ell^\infty\right)_{\ell \in D}\right)$$

represents $\succsim$, then $\overline{D} = R$ and $\tilde{u}$ is a positive affine transformation of u.

Under slightly different assumptions, the fact that the set of expected utilities with respect to all i.i.d. measures can be monotonically aggregated to represent preferences was shown in de Castro and Al-Najjar ([7],[8]). In this regard, the main contribution of Theorem 3.2 is that R generates the *unique closed subset of i.i.d. measures that are essential for such a representation.* It is worth remarking that Theorem 3.2 does *not* imply that the set of i.i.d. measures generated from the relevant measures is the minimal closed set of measures in $\Delta(S^\infty)$ needed to represent preferences. In particular, specific mixtures over these i.i.d. measures may suffice. Formally, this is reflected in the fact that the Bewley set C may be a strict subset of $\Delta\left(\bigcup_{m \in M} \operatorname{supp} m\right)$ in Lemma 3.1.

One aim of the paper is to understand the connection between relevant measures and preference representations. This theorem serves that goal by addressing the issue for the general class of complete and Continuous Symmetric preferences. We see that all decision makers having such preferences will have their preferences fully described by specifying (1) the relevant measures (i.e., the set R), (2) risk attitudes (i.e., u) and (3) how the expectations of (utility) acts with respect to the i.i.d. products of elements of R should be aggregated (i.e., G). This third element may generally depend on some combination of (possibly imprecise) likelihood judgments and any aspect of tastes not captured by vNM risk attitude, for example, ambiguity attitude. To model preferences of decision makers who have a set of i.i.d. measures in mind, one can simply place those i.i.d. measures in the representation (together with u and G) and the theorem guarantees that the resulting relevant measures are exactly those i.i.d. measures.

3.4 Tastes and revealed information

We begin with a fundamental principle that we apply to distinguish tastes from other aspects of preferences. Tastes are aspects that are independent of changing information. In order to formalize this, we need to define what it means for two preferences to reflect the same information. In decision frameworks with a state space, information is modeled as an event in the state space, i.e., a subset of states that, for example, corresponds to the observation of a signal. Therefore different information corresponds to different events. Events that do not intersect with the information are said to be ruled out by that information. In terms of preferences, "ruling out" events means that the outcomes in those events do not matter for preference (i.e., the ruled out events are Savage null). Thus, it is natural to say that two agents act as if they have the same information when their Savage null events coincide. We formalize this as follows:

Definition 3.5. Say $\succsim$ and $\succsim'$ have the *same revealed information* if and only if $\{A \in \Sigma : \forall f, g, h \in \mathcal{F}, f_A h \sim g_A h\} = \{A \in \Sigma : \forall f, g, h \in \mathcal{F}, f_A h \sim' g_A h\}$, i.e., they have the same collection of Savage null events.

One seeming objection to this definition might be to point out that in, for example, SEU, when priors have the same support but different weights on that support, it seems like the different weights may reflect different information, yet our definition says those two preferences have the same revealed information. Upon closer inspection, however, one realizes that if the different weights indeed have their origin in different information then at some point one preference must have ruled out different events (for example, particular signal realizations) than the other. The only reason that this would not generate a difference in the set of null events is if the signals or other ruled out events were left unmodeled. If these events were included in the state space, our definition would indeed conclude that the two preferences reveal different information.

In our analysis we will restrict attention to revealed information that is compatible with symmetry in the sense that the event that an empirical frequency does not converge is Savage null. We call such preferences *symmetrically informed*:

Definition 3.6. $\succsim$ over acts $\mathcal{F}$ is *symmetrically informed* if $\left[\Psi^{-1}(\Delta S)\right]^c$ is Savage null. Say that $\succsim$ is *symmetrically informed of* L, if $L \subseteq \Delta(S)$ is the smallest closed set such that $\left[\Psi^{-1}(L)\right]^c$ is Savage null.

Being symmetrically informed of $L \subset \Delta(S)$ corresponds to ruling out the limiting frequencies associated with measures in L^c. For Continuous Symmetric preferences, the restriction to symmetrically informed preferences is without loss of generality. Recall that each Continuous Symmetric preference has an associated set of relevant measures, R. The next result shows that relevant measures completely capture revealed information for Continuous Symmetric preferences.

Theorem 3.3. *Each Continuous Symmetric preference is symmetrically informed of R. Two Continuous Symmetric preferences have the same R if and only if they have the same revealed information.*

To formalize the idea of tastes as aspects of preference that are unchanging as revealed information varies, it is useful to consider functional forms for numerical representations of preferences. A functional form in this context is a function mapping some arguments (often thought of as the pieces or parameters of the functional form) into a numerical representation of preferences (a function assigning a real number to each act in $\mathcal{F}$). For example, the usual functional form for Continuous Symmetric SEU preferences is

$$V(u, \mu) \equiv \int_{\Delta(S)} \left(\int u(\cdot) d\ell^{\infty} \right) d\mu(\ell) \tag{3.3}$$

with the arguments being a vNM utility function u and a probability measure μ. Notice that for each choice of u and μ, this yields a function mapping acts to real numbers.

Using functional forms, we now define a test to identify when an argument reflects *only* tastes, as opposed to any other aspect of preference such as information or belief. By design the test is conservative – it will not classify an aspect as only taste if there is any possibility of a connection to revealed information. Thus, we make no claim that *all* possible types of tastes will be captured by this test. For example, suppose an individual's ideology leads him to believe that certain states of the world are impossible. Though it is plausible to think of such an ideology as a taste, it would not be picked up by this test since changes in ideology would be behaviorally indistinguishable from changes in information. This test may be applied to any Continuous Symmetric preference. In fact, the symmetric structure is what enables us to separate out pure taste aspects (beyond just risk preferences) from one's behavior. The key is that, under symmetry, different probabilities assigned to an event in S can be distinguished in terms of the limiting frequency events that they make null, and thus in terms of revealed information. This allows a definition identifying a taste aspect as something that is unrelated to the revealed information to be far more powerful than without symmetry, as it is only under symmetry that we have the identification of revealed information with subsets of $\triangle(S)$ as delivered by Theorem 3.3.

Definition 3.7. Fix any functional form $V(\alpha, \beta)$ yielding a numerical representation of Continuous Symmetric preference, with the domains of α and β denoted by $\mathcal{A}$ and $\mathcal{B}$, respectively, where $\mathcal{A}$ has at least two elements and if $V(\alpha, \beta)$ and $V(\alpha', \beta)$ represent the same preference, then $\alpha = \alpha'$.[9] The argument α *reflects only tastes* if the following properties hold:

(1) $V(\alpha, \beta)$ is defined on $\mathcal{A} \times \mathcal{B}$, and

(2) for each β, the preferences represented by $V(\alpha, \beta)$ have the same revealed information for all $\alpha \in \mathcal{A}$.

Consider the requirements of the definition in turn. If α is to be separated out as an aspect of preference, a minimal requirement is that it may be specified freely, regardless of the value of β. This is the content of property (1). Property (2) says that revealed information is not influenced by α. Changing tastes alone should not change revealed information.

Notice that this is not a very discriminating definition when $\mathcal{A}$ and $\mathcal{B}$ are such that the collection, denoted $\mathcal{L}$, of sets that preferences represented by $V(\alpha, \beta)$ can be symmetrically informed of has few elements. For example, consider a collection of Continuous Symmetric SEU preferences represented as in (3.3) such that all the measures μ in the domain share the same support in $\Delta(S)$. All these preferences have the same revealed information. Then the argument μ satisfies property (2), even though it would fail to do so for any domain for μ allowing more than one support (and thus more than one possible revealed information). Therefore the classification of an argument α as reflecting only tastes is most convincing when $\mathcal{L}$ is a rich collection. For this reason, when applying the definition, we consider domains $\mathcal{A}$ and $\mathcal{B}$ that induce very large $\mathcal{L}$, such as the collection of all subsets reflecting ambiguity (i.e., all finite, non-singleton subsets of $\triangle(S)$) or of all finite subsets of $\triangle(S)$.

To illustrate the definition of reflecting only tastes, again consider the Continuous Symmetric SEU representation. It seems reasonable, and is customary, to say that (normalized) u reflects only tastes.[10] In fact, taking $\mathcal{A}$ to be the set of normalized u

[9] Using functional forms with two arguments is without loss of generality, as β can include as many pieces or parameters of the functional form as one wishes.

[10] Here, and for the remainder of the paper, when we refer to normalized u, we mean u such that $u(x_*) = 0$ and $u(x^*) = 1$.

and $\mathcal{B}$ to be the set of μ with finite support satisfies Definition 3.7, and the corresponding $\mathcal{L}$ is the collection of all finite subsets of $\Delta(S)$. What if we swap $\mathcal{A}$ and $\mathcal{B}$ and check if μ reflects only tastes? Since $R = \operatorname{supp} \mu$, Theorem 3.3 implies that property (2) of Definition 3.7 is violated, since changing the support of μ changes the revealed information. Thus μ does not reflect only tastes. Furthermore there is no way to split-off from μ a part that reflects only tastes. To see this, consider separating the weights applied to the $\ell \in \operatorname{supp} \mu$ from $\operatorname{supp} \mu$ itself. Notice that this separation fails property (1) of Definition 3.7 – the weights that may be chosen depend on the size of the support. Furthermore, if one tries to satisfy property (1) by "artificially" changing the domain $\mathcal{A}$ (in this example, for instance, by specifying strictly positive relative weights on the whole of $\Delta(S)$ and using a normalization of these to define the weights applied for any given support) then the required uniqueness of α in the representation will be violated.

What should we make of the fact that the weights applied to the $\ell \in \operatorname{supp} \mu$ are classified as neither reflecting only tastes nor as revealed information? To us, this reflects a true uncertainty in the source of the weights that is not resolvable by the given behavior. On the one hand, one may argue that the individual is "born with" the weights (e.g., an ingrained bias) and therefore the weights are, at least in part, tastes. Equally, one may argue that the individual may have received some unmodeled information which affected the weights. Since this is not modeled, we have no way of knowing. Hence, it is appropriate for the weights to remain unclassified.

This example has shown that our definition works as desired for Continuous Symmetric SEU preferences. The following lemma extends the identification of normalized u as reflecting only tastes to any complete Continuous Symmetric preference using the representation from Theorem 3.2:

Lemma 3.2. *Suppose a Continuous Symmetric $\succsim$ is represented by*

$$V(f) = G\left(\left(\int u(f)\, d\ell^{\infty}\right)_{\ell \in D}\right)$$

where $D \subseteq \Delta(S)$, u is a non-constant vNM utility function and G is a weakly increasing functional. If $\mathcal{A}$ is the set of normalized such u and $\mathcal{B}$ is the set of such G and D, then u reflects only tastes.

Note that G is not classified as reflecting only tastes for the same reasons as the weights applied to the $\ell \in \operatorname{supp} \mu$ were unclassified in SEU.

Neither the conclusions concerning SEU nor the identification of u as taste more generally are particularly novel. The real power of our definitions becomes apparent in the next section, where we apply them to ambiguity models involving tastes beyond risk attitudes.

4 Relevant measures and tastes in specific decision models

In this section, we examine Continuous Symmetric versions of four models from the ambiguity literature. For each, we identify the relevant measures and components of the representation reflecting only tastes.

All of the results in this section are proved using the same basic strategy. Given a closed set of measures $D \subseteq \Delta(S)$ taken from the functional form of the model, we first show that every element of D is a relevant measure. Then, we verify (sometimes aided by Lemma 3.1) that the preferences satisfy Continuous Symmetry. Finally, we either (1) note that each representation is a weakly increasing function of $\left(\int u(f)\, d\ell^{\infty}\right)_{\ell \in D}$ and invoke Theorem 3.2, or (2) prove that all measures outside of D are irrelevant and invoke Theorem 3.1, to conclude that all relevant measures are in D, and thus D is the set of relevant measures. Once the set of relevant measures is identified, we apply Definition 3.7 to show that certain components of the representation reflect only tastes. In light of Lemma 3.2, normalized u reflects only tastes in each of the representations below, and thus we do not repeat this fact in the statements of the individual results and mention only the *additional* components reflecting tastes.

4.1 The α-MEU model

Theorem 4.1. *If $\succsim$ is represented by*

$$V(f) \equiv \alpha \min_{p \in \{\ell^{\infty} : \ell \in D\}} \int u(f)\, dp + (1-\alpha) \max_{p \in \{\ell^{\infty} : \ell \in D\}} \int u(f)\, dp,$$

where $D \subseteq \Delta(S)$ is finite and u is a non-constant vNM utility function and $\alpha \in [0,1]$, then $R = D$. Moreover, if $\mathcal{A}$ is the set of such α and $\mathcal{B}$ is the set of such u and non-singleton such D, then α reflects only tastes.

This demonstrates that when the set of measures in an α-MEU representation is a finite set of i.i.d. products, the marginals generating this set are the relevant measures, R, and that α reflects only tastes. Note that the finiteness restriction is necessary for these α-MEU preferences to satisfy Monotone Continuity of $\succsim^*$, while the restriction to non-singleton D in applying Definition 3.7 is needed to ensure the required uniqueness of α. When D is a singleton, α has no effect on preference and is thus redundant. This is consistent with the idea that the tastes that α reflects are ambiguity attitudes and there is ambiguity only when D is non-singleton.

4.2 The Smooth Ambiguity model

When we normalize ϕ in the following theorem, we set $\phi(u(x_*)) = 0$ and $\phi(u(x^*)) = 1$.

Theorem 4.2. *Assume $\succsim$ is represented by*

$$U(f) \equiv \int_{\Delta(S)} \phi\left(\int u(f) d\ell^{\infty}\right) d\mu(\ell)$$

where u is a non-constant vNM utility function, $\phi : u(X) \to \mathbb{R}$ is a strictly increasing continuous function and $\mu \in \Delta(\Delta(S))$ such that either (i) there are $m, M > 0$ such that $m|a-b| \leq |\phi(a) - \phi(b)| \leq M|a-b|$ for all $a, b \in u(X)$ or, (ii) $\operatorname{supp}\mu$ is finite. Then, $R = \operatorname{supp}\mu$. Moreover, if $\mathcal{A}$ is the set of normalized ϕ satisfying (i) and $\mathcal{B}$ is the set of μ (not necessarily satisfying (ii)) with a non-singleton support and normalized such u, then ϕ reflects only tastes. If $\mathcal{A}$ is the set of normalized ϕ (not necessarily satisfying (i)) and $\mathcal{B}$ is the set of μ (satisfying (ii)) with a non-singleton support and normalized such u, then ϕ reflects only tastes.

Thus, for such smooth ambiguity preferences satisfying either (i) or (ii), the relevant measures are exactly the support of the second-order measure μ and normalized ϕ reflects only tastes. Note that the requirement that either (i) or (ii) is satisfied is necessary for these preferences to satisfy Monotone Continuity of $\succsim^*$, and, similar to the previous theorem, the restriction to μ with non-singleton support in applying Definition 3.7 is needed to ensure the required uniqueness of ϕ. This is again consistent with the idea that ϕ reflects ambiguity attitudes and there is ambiguity only when the support of μ is non-singleton. Note that the weights in μ are not classified as reflecting only tastes. The reason is the same as was discussed for SEU in the previous section.

4.3 The Extended MEU with contraction model

This model has a functional form that is a convex combination of MEU and expected utility.

Theorem 4.3. *If $\succsim$ is represented by*

$$W(f) \equiv \beta \min_{p \in \{\ell^{\infty} : \ell \in D\}} \int u(f)\, dp + (1-\beta) \int u(f)\, dq,$$

where $D \subseteq \Delta(S)$ is finite, $q \in co\{\ell^{\infty} : \ell \in D\}$, $0 < \beta \leq 1$ and u is a non-constant vNM utility function, then $R = D$. Moreover, if $\mathcal{A}$ is the set of such β and $\mathcal{B}$ is the set of normalized such u, such q and non-singleton such D, then β reflects only tastes.

This demonstrates that for an Extended MEU with contraction representation using a finite set of i.i.d. product measures, the marginals generating this set are the relevant measures, R. Furthermore, β reflects only tastes, and this is consistent with the interpretation offered in Gajdos et. al. [21]. Note that the finiteness restriction is sufficient for these preferences to satisfy Monotone Continuity of $\succsim^*$, and the restriction to non-singleton D is for exactly the same reason as in Theorem 4.1.

4.4 The Vector Expected Utility (VEU) model

Theorem 4.4. *Suppose $\succsim$ is represented by a VEU functional, that is,*

$$T(f) \equiv \int u(f)\,dp + A\left(\left(\int \zeta_i u(f)\,dp\right)_{1\leq i\leq n}\right),$$

where p is a probability measure on S^∞, u is a non-constant vNM utility function, $\zeta = (\zeta_1, ..., \zeta_n)$ is a bounded, measurable vector-valued function on S^∞ into $\mathbb{R}^n$ such that $\int \zeta_i dp = 0$, $A(0) = 0$, $A(a) = A(-a)$ for all $a \in \mathbb{R}^n$, and T is weakly monotonic. If n is finite, p and the ζ_i's are symmetric (i.e., $p = \int \ell^\infty dm(\ell)$ for some $m \in \Delta(\Delta(S))$ and, for all $\pi \in \Pi$, $\zeta_i(\omega) = \zeta_i(\pi\omega)$ p almost-everywhere) and A is Lipschitz continuous[11], then $R = \operatorname{supp} m$. Moreover, if $A = \gamma A'$ for some A' normalized such that $\sup_{f\in\mathcal{F}_*}\left|A'\left(\left(\int \zeta_i u(f)\,dp\right)_{1\leq i\leq n}\right)\right| = 1$ where $\mathcal{F}_*$ is the set of acts whose outcome is x_* or x^*, then, if $\mathcal{A}$ is the set of $\gamma \in (0,\infty)$ and $\mathcal{B}$ is the set of such p, ζ, A' and normalized such u, then γ reflects only tastes.

Thus, for VEU preferences with Lipschitz continuous adjustment function A, symmetric baseline probability, p, and a finite number of symmetric adjustment factors, ζ_i, the relevant measures, R, are those $\ell \in \Delta(S)$ given weight by p. The symmetry conditions are imposed to ensure Event Symmetry, while n finite and the Lipschitz condition are imposed to ensure Monotone Continuity of $\succsim^*$. The last part of the result shows that the scale of the adjustment function A, as measured by γ, reflects only tastes. This is consistent with Propositions 2 and 4 of Siniscalchi [43] that imply a greater scale corresponds to stronger ambiguity attitude.

A Appendix A: Relevance under Heterogeneous Environments

A decision maker may face a situation where non-identical experiments are repeated. For example, a doctor faces patients who may differ in ways important for the treatment problem at hand. Another example is an agent who wants to make a decision based on a regression model analysis where different data points may have different values of the regressors. We describe a variation of our model that allows these heterogeneous environments.

Let Ξ be a set of descriptions. We assume $\Xi = \left\{\xi^1, ..., \xi^K\right\}$ is a finite set for simplicity. Descriptions categorize the ordinates (of S^∞) so that it is only ordinates with the same description that are viewed as symmetric by the decision maker. Formally, we augment the state space S^∞ by attaching a description to each ordinate S. Thus, for a doctor facing many patients, each patient has a description $\xi \in \Xi$. A doctor faces a sequence of patients whose descriptions may be different from each other. Let

[11] That is, there is an $M > 0$ such that $|A(a) - A(b)| \leq M \sup_{1\leq i\leq n}|a_i - b_i|$ for all $a, b \in \mathbb{R}^n$.

$\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2, \ldots) \in \Xi^\infty$ be a sequence such that each element of Ξ appears infinitely often. Let $\succsim_{\tilde{\xi}}$ be a preference on $\mathcal{F}$ when faced with ordinates whose descriptions form the sequence $\tilde{\xi}$.

We assume the same axioms as in Section 3.1 on $\succsim_{\tilde{\xi}}$ with the exception of Event Symmetry. Instead we assume Partial Event Symmetry.

Axiom 9 (Partial Event Symmetry)**.** $1_A \sim^*_{\tilde{\xi}} 1_{\pi A}$ *for* $\pi \in \Pi$ *such that* $\tilde{\xi}_i = \tilde{\xi}_{\pi(i)}$ *for all* $i = 1, 2, ...$

Partial Event Symmetry says that an agent views ordinates with the same descriptions in the same way – as long as the descriptions are the same, the order does not matter. In contrast, no restrictions are placed on preferences towards ordinates that have different descriptions. For two ordinates with different descriptions, there is no reason to believe that the two are symmetric. Viewing our earlier framework as one in which there was only one possible description, Partial Event Symmetry is the natural generalization of Event Symmetry.

Formally, therefore, we replace the assumption of *Continuous Symmetry* with *Continuous Partial Symmetry*:

Definition A.1. $\succsim_{\tilde{\xi}}$ satisfies *Continuous Partial Symmetry* if it satisfies C-complete Preorder, Monotonicity, Risk Independence, Non-triviality, Partial Event Symmetry, Mixture Continuity of $\succsim^*_{\tilde{\xi}}$ and Monotone Continuity of $\succsim^*_{\tilde{\xi}}$.

Now, we can define relevant measures under heterogeneous environments. Since beliefs may vary depending on descriptions, a relevant measure is a mapping $\boldsymbol{l}$ from Ξ into $\Delta(S)$. Let $\mathcal{O}_{\boldsymbol{l}}$ denote an open subset of $(\Delta(S))^\Xi$ containing $\boldsymbol{l}$ under the product topology. For $\boldsymbol{l} \in (\Delta(S))^\Xi$, denote by $\boldsymbol{l}\left(\tilde{\xi}\right)$ the product measure on S^∞ whose i-th coordinate marginal is $\boldsymbol{l}\left(\tilde{\xi}_i\right) \in \Delta(S)$. That is, $\boldsymbol{l}\left(\tilde{\xi}\right) = \boldsymbol{l}\left(\tilde{\xi}_1\right) \otimes \boldsymbol{l}\left(\tilde{\xi}_2\right) \otimes ....$

Definition A.2. A mapping $\boldsymbol{l} \in (\Delta(S))^\Xi$ is *relevant* (according to preferences $\succsim_{\tilde{\xi}}$) if, for any $L \in \mathcal{O}_{\boldsymbol{l}}$, there are $f, g \in \mathcal{F}$ such that $f \nsim_{\tilde{\xi}} g$ and $\int f d\hat{\boldsymbol{l}}\left(\tilde{\xi}\right) = \int g d\hat{\boldsymbol{l}}\left(\tilde{\xi}\right)$ for all $\hat{\boldsymbol{l}} \in (\Delta(S))^\Xi \setminus L$.

An irrelevant measure is also defined. Let $\omega^k = \left(\omega^k_1, \omega^k_2, ...\right)$ be the subsequence of ω=$(\omega_1, \omega_2, ...)$ such that ω^k takes all the coordinates having description ξ^k. For $L = L_1 \times ... \times L_K \in \mathcal{O}_{\boldsymbol{l}}$, let

$$\Psi^{-1}_{\tilde{\xi}}(L) \equiv \left\{\omega \in S^\infty : \Psi\left(\omega^k\right) \in L_k, k = 1, 2, ..., K\right\}.$$

Definition A.3. A mapping $\boldsymbol{l} \in (\Delta(S))^\Xi$ is irrelevant (according to preferences $\succsim_{\tilde{\xi}}$) if, for some $L = L_1 \times ... \times L_K \in \mathcal{O}_{\boldsymbol{l}}$, $\Psi^{-1}_{\tilde{\xi}}(L)$ is Savage null i.e., $f_{\Psi^{-1}_{\tilde{\xi}}(L)} g \sim g$ for all $f, g \in \mathcal{F}$.

When $\Xi = \{\xi\}$ is a singleton, $\tilde{\xi} = (\xi, \xi, ...)$ and, therefore, it is as if $L \subseteq \Delta(S)$ and each $\hat{\boldsymbol{l}}\left(\tilde{\xi}\right)$ is i.i.d., and the above definition reduces to our earlier definition of relevant measures (Definition 3.3).

A standard linear regression is the case where the relevant measure is $\boldsymbol{l}$ and $\boldsymbol{l}(\xi_i)$ is normal with mean $\beta\xi_i$ and variance σ^2. Note that the description in this case is simply a vector giving the values of the regressors for a particular observation. An example of a set of relevant measures might be $\{\boldsymbol{l} \in (\Delta(S))^{\Xi} : \boldsymbol{l}(\xi_i)$ is normal with mean $\beta\xi_i$ and variance 1 for $\beta \in [\underline{b}, \overline{b}]^2\}$. This reflects knowledge of normality and the variance, and bounds on the coefficients within which any coefficients are seen as possible.

Relative to the homogeneous case, this framework: (1) allows for ordinates to differ according to Ξ, and (2) allows relevant measures to reflect beliefs about how the marginals for one $\xi \in \Xi$ relate to the marginals for another $\xi' \in \Xi$. This last point is important, for example, in capturing the case, mentioned above, where Ξ is related to S according to a linear regression model.

We provide results similar to those in the homogeneous case:

Lemma A.1. *Suppose $\succsim_{\tilde{\xi}}$ is reflexive and transitive. Then $\succsim_{\tilde{\xi}}$ satisfies Continuous Partial Symmetry if and only if there exist a non-empty compact convex set $M \subseteq \Delta\left((\Delta S)^{\Xi}\right)$ and a non-constant vNM utility function u such that ,*

$$f \succsim_{\tilde{\xi}}^{*} g \text{ if and only if } \int u(f)\,dp \geq \int u(g)\,dp \text{ for all } p \in C, \tag{A.1}$$

where $C = \left\{\int \boldsymbol{l}\left(\tilde{\xi}\right) dm(\boldsymbol{l}) : m \in M\right\}$. Furthermore M is unique.

Define $R \equiv \bigcup_{m \in M} \operatorname{supp} m \subseteq (\Delta S)^{\Xi}$.

Theorem A.1. *Assume $\succsim_{\tilde{\xi}}$ satisfies Continuous Partial Symmetry. Take R accordingly. Then, $R \subseteq (\Delta S)^{\Xi}$ is closed and is the set of all relevant mappings. Moreover, R^c is the set of all irrelevant mappings in $(\Delta S)^{\Xi}$.*

And,

Theorem A.2. *Assume $\succsim_{\tilde{\xi}}$ satisfies Continuous Partial Symmetry, and admits a real-valued representation. Then, there is a non-constant vNM utility function u on X and a weakly increasing functional G on $[u(X)]^R$ such that*

$$f \longmapsto G\left(\left(\int u(f)\,d\boldsymbol{l}\left(\tilde{\xi}\right)\right)_{\boldsymbol{l} \in R}\right)$$

represents $\succsim_{\tilde{\xi}}$. Furthermore, the measures in the representation are essentially unique – if $D \subseteq (\Delta S)^{\Xi}$ and every element in D is relevant, u' is a non-constant vNM utility function, H is a functional on $[u'(X)]^D$ and

$$f \longmapsto H\left(\left(\int (u'(f))d\boldsymbol{l}\left(\tilde{\xi}\right)\right)_{\boldsymbol{l} \in D}\right)$$

represents $\succsim_{\tilde{\xi}}$, then $\overline{D} = R$ and u' is a positive affine transformation of u.

B Appendix B: Proofs

Denote by $B(S)$ the set of bounded measurable functions on S. Similarly for $B(\Delta(S))$ and $B(S^\infty)$.

B.1 Proofs of Lemmas 3.1 and A.1

The first is a special case of the latter and we prove the latter here.

We prove sufficiency of the stated axioms, first. We first show that $\succsim^*_{\tilde{\xi}}$ satisfies the properties assumed in Gilboa et. al. [25, Theorem 1]. Preorder, Monotonicity, Mixture Continuity, Non-triviality, C-Completeness and Independence of $\succsim^*_{\tilde{\xi}}$ follow directly from the axioms we assume and the definition of $\succsim^*_{\tilde{\xi}}$. Therefore, by Gilboa et. al. [25, Theorem 1], there exists a unique non-empty weak* closed and convex set $C \subseteq ba_1^+(S^\infty)$ and a non-constant vNM utility function, $u : X \to \mathbb{R}$, such that

$$f \succsim^*_{\tilde{\xi}} g \text{ if and only if } \int u(f)\, dp \geq \int u(g)\, dp \text{ for all } p \in C.$$

By Alaoglu's Theorem, C is weak* compact. Monotone Continuity of $\succsim^*_{\tilde{\xi}}$ implies $C \subseteq \Delta(S^\infty)$ by Ghirardato, Maccheroni and Marinacci [23, Remark 1]. Moreover, Partial Event Symmetry implies every $p \in C$ is partially symmetric on finite cylinder events.

Next, we prove the claim that every $p \in C$ is of the form $\int \boldsymbol{l}\left(\tilde{\xi}\right) dm(\boldsymbol{l})$ for some $m \in \Delta\left((\Delta S)^\Xi\right)$. (We prove this claim here because we did not find a proof of this claim in the literature.) The proof is based on the idea of Hewitt and Savage [29]. Let $\mathcal{P}_{\tilde{\xi}}$ be the set of partially symmetric measures. Clearly, $\mathcal{P}_{\tilde{\xi}}$ is convex and also weak-convergence compact as $\Delta(S^\infty)$ is. Then, the Choquet Theorem (Phelps [39, p.14]) implies that any element in $\mathcal{P}_{\tilde{\xi}}$ is a mixture of its extreme points. We need to show that each extreme point is of the form $\boldsymbol{l}\left(\tilde{\xi}\right)$. For notational simplicity, let $\tilde{\xi} = (\xi^1, \xi^2, \xi^1, \xi^2, ...)$. Take any extreme point p, $n \geq 1$ and event $A \subseteq S^n$. For each finite cylinder B,

$$p(B) = p(\pi B) = p(A)\, p(\pi B | A) + p(A^c)\, p(\pi B | A^c),$$

where $\pi \in \Pi$ is defined as follows: If n is even,

$$\pi(i) = i + n.$$

If n is odd,

$$\pi(i) = n + i - (-1)^i.$$

(Since B is a finite cylinder, π can be made a finite permutation.) For example, if $B \subset S^2$ and $n = 1$, then $\pi(1) = 3$,$\pi(2) = 2$,$\pi(3) = 1$, and $\pi(k) = k$ for $k \geq 4$, and hence $\pi B = \{\omega : (\omega_3, \omega_2) \in B\}$. Note that A and πB depend on different experiments. Define $q_1, q_2 \in \Delta(S^\infty)$ by

$$q_1(B) = p(\pi B|A) \text{ and}$$
$$q_2(B) = p(\pi B|A^c)$$

for each finite cylinder B. Noting that $\tilde{\xi}_i = \tilde{\xi}_{\pi(i)}$ for all $i = 1, 2, ...$, one can verify that $q_1, q_2 \in \mathcal{P}_{\tilde{\xi}}$. We have just shown that p is a mixture of q_1 and q_2 that lie in $\mathcal{P}_{\tilde{\xi}}$. Since p is an extreme point, $p = q_1 = q_2$. Therefore we have $p(B) = p(A \times \pi B)/p(A)$ where π is defined as above. By the fact that $p(B) = p(\pi B)$, $p(A)p(\pi B) = p(A \times \pi B)$ which proves that p is a product measure. By partial symmetry w.r.t. $\tilde{\xi} = (\xi^1, \xi^2, \xi^1, \xi^2, ...)$, $p = \ell_1 \otimes \ell_2 \otimes \ell_1 \otimes \ell_2 \otimes ...$ and is of the form $\boldsymbol{l}\left(\tilde{\xi}\right)$. Therefore, any element in $\mathcal{P}_{\tilde{\xi}}$ is a mixture of product measures of the form $\boldsymbol{l}\left(\tilde{\xi}\right)$.

Thus, $C = \left\{\int \boldsymbol{l}\left(\tilde{\xi}\right) dm(\boldsymbol{l}) : m \in M\right\}$ for some non-empty $M \subseteq \Delta\left((\Delta S)^{\Xi}\right)$. It is clear that M is convex.

To see that M is weak* compact, take any net $m_\alpha \in M$. Since C is weak* compact, there is $m' \in C$ and a subnet m'_λ of m_α such that

$$\int \left(\int \varphi d\boldsymbol{l}\left(\tilde{\xi}\right)\right) dm'_\lambda(\boldsymbol{l}) \to \int \left(\int \varphi d\boldsymbol{l}\left(\tilde{\xi}\right)\right) dm'(\boldsymbol{l}) \text{ for each } \varphi \in B(S^\infty).$$

It suffices to show that each $\phi \in B\left((\Delta S)^{\Xi}\right)$ can be written as $\boldsymbol{l} \mapsto \int \varphi d\boldsymbol{l}\left(\tilde{\xi}\right)$ for some $\varphi \in B(S^\infty)$. In fact,

$$\phi(\boldsymbol{l}) = \int_{S^\infty} \phi\left(\Psi\left(\omega^1\right), ..., \Psi\left(\omega^K\right)\right) d\boldsymbol{l}\left(\tilde{\xi}\right)(\omega)$$

where $\Psi\left(\omega^k\right)$ gives an empirical frequency limit when considering the experiments of description ξ^k, that is, all coordinates t such that $\tilde{\xi}_t = \xi^k$. Conclude that m'_λ converges to m'.

Uniqueness of M follows from uniqueness of C.

To show necessity, assume such a set M. It is clear that $\succsim^*_{\tilde{\xi}}$ satisfies Monotonicity and Risk Independence and thus $\succsim_{\tilde{\xi}}$ inherits these properties as well. Partial Event Symmetry follows since each element of C is of the form $\int \ell^\infty dm(\ell)$ for some $m \in M$. Non-triviality of $\succsim_{\tilde{\xi}}$ follows from non-constancy of u. Monotone Continuity of $\succsim^*_{\tilde{\xi}}$ follows from weak* compactness of C, which is implied by that of M. Mixture Continuity of $\succsim^*_{\tilde{\xi}}$ follows from Mixture Continuity of expected utility and the fact that intersections of closed sets are closed.

B.2 Proofs of Theorems 3.1 and A.1

The two proofs are essentially the same and we prove the first only. We begin by showing that R is relative weak* closed. The set R is relative weak* closed if it equals $\Delta(S) \cap K$

for some weak* closed $K \subseteq ba(S)$. Consider K equal to the weak* closure of R. That $R \subseteq \Delta(S) \cap K$ is direct. To show $\Delta(S) \cap K \subseteq R$, consider any limit point $\widehat{\ell} \in \Delta(S)$ of R. Lemma 3.1 implies that no ℓ outside of R is relevant – if $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in R$ then $\int f dp = \int g dp$ for all $p \in C$ and thus $f \sim^* g$ implying $f \sim g$. To show that $\widehat{\ell} \in R$, it therefore suffices to show that $\widehat{\ell}$ is relevant. Fix $L \in \mathcal{O}_{\widehat{\ell}}$. Then, $\left(L \backslash \left\{\widehat{\ell}\right\}\right) \cap R \neq \emptyset$. Choose any $\widetilde{\ell} \in \left(L \backslash \left\{\widehat{\ell}\right\}\right) \cap R$. Since $\widetilde{\ell}$ is relevant, for $\widetilde{L} \in \mathcal{O}_{\widetilde{\ell}}$, there are $f, g \in \mathcal{F}$ such that $f \nsim g$ and $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S) \backslash \widetilde{L}$. Note that $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S) \backslash L \subseteq \Delta(S) \backslash \widetilde{L}$. Since L is an arbitrary set in $\mathcal{O}_{\widehat{\ell}}$, $\widehat{\ell}$ is relevant. Thus $R = \Delta(S) \cap K$ and R is relative weak* closed.

We next show that every $\ell \in R^c$ is irrelevant. Since, R^c is open, $\ell \in R^c$ implies there exists $L \in \mathcal{O}_\ell$ such that $L \subseteq R^c$. Note that $\hat{\ell}^\infty(\Psi^{-1}(L)) = 0$ for all $\hat{\ell} \in R$. Thus, $p(\Psi^{-1}(L)) = 0$ for all $p \in C$. By Lemma 3.1, $f_{\Psi^{-1}(L)} g \sim g$ for all $f, g \in \mathcal{F}$, showing that ℓ is irrelevant.

Next we show no $\ell \in R$ is irrelevant. Take any $\ell \in R$ and $L \in \mathcal{O}_\ell$. By Lemma 3.1, $1_{\Psi^{-1}(L)} \succsim^* 1_\emptyset$ since $\int 1_{\Psi^{-1}(L)} dp \geq \int 1_\emptyset dp$ for all $p \in \Delta(S^\infty)$. Now show that $1_{\Psi^{-1}(L)} \not\precsim^* 1_\emptyset$. Note that by definition of R there is $m \in M$ such that $L \cap \operatorname{supp} m \neq \emptyset$. Let $p = \int \hat{\ell}^\infty dm\left(\hat{\ell}\right)$ and compute

$$\int 1_{\Psi^{-1}(L)} dp = m(L) > 0 = \int 1_\emptyset dp.$$

By Lemma 3.1, $1_{\Psi^{-1}(L)} \not\precsim^* 1_\emptyset$. Therefore we have $1_{\Psi^{-1}(L)} \nsim^* 1_\emptyset$, which implies that

$$\alpha 1_{\Psi^{-1}(L)} + (1-\alpha) h \succsim \alpha 1_\emptyset + (1-\alpha) h$$

for some $\alpha \in [0,1]$ and $h \in \mathcal{F}$. Note that both sides coincide outside of $\Psi^{-1}(L)$ and hence $\Psi^{-1}(L)$ is irrelevant.

Show that R is the set of all relevant measures in $\Delta(S)$. Observe that Lemma 3.1 implies that no ℓ outside of R is relevant – if $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in R$ then $\int f dp = \int g dp$ for all $p \in C$ and thus $f \sim^* g$ implying $f \sim g$. We now show that every element of R is relevant. Take any $\widehat{\ell} \in R$. Recall that we proved that $\widehat{\ell}$ is not irrelevant. Thus, for any $L \in \mathcal{O}_{\widehat{\ell}}$, there are acts f and g such that $f_{\Psi^{-1}(L)} g \nsim g$. But for each $\ell \in \Delta(S) \backslash L$, $\int f_{\Psi^{-1}(L)} g d\ell^\infty = \int g d\ell^\infty$. Thus, $\widehat{\ell}$ is relevant. This proves that R is the set of all relevant measures in $\Delta(S)$.

B.3 Proofs of Theorems 3.2 and A.2

Again, we prove the first only. Let $U : \mathcal{F} \to \mathbb{R}$ represent $\succsim$. Recall Lemma 3.1 guarantees the existence of a non-constant affine utility $u : X \to \mathbb{R}$ and a set C derived there from $\succsim$. Define G on $\left\{\tilde{f} \in [u(X)]^R : \tilde{f}(\ell) = \int u(f) d\ell^\infty \text{ for some } f \in \mathcal{F}\right\}$ by $G\left(\left(\int u(f) d\ell^\infty\right)_{\ell \in R}\right) = U(f)$, which is well-defined because $\int u(f) d\ell^\infty = \int u(g) d\ell^\infty$

for all $\ell \in R$ implies $\int u(f)\,dp = \int u(g)\,dp$ for all $p \in C$, which, by Lemma 3.1, implies $f \sim g$. Thus $f \longmapsto G\left(\left(\int u(f)\,d\ell^\infty\right)_{\ell \in R}\right)$ represents $\succsim$. Suppose

$$\hat{f}, \hat{g} \in \left\{\tilde{f} \in [u(X)]^R : \tilde{f}(\ell) = \int u(f)\,d\ell^\infty \text{ for some } f \in \mathcal{F}\right\}$$

are such that $\hat{f}(\ell) \geq \hat{g}(\ell)$ for all $\ell \in R$ and fix some corresponding acts f, g that generate these expected utilities. Since $\int u(f)\,d\ell^\infty \geq \int u(g)\,d\ell^\infty$ for all $\ell \in R$, $\int u(f)\,dp \geq \int u(g)\,dp$ for all $p \in C$. By Lemma 3.1, this implies $f \succsim g$. Therefore $G\left(\tilde{f}\right) = U(f) \geq U(g) = G(\tilde{g})$ which shows G is weakly increasing.

Uniqueness is shown as follows. Since every element in D is relevant, $D \subseteq R$ by Theorem 3.1. Since R is closed, $\overline{D} \subseteq R$. We show that $R \subseteq \overline{D}$. Suppose that $\widehat{\ell} \notin \overline{D}$ for some $\widehat{\ell} \in \Delta(S)$. Since $\overline{D}$ is closed, there exists $L \in \mathcal{O}_{\widehat{\ell}}$ such that $L \subseteq \Delta(S) \backslash \overline{D}$. Since $f \longmapsto H\left(\left(\int (\tilde{u}(f))d\ell^\infty\right)_{\ell \in D}\right)$ represents $\succsim$, if $\int(\tilde{u}(f))d\ell^\infty = \int(\tilde{u}(g))d\ell^\infty$ for all $\ell \in D \subseteq \overline{D} \subseteq \Delta(S) \backslash L$, $f \sim g$. If $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in D \subseteq \overline{D} \subseteq \Delta(S) \backslash L$, then, because $\tilde{u}$ is affine, $\int(\tilde{u}(f))d\ell^\infty = \int(\tilde{u}(g))d\ell^\infty$ for all $\ell \in D \subseteq \overline{D} \subseteq \Delta(S) \backslash L$. Therefore, $\widehat{\ell}$ can't be relevant, and thus $\widehat{\ell} \notin R$ by Theorem 3.1. Uniqueness of u up to positive affine transformation is standard, as $\succsim$ restricted to constant acts is expected utility.

B.4 Proof of Theorem 3.3

We prove the second sentence first. Let R' and R'' be the sets of relevant measures for two Continuous Symmetric preferences, $\succsim'$ and $\succsim''$. If $\ell \in R' \backslash R''$, $\Psi^{-1}(L)$ is Savage null according to $\succsim''$ for some $L \in \mathcal{O}_\ell$ by Theorem 3.1, but it is not Savage null according to $\succsim'$ by definition. Therefore, $R' \neq R''$ implies the two preferences do not have the same revealed information.

The following claim is sufficient for the other direction, as it shows that $R' = R''$ implies that Savage null events coincide:

Claim: $A \in \Sigma$ is Savage null if and only if $\ell^\infty(A) = 0$ for all $\ell \in R$.

Proof of Claim: Take the sets C and M defined in Lemma 3.1. Suppose $\ell^\infty(A) > 0$ for some $\ell \in R$. Then, we can take $L \in \mathcal{O}_\ell$ such that $\hat{\ell}^\infty(A) > 0$ for all $\hat{\ell} \in L$. Since $R \equiv \bigcup_{m \in M} \operatorname{supp} m \subseteq \Delta(S)$, $p(A) > 0$ for some $p = \int \ell^\infty dm(\ell) \in C$ and $m \in M$. Therefore, by Lemma 3.1, $\lambda 1_A + (1-\lambda)h \succ \lambda 1_\emptyset + (1-\lambda)h$ for some $h \in \mathcal{F}$. Since these two acts coincide on A^c, A is not Savage null. Now, for the other direction, suppose that $\ell^\infty(A) = 0$ for all $\ell \in R$. Then, $\int u(f_A h)\,dp = \int_{A^c} u(h)\,dp = \int u(g_A h)\,dp$ for all $p \in C$, by the definition of R and C. Lemma 3.1 implies that $f_A h \sim g_A h$, and thus A is Savage null.

We conclude by proving the first sentence, that any Continuous Symmetric $\succsim$ is symmetrically informed of R. Let $A = [\Psi^{-1}(R)]^c$. From Theorem 3.1, we know that R is relative weak* closed. Observe that $\int u(f_A g)\,dp = \int u(g)\,dp$ for any acts f and g, and $p = \int \ell^\infty dm(\ell)$ with $\operatorname{supp} m \subseteq R$. Then by Lemma 3.1, $f_A g \sim^* g$ which implies

$f_A g \sim g$. This shows that A is Savage null according to $\succsim$. Now we show that there is no relative weak* closed set $L \subset R$ such that $[\Psi^{-1}(L)]^c$ is Savage null. Suppose $R' \subset R$ is such a set and take $\ell \in R \backslash R'$. Since R' is relative weak* closed, there is some $L \in \mathcal{O}_\ell$ satisfying $L \subset R \backslash R'$. Then, $[\Psi^{-1}(L)] \subset [\Psi^{-1}(R')]^c$ is Savage null. Thus, ℓ is irrelevant. By Theorem 3.1, $\ell \in R^c$ which is a contradiction.

B.5 Proof of Lemma 3.2

We check the definition of reflecting only tastes (Definition 3.7). First, given G and D, normalized u is unique. Second, the utility function is defined for any $(u, G, D) \in \mathcal{A} \times \mathcal{B}$ so that property (1) of the definition is satisfied. Third, since changing normalized u while holding G and D fixed does not change the set of Savage null events, property (2) is satisfied.

B.6 Proof of Theorem 4.1

Suppose $\succsim$ is represented by such a $V(f)$. We first show that all measures in D are relevant. Suppose $\hat{\ell} \in D$ and fix any open $K \subseteq \Delta(S)$ such that $\hat{\ell} \in K$. Consider $f = 1_{\Psi^{-1}(K)}$ and $g = 1_\emptyset$ and observe that $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S) \backslash K$. Note that $\int u(f) d\ell^\infty > \int u(g) d\ell^\infty$ for all $\ell \in K$ while $\int u(f) d\ell^\infty \geq \int u(g) d\ell^\infty$ for all $\ell \in D$. Thus, if $\alpha \in [0,1)$, $f \succ g$ and $\hat{\ell}$ is relevant. If $\alpha = 1$, consider instead $f = \frac{1}{2} 1_{\Psi^{-1}(K)} + \frac{1}{2} 1_{\Psi^{-1}(\Delta(S) \backslash K)}$ and $g = \frac{1}{2} 1_\emptyset + \frac{1}{2} 1_{\Psi^{-1}(\Delta(S) \backslash K)}$ and observe that $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S) \backslash K$ while $\min_{\ell \in D} \int u(f) d\ell^\infty = \frac{1}{2} u(x^*) + \frac{1}{2} u(x_*) > u(x_*) = \min_{\ell \in D} \int u(g) d\ell^\infty$ so that $f \succ g$ and again $\hat{\ell}$ is relevant.

We show that $\succsim$ is Continuous Symmetric. All axioms except Monotone Continuity of $\succsim^*$ are straightforward. To check the latter, consider $V_1(f) \equiv \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(f) dp$ first. The Bewley set of V_1 is $co(\{\ell^\infty : \ell \in D\})$ and it is weak* compact since D is finite. Thus, V_1 satisfies Monotone Continuity of $\succsim^*$. Similarly, $V_0(f) = \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(f) dp$ also satisfies Monotone Continuity of $\succsim^*$. Take $A_n \searrow \emptyset$ and $x, x', x'' \in X$ such that $u(x') > u(x'')$. Then, there is $\bar{n}_1$ and $\bar{n}_0$ such that

$$V_1(\lambda x' + (1-\lambda) h) \geq V_1(\lambda x A_n x'' + (1-\lambda) h)$$

for all $\lambda \in [0,1]$, $h \in \mathcal{F}$ and $n \geq \bar{n}_1$, and

$$V_0(\lambda x' + (1-\lambda) h) \geq V_0(\lambda x A_n x'' + (1-\lambda) h)$$

for all $\lambda \in [0,1]$, $h \in \mathcal{F}$ and $n \geq \bar{n}_2$. Since $V = \alpha V_1 + (1-\alpha) V_0$,

$$V(\lambda x' + (1-\lambda) h) \geq V(\lambda x A_n x'' + (1-\lambda) h) \text{ for } n = \max(\bar{n}_1, \bar{n}_2).$$

Thus, Monotone Continuity of $\succsim^*$ is satisfied.

Observe that $V(f)$ can be re-written as

$$H\left(\left(\int u(f)\, d\ell^\infty\right)_{\ell\in D}\right)$$
$$\equiv \alpha \min_{\ell\in D} \int u(f)\, d\ell^\infty + (1-\alpha)\max_{\ell\in D}\int u(f)\, d\ell^\infty,$$

and H so defined is weakly increasing. Therefore we may apply the uniqueness result in Theorem 3.2 to conclude $\overline{D} = R$. Since D is finite, $\overline{D} = D$.

We check that α reflects only tastes. The only non-trivial part is uniqueness of α, given u and non-singleton D. Note that increasing α strictly decreases the utility for the bet $1_{\Psi^{-1}(\ell)}$ for $\ell \in D$ and the utility of a lottery that is indifferent to $1_{\Psi^{-1}(\ell)}$ does not depend on α. Thus α is unique, given u and non-singleton D.

B.7 Proof of Theorem 4.2

Suppose $\succsim$ is represented by such a $U(f)$. We first show that all measures in $\operatorname{supp}\mu$ are relevant. Suppose $\hat{\ell} \in \operatorname{supp}\mu$ and fix any open $L \subseteq \Delta(S)$ such that $\hat{\ell} \in L$. Consider $f = 1_{\Psi^{-1}(L)}$ and $g = 1_{\emptyset}$ and observe that $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S)\setminus L$. Since ϕ is strictly increasing, $\phi\left(\int u(f)\, d\ell^\infty\right) > \phi\left(\int u(g)\, d\ell^\infty\right)$ for all $\ell \in L$ and $\phi\left(\int u(f)\, d\ell^\infty\right) \geq \phi\left(\int u(g)\, d\ell^\infty\right)$ for all $\ell \in \operatorname{supp}\mu$. By the definition of $\operatorname{supp}\mu$, $\mu(L) > 0$. Thus, $f \succ g$ and $\hat{\ell}$ is relevant.

We next show that U satisfies Continuous Symmetry. We directly verify only the following axioms: Monotone Continuity of $\succsim^*$ and Mixture Continuity of $\succsim$ (and thus Mixture Continuity of $\succsim^*$). That the remaining axioms are satisfied is straightforward.

Monotone Continuity of $\succsim^*$: Suppose that case (i) holds, so there are $m, M > 0$ such that $m|a-b| \leq |\phi(a) - \phi(b)| \leq M|a-b|$ for all $a, b \in u(X)$. Fix any $x, x', x'' \in X$ with $x' \succ x''$. The only non-trivial case is $x \succ x'$. Without loss of generality, assume $u(x) = 1 > u(x') = t' > u(x'') = 0$ and $[0,1] \subseteq u(X)$. Suppose $A_n \searrow \emptyset$. Take $\varepsilon', \varepsilon > 0$ so that

$$\varepsilon' < t' \text{ and } m(t'-\varepsilon')(1-\varepsilon) \geq M(1-t')\varepsilon.$$

Define $\zeta_n : \Delta(S) \to \mathbb{R}$ by $\zeta_n(\ell) = \ell^\infty(A_n)$, and temporarily equip $\Delta(S)$ with the wc topology. Since wc open sets are weak* open, μ is well-defined on the Borel σ-algebra generated by wc open sets. Then, by Lusin's theorem (Aliprantis and Border [2, Theorem 12.8]), there is a wc compact set $L \subseteq \Delta(S)$ such that $\mu(L) > 1-\varepsilon$ and all ζ_n are wc continuous. Note that ζ_n converges monotonically to 0 pointwise. Then by Dini's Theorem (Aliprantis and Border [2, Theorem 2.66]), ζ_n on L converges uniformly to 0. Hence there is $N > 0$ such that $\zeta_N = \ell^\infty(A_N) < \varepsilon'$ for all $\ell \in L$. To see $x' \succsim^* xA_Nx''$, and thus Monotone Continuity of $\succsim^*$, compute, for any $\alpha \in [0,1]$ and

$h \in \mathcal{F}$,

$$
\begin{aligned}
&U\left(\alpha x'+(1-\alpha)h\right)-U\left(\alpha x A_N x''+(1-\alpha)h\right)\\
&=\int_L \phi\left(\alpha t'+(1-\alpha)\int h d\ell^\infty\right)-\phi\left(\alpha \ell^\infty\left(A_N\right)+(1-\alpha)\int h d\ell^\infty\right)d\mu\left(\ell\right)\\
&+\int_{\Delta(S)\backslash L}\phi\left(\alpha t'+(1-\alpha)\int h d\ell^\infty\right)-\phi\left(\alpha \ell^\infty\left(A_N\right)+(1-\alpha)\int h d\ell^\infty\right)d\mu\left(\ell\right)\\
&>\int_L \phi\left(\alpha t'+(1-\alpha)\int h d\ell^\infty\right)-\phi\left(\alpha \varepsilon'+(1-\alpha)\int h d\ell^\infty\right)d\mu\left(\ell\right)\\
&+\int_{\Delta(S)\backslash L}\phi\left(\alpha t'+(1-\alpha)\int h d\ell^\infty\right)-\phi\left(\alpha+(1-\alpha)\int h d\ell^\infty\right)d\mu\left(\ell\right)\\
&\geq \int_L \alpha m\left(t'-\varepsilon'\right)d\mu\left(\ell\right)+\int_{\Delta(S)\backslash L}\alpha M\left(t'-1\right)d\mu\left(\ell\right)\\
&=\alpha\left[m\left(t'-\varepsilon'\right)\mu\left(L\right)-M\left(1-t'\right)\left(1-\mu\left(L\right)\right)\right]\\
&\geq \alpha\left[m\left(t'-\varepsilon'\right)\left(1-\varepsilon\right)-M\left(1-t'\right)\varepsilon\right]\geq 0.
\end{aligned}
$$

Turn to the case where (ii) holds, so that $\operatorname{supp}\mu$ is finite. Again suppose $A_n \searrow \emptyset$ and $x \succ x' \succ x''$. Since $\operatorname{supp}\mu$ is finite, $\sup_{\ell\in\operatorname{supp}\mu}\ell^\infty\left(A_n\right)\to 0$. Thus, for $\varepsilon>0$ satisfying $u\left(x'\right)>\varepsilon u\left(x\right)+\left(1-\varepsilon\right)u\left(x''\right)$, there is $n>0$ such that $\ell^\infty\left(A_n\right)<\varepsilon$ for all $\ell\in\operatorname{supp}\mu$. This implies

$$
\begin{aligned}
&U\left(\alpha x'+(1-\alpha)h\right)-U\left(\alpha x A_n x''+(1-\alpha)h\right)\\
&=\int \phi\left(\alpha u\left(x'\right)+(1-\alpha)\int u\left(h\right)d\ell^\infty\right)\\
&-\phi\left(\alpha\left(\ell^\infty\left(A_n\right)u\left(x\right)+\left(1-\ell^\infty\left(A_n\right)\right)u\left(x''\right)\right)+(1-\alpha)\int u\left(h\right)d\ell^\infty\right)d\mu\left(\ell\right)\\
&\geq 0
\end{aligned}
$$

for all $\alpha\in[0,1]$, $h\in\mathcal{F}$, and $\ell\in\operatorname{supp}\mu$. Therefore, $x' \succsim^* xA_nx''$ and Monotone Continuity of $\succsim^*$ holds.

Mixture Continuity of $\succsim$ (and thus Mixture Continuity of $\succsim^*$): Fix acts $f,g,h\in\mathcal{F}$ and consider a sequence $\lambda_n\in[0,1]$ such that $\lambda_n\to\lambda$ and $\lambda_n f+(1-\lambda_n)g\succsim h$ for all n. Therefore, for all n,

$$
\begin{aligned}
&\int_{\Delta(S)}\phi\left(\lambda_n\int_{S^\infty}u(f)d\ell^\infty+(1-\lambda_n)\int_{S^\infty}u(g)d\ell^\infty\right)d\mu\left(\ell\right)\\
&\geq\int_{\Delta(S)}\phi\left(\int_{S^\infty}u(h)d\ell^\infty\right)d\mu\left(\ell\right).
\end{aligned}
$$

Since ϕ is continuous, by the Dominated Convergence Theorem (e.g., Aliprantis and

Border [2, Theorem 11.21])

$$\int_{\Delta(S)} \phi\left(\lambda_n \int_{S^\infty} u(f)d\ell^\infty + (1-\lambda_n)\int_{S^\infty} u(g)d\ell^\infty\right) d\mu(\ell)$$
$$\to \int_{\Delta(S)} \phi\left(\lambda \int_{S^\infty} u(f)d\ell^\infty + (1-\lambda)\int_{S^\infty} u(g)d\ell^\infty\right) d\mu(\ell)$$

so that $\lambda f + (1-\lambda)g \succsim h$.

Next, observe that $U(f)$ can be re-written as

$$H\left(\left(\int u(f)\, d\ell^\infty\right)_{\ell \in \operatorname{supp}\mu}\right)$$
$$\equiv \int_{\Delta(S)} \phi\left(\int_{S^\infty} u(f)d\ell^\infty\right) d\mu(\ell),$$

and H so defined is weakly increasing. Therefore we may apply the uniqueness result in Theorem 3.2 to conclude $R = \overline{\operatorname{supp}\mu}$. Since $\operatorname{supp}\mu$ is relative weak* closed by definition, $\overline{\operatorname{supp}\mu} = D$.

We check ϕ reflects only tastes under the two specifications of $\mathcal{A}$ and $\mathcal{B}$. All except uniqueness of normalized ϕ, given u and μ with a non-singleton support is straightforward. Observe that the preference restricted to acts measurable with respect to the σ-algebra generated by the limiting frequency events $\Psi^{-1}(\ell)$ is a subjective utility preference with the belief μ and utility $\phi \circ u$. Since $\operatorname{supp}\mu$ is non-singleton, $\phi \circ u$ is unique up to normalization. Since ϕ is normalized and u is given, ϕ is unique. This shows that ϕ reflects only tastes.

B.8 Proof of Theorem 4.3

Suppose $\succsim$ is represented by such a $W(f)$. We first show that all measures in D are relevant. Suppose $\hat{\ell} \in D$ and fix any open $L \subseteq \Delta(S)$ such that $\hat{\ell} \in L$. Consider $f = 1_{\Psi^{-1}(L)}$ and $g = 1_\emptyset$ and observe that $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S) \setminus L$. Observe that $\int u(f)\, d\ell^\infty > \int u(g)\, d\ell^\infty$ for all $\ell \in L$ while $\int u(f)\, d\ell^\infty \geq \int u(g)\, d\ell^\infty$ for all $\ell \in D$ and thus also $\int u(f)\, dq \geq \int u(g)\, dq$. Therefore, if $\beta < 1$ and $q(\Psi^{-1}(L)) > 0$, $f \succ g$ and $\hat{\ell}$ is relevant. If either $\beta = 1$ or $q(\Psi^{-1}(L)) = 0$, consider instead $f = \frac{1}{2}1_{\Psi^{-1}(L)} + \frac{1}{2}1_{\Psi^{-1}(\Delta(S)\setminus L)}$ and $g = \frac{1}{2}1_\emptyset + \frac{1}{2}1_{\Psi^{-1}(\Delta(S)\setminus L)}$ and observe that $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S) \setminus L$ while $\min_{\ell \in D} \int u(f)\, d\ell^\infty = \frac{1}{2}u(x^*) + \frac{1}{2}u(x_*) > u(x_*) = \min_{\ell \in D} \int u(g)\, d\ell^\infty$ so that $f \succ g$ and again $\hat{\ell}$ is relevant.

We now show that $\succsim$ satisfies Continuous Symmetry. To invoke Lemma 3.1, we demonstrate that $\succsim^*$ may be represented as in (3.2). Suppose $\int u(f)\, dp \geq \int u(g)\, dp$ for all $p \in co\{\beta\ell^\infty + (1-\beta)q : \ell \in D\}$. Fix any $\lambda \in [0,1]$ and acts $f, g, h \in \mathcal{F}$, and let

$\hat{\ell}^\infty \in \arg\min_{p\in\{\ell^\infty:\ell\in D\}} \int u(\lambda f + (1-\lambda)h)\, dp$. Then

$$\begin{aligned}
& W(\lambda f + (1-\lambda)h) \\
&= \int u(\lambda f + (1-\lambda)h)\, d\left(\beta\hat{\ell}^\infty + (1-\beta)q\right) \\
&\geq \int u(\lambda g + (1-\lambda)h)\, d\left(\beta\hat{\ell}^\infty + (1-\beta)q\right) \\
&\geq W(\lambda g + (1-\lambda)h)
\end{aligned}$$

so that $f \succsim^* g$. Going the other direction, suppose $f \succsim^* g$ and that there exists a $\hat{p} \in co\{\beta\ell^\infty + (1-\beta)q : \ell \in D\}$ such that $\int u(f)\, d\hat{p} < \int u(g)\, d\hat{p}$. This implies that there exists an $\hat{\ell} \in D$ such that $\int u(f)\, d\left(\beta\hat{\ell}^\infty + (1-\beta)q\right) < \int u(g)\, d\left(\beta\hat{\ell}^\infty + (1-\beta)q\right)$. Let $\hat{h} = 1_{\Psi^{-1}(D\setminus\hat{\ell})}$. Choose $\hat{\lambda} \in (0,1)$ small enough to satisfy

$$\begin{aligned}
& (1-\hat{\lambda})(u(x^*) - u(x_*)) \\
& > \hat{\lambda}\max[\int u(f)\, d\hat{\ell}^\infty - \min_{p\in\{\ell^\infty:\ell\in D\}} \int u(f)\, dp, \int u(g)\, d\hat{\ell}^\infty - \min_{p\in\{\ell^\infty:\ell\in D\}} \int u(g)\, dp].
\end{aligned}$$

Then

$$\begin{aligned}
& \min_{p\in\{\ell^\infty:\ell\in D\}} \int u\left(\hat{\lambda} f + (1-\hat{\lambda})\hat{h}\right) dp \\
&= \int u\left(\hat{\lambda} f + (1-\hat{\lambda})\hat{h}\right) d\hat{\ell}^\infty \\
&< \int u\left(\hat{\lambda} g + (1-\hat{\lambda})\hat{h}\right) d\hat{\ell}^\infty \\
&= \min_{p\in\{\ell^\infty:\ell\in D\}} \int u\left(\hat{\lambda} g + (1-\hat{\lambda})\hat{h}\right) dp.
\end{aligned}$$

Therefore, as $\beta > 0$,

$$\begin{aligned}
& W\left(\hat{\lambda} f + (1-\hat{\lambda})\hat{h}\right) \\
&= \int u\left(\hat{\lambda} f + (1-\hat{\lambda})\hat{h}\right) d\left(\beta\hat{\ell}^\infty + (1-\beta)q\right) \\
&< \int u\left(\hat{\lambda} g + (1-\hat{\lambda})\hat{h}\right) d\left(\beta\hat{\ell}^\infty + (1-\beta)q\right) \\
&= W(\lambda g + (1-\lambda)h)
\end{aligned}$$

contradicting $f \succsim^* g$. Summarizing, we have shown that

$$f \succsim^* g \text{ if and only if } \int u(f)\, dp \geq \int u(g)\, dp \text{ for all } p \in co\{\beta\ell^\infty + (1-\beta)q : \ell \in D\}.$$

Therefore, applying Lemma 3.1 and noting that $co\{\beta\ell^{\infty}+(1-\beta)q:\ell\in D\}$ is weak* compact because D is finite, $\succsim$ represented by $W(f)$ satisfies Continuous Symmetry.

Observe that, since $q\in co\{\ell^{\infty}:\ell\in D\}$, $W(f)$ can be re-written as

$$\begin{aligned}&H\left(\left(\int u(f)\,d\ell^{\infty}\right)_{\ell\in D}\right)\\&\equiv\beta\min_{p\in\{\ell^{\infty}:\ell\in D\}}\int u(f)\,dp+(1-\beta)\int u(f)\,dq,\end{aligned}$$

and H so defined is weakly increasing. Therefore we may apply the uniqueness result in Theorem 3.2 to conclude $\overline{D}=R$. Since D is finite, $\overline{D}=D$.

To show that β reflects only tastes, we show that β is unique, given u, q and non-singleton D, as properties (1) and (2) of Definition 3.7 are straightforward. Note that increasing β strictly decreases the utility for the bet $1_{\Psi^{-1}(\ell)}$ with $q(\Psi^{-1}(\ell))>0$ and the utility for a lottery indifferent to $1_{\Psi^{-1}(\ell)}$ does not depends on β. Thus, β is unique, given u, q and non-singleton D.

B.9 Proof of Theorem 4.4

First we show that each measure in $\operatorname{supp} m$ is relevant. Suppose $\hat{\ell}\in\operatorname{supp} m$ and fix any open $L\subseteq\Delta(S)$ such that $\hat{\ell}\in L$. Take $x_1,x_2,x_3\in X$ such that $x_2\sim\frac{1}{2}x_1+\frac{1}{2}x_3$ and $x_1\succ x_3$. Define two acts f and g by

$$f(\omega)=\begin{cases}x_1 & \text{if } \Psi(\omega)\in L\\ x_2 & \text{otherwise}\end{cases}\quad\text{and } g(\omega)=\begin{cases}x_3 & \text{if } \Psi(\omega)\in L\\ x_2 & \text{otherwise}\end{cases}.$$

Since $\int f d\hat{\ell}^{\infty}=\int g d\hat{\ell}^{\infty}$ for all $\widehat{\ell}\in\Delta(S)\backslash L$, it suffices to show that $f\nsim g$. Assume $f\sim g$. Then, for each $i=1,...,n$,

$$\begin{aligned}\int\zeta_i u(f)\,dp&=\int_{\Psi^{-1}(L)}\zeta_i u(x_1)\,dp+\int_{\Omega\backslash\Psi^{-1}(L)}\zeta_i u(x_2)\,dp\\&=\int_{\Psi^{-1}(L)}\zeta_i\left[u(x_1)-u(x_2)\right]dp+\int_{\Omega}\zeta_i u(x_2)\,dp\\&=\int_{\Psi^{-1}(L)}\zeta_i\left[u(x_1)-u(x_2)\right]dp\\&=\int_{\Psi^{-1}(L)}\zeta_i\left[u(x_2)-u(x_3)\right]dp=-\int\zeta_i u(g)\,dp.\end{aligned}$$

The third equality follows because $\int\zeta_i dp=0$, and the fourth comes from $x_2\sim\frac{1}{2}x_1+\frac{1}{2}x_3$. Then, $f\sim g$ implies

$$\int u(f)\,dp+A\left(\left(\int\zeta_i u(f)\,dp\right)_{1\leq i\leq n}\right)=\int u(g)\,dp+A\left(\left(\int\zeta_i u(g)\,dp\right)_{1\leq i\leq n}\right).$$

As $A(a) = A(-a)$, this yields $\int u(f)\, dp = \int u(g)\, dp$ which contradicts $m(L) > 0$ since $x_1 \succ x_3$. Thus, $f \nsim g$ and each measure in $\operatorname{supp} m$ is relevant.

Next, we show that all measures in $\Delta(S)\setminus \operatorname{supp} m$ are irrelevant. Suppose $\hat{\ell} \in \Delta(S)\setminus \operatorname{supp} m$. There exists an open $L \subseteq \Delta(S)$ such that $\hat{\ell} \in L$ and $L \subseteq \Delta(S)\setminus \operatorname{supp} m$. We take arbitrary acts f and g, and show that $f_{\Psi^{-1}(L)}g \sim g$. Since $\int 1_{\Psi^{-1}(L)} d\ell^{\infty}$ for any $\ell \in \operatorname{supp} m$, we have

$$\int u\left(f_{\Psi^{-1}(L)}g\right) d\ell^{\infty} = \int u(g)\, d\ell^{\infty} \text{ and } \int \zeta_i u\left(f_{\Psi^{-1}(L)}g\right) d\ell^{\infty} = \int \zeta_i u(g)\, d\ell^{\infty}$$

for each $i = 1, ..., n$. Thus,

$$\int u\left(f_{\Psi^{-1}(L)}g\right) dp = \int u(g)\, dp \text{ and } \int \zeta_i u\left(f_{\Psi^{-1}(L)}g\right) dp = \int \zeta_i u(g)\, dp$$

for each $i = 1, ..., n$. This implies $T(f_{\Psi^{-1}(L)}g) = T(g)$. Therefore, all measures in $\Delta(S)\setminus \operatorname{supp} m$ are irrelevant.

Next, we show that $\succsim$ satisfies Continuous Symmetry. The form assumed for p and the symmetry property assumed for each ζ_i ensure that Event Symmetry is satisfied. The other properties in Symmetry along with Mixture Continuity of $\succsim$ follow directly from the properties of VEU (see Siniscalchi [43]). Mixture Continuity of $\succsim$ implies Mixture Continuity of $\succsim^*$. To see Monotone Continuity of $\succsim^*$, observe that $x' \succsim^* xA_kx''$ if and only if, for all $\alpha \in [0,1]$ and $h \in \mathcal{F}$,

$$\begin{aligned}
&\alpha u(x') + A\left(\left((1-\alpha)\int u(h)\zeta_i dp\right)_{1\le i\le n}\right) \\
&\geq \alpha\left(p(A_k)u(x) + (1 - p(A_k)u(x''))\right) \\
&+ A\left(\left(\alpha\left[u(x)\int_{A_k}\zeta_i dp + u(x'')\int_{A_k^c}\zeta_i dp\right] + (1-\alpha)\int u(h)\zeta_i dp\right)_{1\le i\le n}\right).
\end{aligned}$$

Since p is countably additive and ζ_i is bounded and measurable, $A_k \searrow \emptyset$ implies $p(A_k) \to 0$ and $\int_{A_k}\zeta_i dp \to 0$ and $\int_{A_k^c}\zeta_i dp \to \int_{S^\infty}\zeta_i dp = 0$. Therefore, since n is finite and A is Lipschitz continuous, there exists a k such that A_k is small enough so that $x' \succsim^* xA_kx''$. This proves Monotone Continuity of $\succsim^*$.

Finally, applying Theorem 3.1, the fact that all measures in $\Delta(S)\setminus \operatorname{supp} m$ are irrelevant implies no measures in $\Delta(S)\setminus \operatorname{supp} m$ are relevant. Therefore $R = \operatorname{supp} m$.

We now show that γ reflects only tastes. Again, uniqueness is the only part that may not be straightforward. By Theorem 1 of Siniscalchi [43], the scale of A is uniquely determined, given p, ζ, A' and u.

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