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MANAGEMENT SCIENCE
Vol. 67, No. 12, December 2021, pp. 7619–7636
ISSN 0025-1909 (print), ISSN 1526-5501 (online)

# If It Is Surely Better, Do It More? Implications for Preferences Under Ambiguity

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**Received:** August 20, 2018
**Revised:** August 21, 2020
**Accepted:** October 12, 2020
**Published Online in Articles in Advance:** March 1, 2021

**https://doi.org/10.1287/mnsc.2020.3890**



**Abstract.** Consider a canonical problem in choice under uncertainty: choosing from a convex feasible set consisting of all (Anscombe–Aumann) mixtures of two acts f and g, $\{\alpha f + (1-\alpha)g : \alpha \in [0,1]\}$. We propose a preference condition, *monotonicity in optimal mixtures*, which says that surely improving the act f (in the sense of weak dominance) makes the optimal weight(s) on f weakly higher. We use a stylized model of a sales agent reacting to incentives to illustrate the tight connection between monotonicity in optimal mixtures and a monotone comparative static of interest in applications. We then explore more generally the relation between this condition and preferences exhibiting ambiguity-sensitive behavior as in the classic Ellsberg paradoxes. We find that monotonicity in optimal mixtures and ambiguity aversion (even only local to an event) are incompatible for a large and popular class of ambiguity-sensitive preferences (the c-linearly biseparable class. This implies, for example, that maxmin expected utility preferences are consistent with monotonicity in optimal mixtures if and only if they are subjective expected utility preferences. This incompatibility is not between monotonicity in optimal mixtures and ambiguity aversion per se. For example, we show that smooth ambiguity preferences can satisfy both properties as long as they are not too ambiguity averse. Our most general result, applying to an extremely broad universe of preferences, shows a sense in which monotonicity in optimal mixtures places upper bounds on the intensity of ambiguity-averse behavior.

**History:** Accepted by Manel Baucells, decision analysis.
**Supplemental Material:** Data files are available at https://doi.org/10.1287/mnsc.2020.3890

**Keywords:** ambiguity • uncertainty • decision analysis theory • utility-preference theory • ambiguity aversion

## 1. Introduction

This paper proposes and investigates a preference condition, *monotonicity in optimal mixtures*, having particular relevance for comparative statics of behavior under ambiguity-sensitive preferences. The canonical way to represent the options a decision maker considers under uncertainty is to use *acts*, functions mapping states of the world to outcomes (which, in the standard Anscombe–Aumann setting adopted by this paper, may themselves be lotteries over more basic outcomes). Consider the set of acts generated from all (Anscombe–Aumann) mixtures of two acts $f$ and $g$: $\{\alpha f + (1-\alpha)g : \alpha \in [0,1]\}$ and think of preferences over this set as inducing preferences over $\alpha$. As one varies the acts $f$ and $g$ under consideration, the resulting preferences over $\alpha$ would be expected to change. A natural monotone comparative static to consider is that surely improving one of the acts (say, $f$), in the sense of state-by-state (weak) dominance, raises the optimal weight $\alpha$ placed on it. Monotonicity in optimal mixtures says that improving the act $f$ in the sense of weak dominance makes the (set of) optimal weight(s) on $f$ weakly higher. All subjective expected utility (SEU) preferences satisfy monotonicity in optimal mixtures (see Section 5 or Proposition 1).

For preferences as in Ellsberg's (1961) classic paradoxes, there is another force that might influence preferences over $\alpha$. Acts corresponding to intermediate weights $\alpha$ may have value as a hedge against ambiguity when $f$ and $g$ perform well under different distributions as, for example, when $f$ corresponds to winning a prize only if a red ball is drawn, $g$ corresponds to winning only if a blue ball is drawn, the composition of red versus blue balls is unknown, and $\frac{1}{2}f + \frac{1}{2}g$ corresponds to a sure 50% chance of winning a prize.

What are the implications of monotonicity in optimal mixtures for preferences? We address this in the context of a broad and axiomatically well understood universe of preferences (the monotonic, Bernoullian, and Archimedean or MBA preferences of Cerreia-Vioglio et al. 2011) that contains the vast majority of extant models of ambiguity-sensitive preferences (as well as the standard, ambiguity-neutral SEU model). We first show that the implications are stark for a large and popularly applied subclass of these preferences—the c-linearly biseparable preferences of Ghirardato and Marinacci (2001), which encompass maxmin expected utility (MEU) (Gilboa and Schmeidler (1989), Choquet expected utility (Schmeidler 1989), $\alpha$-MEU (Ghirardato et al. 2004) preferences, and more—for example, monotonicity in optimal mixtures and ambiguity aversion (even only local to a particular event) are shown to be incompatible for all such preferences. We show they are compatible for some MBA preferences, which we establish using the smooth ambiguity model (Klibanoff et al. 2005). For these preferences, monotonicity in optimal mixtures is satisfied when relative ambiguity aversion is not too large (Theorems 3 and 4). Finally, our most general result examines all MBA preferences and shows a sense in which monotonicity in optimal mixtures places upper bounds on the intensity of ambiguity-averse behavior (Theorem 5). When utility is unbounded above, these bounds are violated, for example, by all variational preferences (Maccheroni et al. 2006) that are not SEU.

Though we view monotonicity in optimal mixtures as a reasonable property of preferences, the purpose of the paper is not to advocate for it as either a normative or descriptive *requirement*. Rather, we see it primarily as an informative comparative static property. Our theoretical results, thus, give insight into the comparative static consequences of different models of ambiguity-sensitive behavior relevant in a variety of managerial settings.

As an example of such insight, our compatibility results for the smooth ambiguity model are closely related to work on comparative statics of portfolios of random variables (risky assets) under expected utility, addressing the question of when any first-order stochastically dominant shift in the (conditional on any realization of the other assets) distribution of an asset will result in a risk-averse expected utility investor increasing that asset's share in the optimal portfolio (see Fishburn and Burr Porter 1976, Hadar and Seo 1990, Meyer and Ormiston 1994, Mitchell and Douglas 1997, and the discussion following Remark 3 in Section 6.2). Moreover, our results on incompatibilities could be applied back to that literature to yield new results on comparative statics for various nonexpected utility models of choice under risk. For instance, any nonexpected utility model relying on convex preferences with kinks—for example, rank-dependent expected utility (Quiggin 1982) with concave utility and probability transformation function—must sometimes lead a first-order stochastic improvement to reduce that asset's share in the optimal portfolio. Furthermore, our results imply that, in a more realistic setting in which asset payoffs depend on events for which objective probabilities are not given, even risk-neutral investors cannot be too ambiguity averse if such reductions in share are never to occur. Though all of our results are shown independently of the risk aversion (or lack thereof) of the individual, in the context of this portfolio application, it is interesting to note that Fox et al. (1996) find evidence of the combination of risk neutrality with sensitivity to ambiguity among professional options traders.

Another domain of insight from our results can be seen in Auster (2014, 2018), concerning bilateral trade under ambiguity about quality. Optimal offer behavior on the part of an ambiguity-averse buyer derived there involves hedging-motivated mixing between a pooling price and a price that will be accepted only by a low-quality seller. One comparative static Auster examines is what happens to the mixing weight as the buyer's valuation of the high-quality seller's good increases. This corresponds to an improvement in the payoff to the pooling price in the sense of weak dominance. When the buyer has MEU preferences that are not SEU, in line with our result (Proposition 1) on incompatibility with monotonicity of optimal mixtures, there are many cases in which the optimal response is to offer the pooling price less often. Our upper bound result (Theorem 5) and our results on the smooth ambiguity model (Theorems 3 and 4) explain why such behavior could occur only with sufficiently strong ambiguity aversion.

As a further illustration of the link between monotonicity in optimal mixtures and behavior in managerial settings, we present a stylized model of a sales agent reacting to incentives designed to guide the agent's choice of which sales prospect to work on. In Section 2, we describe the sales agent model and illustrate how a key comparative static may or may not hold depending on the ambiguity-averse preferences attributed to the agent.

After describing the formal setting, notation, and MBA preferences in Sections 3 and 4, we define and discuss the monotonicity in optimal mixtures condition in Section 5, in which we also return to the sales agent application and show there is an equivalence between the key comparative static previously mentioned and monotonicity in optimal mixtures (see Theorem 1). The main results on the implications of monotonicity in optimal mixtures are in Section 6. A brief final section concludes. An appendix contains the axioms characterizing MBA preferences.

## 2. A Sales Agent Model: A Motivating Example

A firm employs a sales agent who can devote effort toward completing one of two possible sales. The returns to effort are such that it takes the agent's full effort to generate a chance that a sale will be successful so that it is never optimal for the agent to work on both sales. Sale 1, if successfully completed, leads to revenue of $v_1 > 0$ for the firm. If unsuccessful, the revenue for the firm would be zero. Similarly, the firm makes $v_2 > 0$ if sale 2 is successfully completed and otherwise zero. The model includes four possible states of the world $\{s_1, s_2, s_3, s_4\}$ corresponding to the four possible combinations of success or failure of the two sales if worked on. If the true state of the world $s$ is equal to $s_1$, it means sale 1 will be successful when the sales agent works on sale 1 and sale 2 will not be successful even if the agent works on it. Similarly, $s = s_2$ indicates the opposite, $s = s_3$ indicates each sale would be successful if worked on, and $s = s_4$ correspond to neither sale being successful if worked on. The firm uses commissions/bonuses to encourage the agent to pursue those sales prospects that are most attractive from the firm's point of view. Specifically, the compensation for the agent is as follows: if the agent completes sale $i$ successfully, the agent receives a payment $w(v_i)$, where $w(\cdot)$ is the nonnegative compensation scheme chosen by the firm. Table 1 summarizes the agent's payoffs.

A strategy for a sales agent specifies a probability $q$ of working on sale 1 (and, thus, $1 - q$ of working on sale 2). The best strategy for the agent varies with the compensation levels $w(v_1)$ and $w(v_2)$ and the preferences (including beliefs) of the agent. Taking these preferences as given, let $q^*(w(v_1), w(v_2))$ denote the optimal strategy (or, in cases of nonuniqueness, the set of all optimal strategies) of the sales agent as a function of $w(v_1)$ and $w(v_2)$ as they vary across all nonnegative payment levels. Assuming that the agent prefers more compensation to less, $q^*$ is nondecreasing in $w(v_1)$ and nonincreasing in $w(v_2)$ for any agent with SEU preferences. Will these comparative statics continue to hold for an ambiguity-sensitive agent? In the next two sections, we provide examples that show the answer depends on aspects of the agent's preferences beyond simply whether the agent is ambiguity averse. Later in the paper, after the relevant concepts have been formally introduced, we return to these comparative statics. Assuming more compensation is preferred to less, we show in Theorem 1 in Section 5 that the sales agent is responsive in this manner to compensation if and only if the agent's preferences satisfy monotonicity in optimal mixtures on the space of feasible acts for the agent. Thus, these examples and Theorem 1 further motivate interest in the monotonicity in optimal mixtures condition.

**Table 1.** Sales Agent's Payoff as a Function of the Sales Prospect Worked on, the Realized State of the World, and the Compensation Scheme $w$

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| Sale 1 | $w(v_1)$ | 0 | $w(v_1)$ | 0 |
| Sale 2 | 0 | $w(v_2)$ | $w(v_2)$ | 0 |

For the examples in this section, we examine sales agents whose preferences are given by two of the most popular models of ambiguity-averse behavior: the MEU and the smooth ambiguity model, respectively.

## 2.1. An MEU Agent

Here, we model the sales agent using a seminal model of ambiguity-averse preferences: the MEU model (Gilboa and Schmeidler 1989). Each MEU preference over acts $f$ can be represented by a functional of the following form:

$$\min_{p\in C} \sum_{s} u\big(f(s)\big)p(s), \tag{1}$$

where $u$ is a nonconstant von Neumann–Morgenstern utility function and $C$ is a nonempty, closed, and convex set of probability measures over states. As was described, the state space for this example is $S = \{s_1, s_2, s_3, s_4\}$. Notice that, when the set $C$ contains only one probability measure, preferences are SEU. Gilboa and Schmeidler (1989) show that MEU preferences are characterized by dropping the Anscombe–Aumann independence axiom of SEU and replacing it with two weaker axioms: certainty independence and uncertainty aversion.

For a given probability $q$ of working on sale 1 (and, thus, $1 - q$ of working on sale 2), nonnegative compensation function $w$, strictly increasing utility function $u$ normalized so that $u(0) = 0$, and probability distribution $p$ over states, the expected utility of the agent is

$$q\big[\big(p(s_1) + p(s_3)\big)u(w(v_1))\big] + \big(1 - q\big)\big[\big(p(s_2) + p(s_3)\big)u(w(v_2))\big].$$

The optimal strategy (or set of optimal strategies) $q^*$, therefore, satisfies

$$q^*(w(v_1), w(v_2)) = \arg\max_{q\in[0,1]}\left[\min_{p\in C} q\big[\big(p(s_1) + p(s_3)\big)u(w(v_1))\big] + (1 - q)\big[\big(p(s_2) + p(s_3)\big)u(w(v_2))\big]\right].$$

To facilitate an explicit solution while still maintaining a good deal of flexibility, we consider in this example sets of probability distributions of the following parametric form: $C = \{(r(1-\delta)(1-\kappa), (1-r)(1-\delta)(1-\kappa), \delta(1-\kappa), \kappa) : \kappa \in K, \delta \in D, r \in R\}$, where $K = [\underline{\kappa}, \overline{\kappa}] \subseteq [0, 1)$, $D = [\underline{\delta}, \overline{\delta}] \subseteq [0, 1)$, and $R = [\underline{r}, \overline{r}] \subseteq [0, 1]$. Note that $1 - \kappa$ is the probability that at least one of the prospects will sell if the agent works on it. This probability could be considered a measure of overall market conditions. Conditional on at least one of the prospects selling if worked on, $\delta$ is the probability that both would sell if worked on and so is a measure of the positive association between the success of the two prospects. Finally, conditional on exactly one of the two prospects selling if worked on, $r$ is the probability that it is the first prospect that will sell if worked on. The set $C$ is constructed so that ambiguity about each of these parameters is fixed no matter what value is taken by the other two parameters. Given MEU preferences with such a $C$, observe that, independent of choice of $q$, the minimizing measure $p$ has $\kappa = \overline{\kappa}$ and $\delta = \underline{\delta}$. Thus, the optimal $q^*$ satisfies

$$q^*(w(v_1), w(v_2)) = \arg\max_{q\in[0,1]}(1-\overline{\kappa})\Bigg[\min_{r\in[\underline{r},\overline{r}]} q[(r(1-\underline{\delta})+\underline{\delta}) \times u(w(v_1))] + (1-q)[((1-r)(1-\underline{\delta})+\underline{\delta})u(w(v_2))]\Bigg]. \quad (2)$$

Let $\hat{q} \equiv \frac{u(w(v_2))}{u(w(v_1))+u(w(v_2))} \in [0,1]$. Observe that, for $q < \hat{q}$, the min in (2) is attained only at $r = \overline{r}$. Similarly, for $q > \hat{q}$, it is attained only at $r = \underline{r}$. For $q = \hat{q}$, all $r \in [\underline{r}, \overline{r}]$ are minimizers. Therefore, if $(\overline{r}(1-\underline{\delta})+ \underline{\delta})u(w(v_1)) < ((1-\overline{r})(1-\underline{\delta})+ \underline{\delta})u(w(v_2))$, then $q^*(w(v_1), w(v_2)) = 0$, and similarly, if $(\underline{r}(1-\underline{\delta})+\underline{\delta})u(w(v_1)) > ((1-\underline{r})(1-\underline{\delta})+\underline{\delta})u(w(v_2))$, then $q^*(w(v_1), w(v_2)) = 1$. If $(\overline{r}(1-\underline{\delta})+ \underline{\delta})u(w(v_1)) = ((1-\overline{r})(1-\underline{\delta})+\underline{\delta})u(w(v_2))$, then multiplicity occurs and $q^*(w(v_1), w(v_2)) = [0, \hat{q}]$, and if $(\underline{r}(1-\underline{\delta})+\underline{\delta})u(w(v_1)) = ((1-\underline{r})(1-\underline{\delta})+ \underline{\delta})u(w(v_2))$, then $q^*(w(v_1), w(v_2)) = [\hat{q}, 1]$.[1] In all other cases,

$$q^*(w(v_1), w(v_2)) = \frac{u(w(v_2))}{u(w(v_1)) + u(w(v_2))}. \quad (3)$$

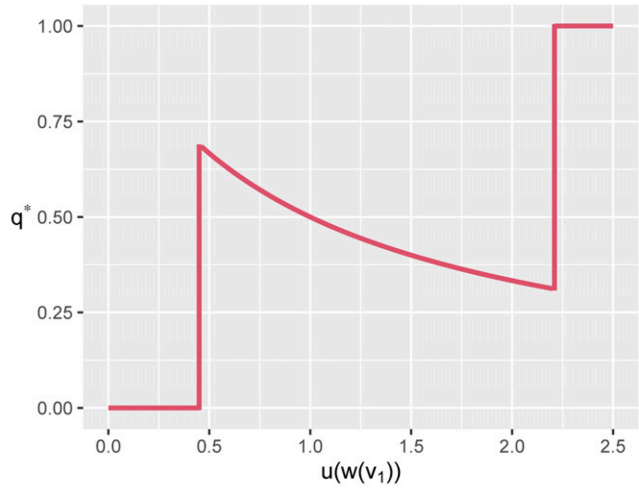
Observe that (3) defines $q^*$ when

$$\frac{(1-\underline{r})(1-\underline{\delta})+\underline{\delta}}{\underline{r}(1-\underline{\delta})+\underline{\delta}} > \frac{u(w(v_1))}{u(w(v_2))} > \frac{(1-\overline{r})(1-\underline{\delta})+\underline{\delta}}{\overline{r}(1-\underline{\delta})+\underline{\delta}}, \quad (4)$$

which defines a nonempty range of $\frac{u(w(v_1))}{u(w(v_2))}$ values if and only if $\underline{r} < \overline{r}$ (i.e., there is some ambiguity about which prospect will sell if worked on conditional on there being exactly one such prospect). From (3), we see that, for any $(w(v_1), w(v_2))$ in this range, $q^*(w(v_1), w(v_2))$ is such that increasing the payment for a given sales prospect *decreases* the likelihood that the agent works on that sales prospect. For example, if $\underline{r} = \frac{1}{8}$, $\overline{r} = \frac{7}{8}$, and $\underline{\delta} = \frac{1}{3}$, this is true for any $(u(w(v_1)), u(w(v_2)))$ such that $\frac{5}{11} < \frac{u(w(v_1))}{u(w(v_2))} < \frac{11}{5}$ (Figure 1).

From the MEU agent's point of view, under any increasing payment scheme $w$, in the region in which positive probability is placed on both prospects, the agent becomes *less likely* to work toward sale $i$ as its value $v_i$ increases. Similarly, any scheme in which payment is *decreasing* in the value of the sale implies that, in such a region, the agent is *more likely* to make an effort toward sale $i$ as its value increases.

**Figure 1.** (Color online) MEU Agent's Optimal Probabilities, $q^*$, of Working on Sale 1 as the Payoff from a Successful Sale 1, $u(w(v_1))$, Varies, Holding the Payoff from a Successful Sale 2 Fixed at $u(w(v_2)) = 1$ with Parameters $\underline{r} = \frac{1}{8}$, $\overline{r} = \frac{7}{8}$, $\underline{\delta} = \frac{1}{3}$




*Note.* Parameters $\kappa$ and $\overline{\delta}$ do not affect $q^*$ for an MEU agent.

## 2.2. A Smooth Ambiguity Agent

Now, suppose that, instead of MEU, the agent's preferences can be described by the smooth ambiguity model (Klibanoff et al. 2005). Each smooth ambiguity preference over acts $f$ can be represented by a functional of the following form:

$$\int \phi\left(\sum_s u(f(s))p(s)\right)d\mu(p), \quad (5)$$

where $u$ is a nonconstant von Neumann–Morgenstern utility function, $\phi$ is a continuous and strictly increasing function on the range of $u$, and $\mu$ is a countably additive probability measure over probability measures over states $S$. To maintain comparability to the MEU example, we assume that the support of $\mu$ is contained in the set $C$. Thus, $\mu$ can be written as a probability measure over $[\underline{\kappa}, \overline{\kappa}] \times [\underline{\delta}, \overline{\delta}] \times [\underline{r}, \overline{r}]$. We also continue to assume that the compensation scheme $w(\cdot)$ is nonnegative and that $u$ is strictly increasing and normalized so that $u(0) = 0$. The curvature of $\phi$ reflects attitude toward ambiguity with concavity (respectively, convexity) of $\phi$ corresponding to ambiguity aversion (love), more concave corresponding to more ambiguity averse, and an affine $\phi$ implying ambiguity neutrality and SEU (see Klibanoff et al. 2005). For this example, we assume ambiguity aversion and specifically that $\phi$ is twice continuously differentiable with $\phi' > 0$ and $\phi'' < 0$.

The optimal strategies $q^*$ are the following:

$$\begin{aligned} &q^*(w(v_1), w(v_2)) \\ &\quad = \arg\max_{q \in [0,1]} \int_{[\underline{\kappa},\overline{\kappa}]\times[\underline{\delta},\overline{\delta}]\times[\underline{r},\overline{r}]} \\ &\qquad \times \phi\left( \begin{array}{l} (1-\kappa)\big[q(r(1-\delta)+\delta)u(w(v_1)) \\ +(1-q)((1-r)(1-\delta)+\delta)u(w(v_2))\big] \end{array} \right) \\ &\qquad \times d\mu(\kappa,\delta,r). \end{aligned}$$

If $(r(1-\delta)+\delta)u(w(v_1)) \le ((1-r)(1-\delta)+\delta)u(w(v_2))$ for all $\delta, r \in [\underline{\delta},\overline{\delta}] \times [\underline{r},\overline{r}]$, then $q^*(w(v_1), w(v_2)) = 0$. Similarly, if $(r(1-\delta)+\delta)u(w(v_1)) \ge ((1-r)(1-\delta)+\delta)u(w(v_2))$ for all $\delta, r \in [\underline{\delta},\overline{\delta}] \times [\underline{r},\overline{r}]$, then $q^*(w(v_1), w(v_2)) = 1$.[2]

Between these boundary values, one can show that, if $x\frac{-\phi''(x)}{\phi'(x)} \le 1$ for all $x > 0$, then $q^*$ is nondecreasing in $w(v_1)$ and nonincreasing in $w(v_2)$. Thus, because $\frac{-\phi''(x)}{\phi'(x)}$ is an (Arrow–Pratt) index of ambiguity aversion (see Klibanoff et al. 2005), we see that, if ambiguity aversion is not too severe, in contrast to the MEU case, more compensation for a prospect pushes the agent with smooth ambiguity preferences toward that prospect.[3] This monotonicity may be violated at higher levels of ambiguity aversion. See Figure 2 for an illustration, in which $x\frac{-\phi''(x)}{\phi'(x)}$ is parameterized by $\theta$.

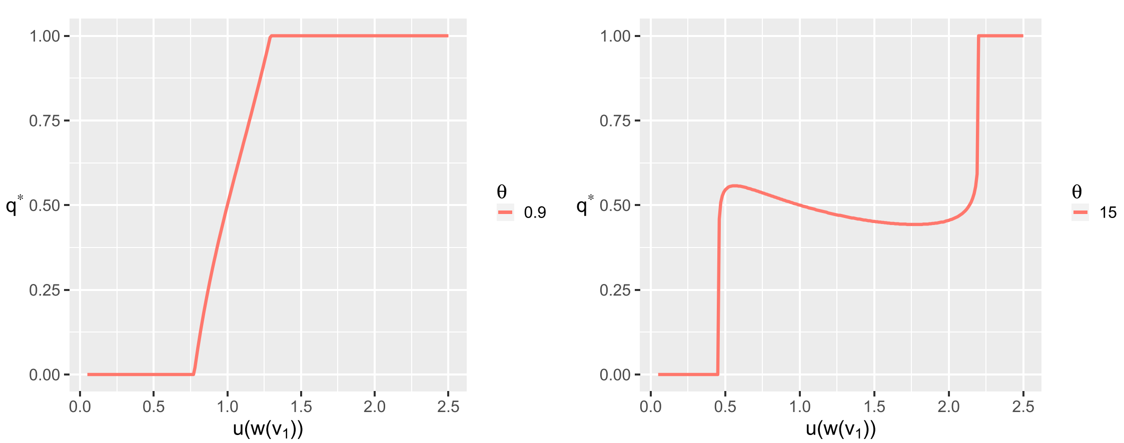
Thus, describing the agent using smooth ambiguity preferences has enough flexibility to allow for either monotone (as in the left panel) or nonmonotone behavior (as in the right panel), and the intensity of ambiguity aversion is a key determinant of which of the two behaviors is exhibited.

### 2.3. Discussion of the Sales Agent Example

There is evidence that uncertainty/conflict can lead individuals to have a strict preference for payoffs that mixed acts generate (Dwenger et al. 2018, Lin and Reich 2018). Having models that capture such preferences is important if one is to study mechanisms through which an individual's behavior under uncertainty can be tilted toward an organizationally/socially desired one. Unlike the subjective expected utility model, models of ambiguity-averse preferences explicitly allow for preference for hedging as modeled by preference for mixed acts and are often even *defined* by such a property, most prominently, for example, in the form of Schmeidler's uncertainty aversion axiom (Schmeidler 1989), which says that a mixture of two indifferent acts is never worse and may be strictly better than the original acts.[4] As demonstrated, however, various ambiguity aversion models have vastly different implications for how individuals in an organization respond to incentive schemes. We believe monotonicity in optimal mixtures is a key condition that can help categorize ambiguity-aversion frameworks related to their implications for such incentive problems.

Our analysis of the sales agent model is not aimed at drawing normative conclusions about monotonicity in optimal mixtures or the MEU and smooth ambiguity models. The main purpose is, rather, to highlight an application in which researchers in economics and management science need to be aware of whether the ambiguity-aversion framework they are using satisfies monotonicity in optimal mixtures or not. Indeed, MEU (or other models that do not satisfy monotonicity in optimal mixtures) might be

**Figure 2.** (Color online) Smooth Ambiguity Agent's Optimal Probability, $q^*$, of Working on Sale 1 as the Payoff from a Successful Sale 1, $u(w(v_1))$, Varies, Holding the Payoff from a Successful Sale 2 Fixed at $u(w(v_2)) = 1$ with Parameters $\underline{r} = \frac{1}{8}$, $\overline{r} = \frac{7}{8}$, $\underline{\delta} = \overline{\delta} = \frac{1}{3}$, $\mu(\underline{r},\underline{\delta},\underline{\kappa}) = \mu(\overline{r},\overline{\delta},\overline{\kappa}) = \frac{1}{2}$, and $\phi(x) \equiv \frac{x^{1-\theta}}{1-\theta}$ with $\theta = 0.9$ (Left Panel) or $\theta = 15$ (Right Panel)




*Note.* Parameter κ does not affect q* for such an agent.

more suitable than the smooth ambiguity model (or other frameworks that can accommodate monotonicity in optimal mixtures) in settings in which its relative simplicity and a hedging motive strong enough to drive nonmonotonicity are important and judged reasonable.

We now leave the confines of this example and turn to describing the general setting of the paper.

## 3. Setting and Preliminaries

We operate within a standard Fishburn (1970)-style version of an Anscombe and Aumann (1963) setting. Let $S$ be the finite set of states. An event $E$ is a subset of $S$. Let $Z$ be the set of prizes or outcomes, and $X$ is the set of all simple lotteries over prizes (i.e., the set of all finite-support probability distributions on $Z$). Observe that $X$ is a convex set with respect to (w.r.t.) the following mixture operation: for $\alpha \in [0,1]$ and $x, y \in X$, $\alpha x + (1-\alpha)y$ is the element of $X$ defined, for all $z \in Z$, by

$$\big(\alpha x + (1-\alpha)y\big)(z) \equiv \alpha x(z) + (1-\alpha)y(z).$$

Acts are functions from $S$ to $X$. Let $\mathcal{F}$ denote the set of all acts. Acts are the objects of choice. Preferences are defined by a binary relation $\succsim$ over acts. The symmetric and asymmetric parts of $\succsim$ are denoted by $\sim$ and $\succ$, respectively. Mixtures over acts are defined through statewise mixing of the resulting lotteries: for $\alpha \in [0,1]$ and $f, g \in \mathcal{F}$, $\alpha f + (1-\alpha)g$ is the act defined, for all $s \in S$, by

$$\big(\alpha f + (1-\alpha)g\big)(s) \equiv \alpha f(s) + (1-\alpha)g(s).$$

For $x, y \in X$ and an event $E$, let $xEy$ denote the act $f$ such that (s.t.) $\forall s \in E, f(s) = x$ and $\forall s \notin E, f(s) = y$. Constant acts are those that give the same lottery in all states (i.e., $f(s) = f(s'), \forall s, s' \in S$). In a standard abuse of notation, we sometimes use $x$ to denote the constant act giving $x \in X$ in each state. An act $f$ is an interior act if, for each state $s$, there exist $\overline{x}(s), \underline{x}(s) \in X$ such that $\overline{x}(s) \succ f(s) \succ \underline{x}(s)$.

A set function $\rho : 2^S \to \mathbb{R}$ is a *capacity* if $\rho(\emptyset) = 0$, $\rho(S) = 1$, and for all $E, F \subseteq S$ with $E \subseteq F$, $\rho(E) \le \rho(F)$.

## 4. Preferences

Throughout, we restrict attention to preferences in the MBA class defined and axiomatized by Cerreia-Vioglio et al. (2011).[5] In terms of numerical representation, this is equivalent to assuming $\succsim$ can be represented by

$$V\Big(\big(u\big(f(s)\big)\big)_{s\in S}\Big), \tag{6}$$

where $u : X \to \mathbb{R}$ is a nonconstant, affine utility function and $V : u(X)^S \to \mathbb{R}$ is normalized, monotonic, and sup-norm continuous. (Note that $u(f(s)) \equiv \sum_z u(z) f(s)(z)$.)

This is a very general class of preferences and has the virtue of including most of the models of decision making under ambiguity in the literature. The most important restriction imposed by MBA preferences is that nonexpected utility behavior with respect to lotteries (i.e., constant acts) is ruled out. Thus, the departures from expected utility that are allowed by MBA preferences concern aggregation across states. In this sense, we restrict attention to preferences that may violate subjective expected utility but obey expected utility under "objective" risk. An advantage of doing so is that our analysis may be carried out in utility space, greatly facilitating our arguments.

Subjective expected utility preferences are exactly the MBA preferences that additionally satisfy the (Anscombe–Aumann) independence axiom:

**Axiom 1.** (Independence). *For all acts $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1]$, $f \succsim g$ if and only if $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$.*

## 5. A Monotonicity Consideration: Monotonicity in Optimal Mixtures

Before we introduce the main definition of the paper, we remind the reader of a standard definition of *set order* (see, e.g., Milgrom and Shannon 1994).

**Definition 1.** For any two sets $A, B \subseteq \mathbb{R}$, we say $A$ is smaller in the set order than $B$ and write $A \le_s B$ if $a \in A$ and $b \in B$ imply $\min\{a, b\} \in A$ and $\max\{a, b\} \in B$.

It is straightforward to verify that, when $A$ and $B$ are singletons, the relationship $\le_s$ on sets collapses to the usual ordering $\le$ on numbers. Symmetrically, we say $B$ is larger in the set order than $A$, denoted $B \ge_s A$, if $A \le_s B$.

The main novel property we introduce is the following:

**Definition 2.** (Monotonicity in Optimal Mixtures). For all acts $f, f', g$ such that $f'(s) \succsim f(s)$ for all $s \in S$, the set of all $\alpha^* \in [0,1]$ such that $\alpha^* f + (1-\alpha^*)g$ is optimal in $\{\alpha f + (1-\alpha)g : \alpha \in [0,1]\}$ is smaller in the set order than the set of all $\alpha' \in [0,1]$ such that $\alpha' f' + (1-\alpha')g$ is optimal in $\{\alpha f' + (1-\alpha)g : \alpha \in [0,1]\}$.

Monotonicity in optimal mixtures says that improving the act $f$ via weak dominance at least weakly enlarges (in the set order) the optimal weight(s) placed on it when mixing with $g$. If optima in both cases are unique, it says that this optimal weight must at least weakly increase when improving $f$ to the dominant $f'$. The definition has a monotone comparative static flavor, which gives it a natural connection to comparative statics in applications. It also lends itself to simple revealed preference tests for any given triple of acts $f, f', g$ such that $f'$ weakly dominates $f$. For example, fixing $f, f', g$, one can first have an individual choose from $\{\alpha f + (1-\alpha)g : \alpha \in [0,1]\}$, yielding an $\alpha^*$, and then also choose from $\{\alpha f' + (1-\alpha)g : \alpha \in [0,1]\}$, yielding an $\alpha'$. If $\alpha^* > \alpha'$, then offer (a) to trade $\alpha' f + (1-\alpha')g$ in exchange for $\alpha^* f + (1-\alpha^*)g$ and (b) to trade $\alpha^* f' + (1-\alpha^*)g$ in exchange for $\alpha' f' + (1-\alpha')g$. A violation of monotonicity in optimal mixtures occurs when the individual is (strictly) unwilling to make at least one of these trades.

Monotonicity in optimal mixtures can be related to two well-known preference conditions: independence (see Axiom 1 in the previous section) and the following property satisfied by all MBA preferences:

**Axiom 2.** (State-by-State Monotonicity). *For all acts $f', f$, if $f'(s) \succsim f(s)$ for all $s \in S$, then $f' \succsim f$.*

Monotonicity in optimal mixtures is a strengthening of state-by-state monotonicity. To see this, suppose $f'(s) \succsim f(s)$ for all $s \in S$ and take $g = f$ in the statement of monotonicity in optimal mixtures. Because any mixture of $f$ with itself is optimal, consider $\alpha^* = 1$. Then, monotonicity in optimal mixtures requires that $\alpha' = 1$ is optimal in $\{\alpha f' + (1-\alpha)f : \alpha \in [0,1]\}$, implying $f' \succsim f$ and, thus, state-by-state monotonicity.

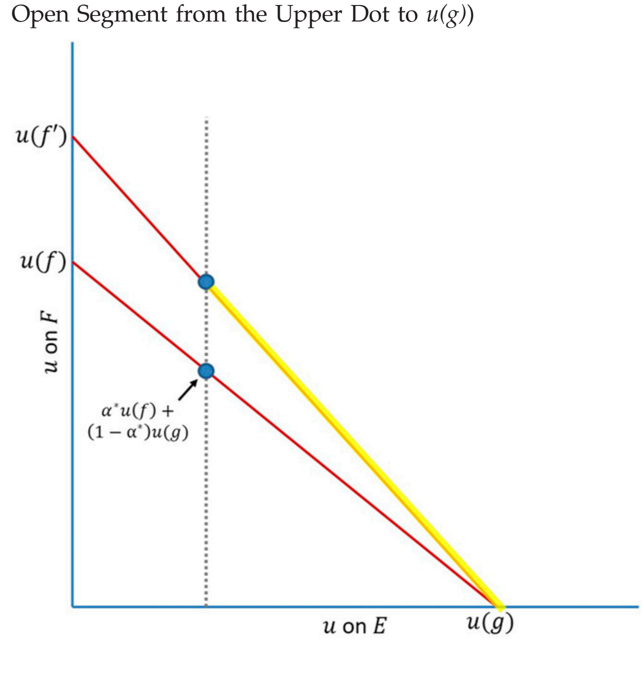
Given state-by-state monotonicity, the independence axiom is a strengthening of monotonicity in optimal mixtures. To see this, denote, for any pair of acts $f, g$, the set of all optimal $\alpha$ mixtures in $\{\alpha f + (1-\alpha)g : \alpha \in [0,1]\}$ by $\alpha^*_{f,g}$. By the independence axiom, one can show $\alpha^*_{f,g} = \begin{cases} 0 & f \prec g \\ [0,1] & f \sim g \\ 1 & f \succ g \end{cases}$. The analogous property can be shown for the set $\alpha^*_{f',g}$ of optimal mixtures between $f'$ and $g$. Now, suppose $\alpha \in \alpha^*_{f,g}$ and $\alpha' \in \alpha^*_{f',g}$. If $\alpha \le \alpha'$, then, by construction, we have $\min\{\alpha, \alpha'\} \in \alpha^*_{f,g}$ and $\max\{\alpha, \alpha'\} \in \alpha^*_{f',g}$. If $\alpha > \alpha'$, then it has to be that $\alpha > 0$ and $\alpha' < 1$. From $\alpha > 0$, it follows that $f \succsim g$. From $\alpha' < 1$, it follows that $g \succsim f'$. But, from state-by-state monotonicity, we know $f' \succsim f$. Therefore, $f \sim f' \sim g$. This implies that $\alpha^*_{f,g} = \alpha^*_{f',g} = [0,1]$, in turn implying $\min\{\alpha, \alpha'\} \in \alpha^*_{f,g}$ and $\max\{\alpha, \alpha'\} \in \alpha^*_{f',g}$. Thus, monotonicity in optimal mixtures is satisfied. An immediate corollary of this relationship with independence is that all subjective expected utility preferences satisfy monotonicity in optimal mixtures.

An analogy with consumer theory can give further insight into monotonicity in optimal mixtures. Consider the special case in which $g$ yields a fixed, positive utility level on an event $E$ and zero utility elsewhere; $f$ yields a fixed, positive utility level on an event $F$ and zero utility elsewhere; $E$ and $F$ are disjoint; and $f'$ strictly improves $f$ only on $F$ (and does so by a fixed amount of utility). One can then view preferences over mixtures between $f$ and $g$ as preferences over consumption bundles of two goods—utility in event $E$ and utility in event $F$—in which the feasible bundles lie on the line segment in consumption space connecting the points $(0, u(f(F)))$ and $(u(g(E)), 0)$. Replacing $f$ by $f'$ rotates this budget set outward as utility in event $F$ has effectively become cheaper. As depicted in Figure 3, monotonicity in optimal mixtures implies that there are optimal choices such that consumption of utility in event $E$ does not rise as a result of this price decrease on utility in event $F$. In the language of consumer theory, there are optimal choices such that the substitution effect on consumption of utility in $E$ of such a price change (nonpositive) must be at least as large in magnitude as the corresponding income effect (nonnegative): monotonicity in optimal mixtures implies the existence of optimal choices such that utilities in $E$ and $F$ must be gross substitutes. Observe that the linear indifference curves of subjective expected utility preferences imply a constant marginal rate of substitution in utility space and, thus, that utility in $E$ and $F$ are perfect substitutes and, thus, certainly gross substitutes.

Finally, in our analysis and for applications, it is sometimes useful to consider monotonicity in optimal mixtures restricted to particular acts. Formally, for a set of acts $\mathcal{G}$, when we write monotonicity in optimal mixtures restricted to $\mathcal{G}$, we mean adding the requirement that $f, f', g \in \mathcal{G}$ to Definition 2.

Returning to our sales agent model, the following result shows the tight link between monotonicity in optimal mixtures and the monotonicity of the sales agent's reaction to compensation. In reading it, recall that $q^*(\cdot,\cdot)$ denotes the agent's optimal probability (or all optimal probabilities in the case of multiplicity) of working on sale 1 (with the remainder assigned to

**Figure 3.** (Color online) Monotonicity in Optimal Mixtures Implies That, if $\alpha^*$ Is an Optimal Mixture of $f$ and $g$, Then Some Optimal Mixture of $f'$ and $g$ Must Lie Weakly to the Left of the Upper Dot (i.e., Outside the Thickened Half-Open Segment from the Upper Dot to $u(g)$)



working on sale 2). The two arguments of $q^*$ are the compensation for closing sales 1 and 2, respectively. Furthermore, the set of acts $H$ in the theorem are exactly the acts that have a utility profile that can be generated by some specification of nonnegative compensation levels $w(v_1)$ and $w(v_2)$ and a sales prospect (i.e., sale 1 or 2) on which the agent works.

**Theorem 1.** *Suppose $\succsim$ are MBA preferences, $u$ is strictly increasing in payment and normalized so that $u(0) = 0$, and the sales agent's optimal strategy correspondence $q^*(\cdot,\cdot)$ exists. Then, satisfying monotonicity in optimal mixtures restricted to the set of acts $H \equiv \{h : u(h(s)) \geq 0$ for all $s, u(h(s_4)) = 0$ and $h$ is measurable either with respect to $\{\{s_1, s_3\}, \{s_2, s_4\}\}$ or $\{\{s_2, s_3\}, \{s_1, s_4\}\}\}$ is equivalent to $q^*$ weakly increasing (in the set order) in its first argument and weakly decreasing (in the set order) in its second argument.*

**Proof.** Observe that, if act $f$ corresponds to working on sale 1, $g$ corresponds to working on sale 2, and $f'$ corresponds to working on sale 1 with an increased level of compensation $w(v_1)$ if successful, these each belong to the set $H$, and applying monotonicity in optimal mixtures to such acts implies $q^*$ must be weakly increasing in the set order sense in its first argument. Reversing the roles of $f$ and $g$ and increasing $w(v_2)$, monotonicity in optimal mixtures similarly implies that $1 - q^*$ is weakly increasing, and so $q^*$ is weakly decreasing in its second argument in the set order sense. It remains to show that these properties of $q^*$ imply that monotonicity in optimal mixtures holds when restricted to acts in $H$. We proceed by showing the contrapositive: that any violation of monotonicity in optimal mixtures on $H$ implies a violation of at least one of the properties of $q^*$. Suppose there is a violation of this restricted monotonicity in optimal mixtures. Fix some $f, f', g$ in $H$ that generate the violation. If $f$ and $g$ are measurable with respect to the same partition as each other, all three acts can be ordered by weak dominance, and the conclusion of monotonicity in optimal mixtures follows because all MBA preferences satisfy state-by-state monotonicity. Thus, it remains to consider the cases in which the acts involved have state-by-state expected utilities of the following form:

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| $f$ | $u(f(s_1))$ | 0 | $u(f(s_1))$ | 0 |
| $f'$ | $u(f(s_1)) + x$ | 0 | $u(f(s_1)) + x$ | 0 |
| $g$ | 0 | $u(g(s_2))$ | $u(g(s_2))$ | 0 |

with $x \geq 0$, or

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| $f$ | 0 | $u(f(s_2))$ | $u(f(s_2))$ | 0 |
| $f'$ | 0 | $u(f(s_2)) + y$ | $u(f(s_2)) + y$ | 0 |
| $g$ | $u(g(s_1))$ | 0 | $u(g(s_1))$ | 0 |

with $y \geq 0$. Violation of monotonicity in optimal mixtures implies that there exists an $\alpha^* \in \arg\max_{\lambda \in [0,1]} V(u(\lambda f + (1-\lambda)g))$ and an $\alpha' \in \arg\max_{\lambda \in [0,1]} V(u(\lambda f' + (1-\lambda)g))$ such that $\alpha^* > \alpha'$ and either $\alpha' \notin \arg\max_{\lambda \in [0,1]} V(u(\lambda f + (1-\lambda)g))$ or $\alpha^* \notin \arg\max_{\lambda \in [0,1]} V(u(\lambda f' + (1-\lambda)g))$. In the case in which $f$ is measurable with respect to $\{\{s_1, s_3\}, \{s_2, s_4\}\}$, this implies $\alpha^* \in q^*(u^{-1}(u(f(s_1))), u^{-1}(u(g(s_2))))$, $\alpha' \in q^*(u^{-1}(u(f(s_1)) + x), u^{-1}(u(g(s_2))))$, and either $\alpha' \notin q^*(u^{-1}(u(f(s_1))), u^{-1}(u(g(s_2))))$ or $\alpha^* \notin q^*(u^{-1}(u(f(s_1)) + x), u^{-1}(u(g(s_2))))$. Because $u$ is strictly increasing and $x \geq 0$, it follows that $q^*$ is not weakly increasing in its first argument in the set order sense. In the case in which $f$ is measurable with respect to $\{\{s_2, s_3\}, \{s_1, s_4\}\}$, an analogous argument implies $q^*$ is not weakly decreasing in its second argument in the set order sense. Thus, violation of monotonicity in optimal mixtures restricted to $H$ implies a violation of at least one of the conditions on $q^*$, and this completes the proof. □

Given this connection of monotonicity in optimal mixtures with behavior in such managerially relevant contexts, we next turn to understanding the relation between the choice of preference model and monotonicity in optimal mixtures.

## 6. Implications of Monotonicity in Optimal Mixtures

How does monotonicity in optimal mixtures relate to some popular models of ambiguity-sensitive preferences as well as to MBA preferences generally? We first show (Theorem 2) that, for a large and popular subclass of MBA preferences—the c-linearly biseparable preferences of Ghirardato and Marinacci (2001), which encompass MEU (Gilboa and Schmeidler 1989), Choquet expected utility (Schmeidler 1989), $\alpha$-MEU (Ghirardato et al. 2004) preferences, and more—there is a tight relation between ambiguity aversion toward bets concerning an event and violations of monotonicity in optimal mixtures for such bets. The former implies the latter, and under some mild additional conditions, they are equivalent. When specialized to MEU preferences, this allows us to show that MEU preferences satisfy monotonicity in optimal mixtures if and only if they are SEU preferences (Proposition 1). These results illuminate the behavior we saw under MEU in the sales agent model and show that the type of nonmonotone behavior exhibited there is inherent in using MEU to model ambiguity-averse departures from SEU.

Turning to the other popular subclass of MBA preferences used in the sales agent example, the smooth ambiguity model (Klibanoff et al. 2005), we show that these preferences satisfy monotonicity in optimal mixtures whenever relative ambiguity aversion is not too large (Theorems 3 and 4). This, again, illuminates and generalizes the behavior exhibited in the sales agent context.

Finally, our most general result examines all MBA preferences and shows a sense in which monotonicity in optimal mixtures places upper bounds on the intensity of ambiguity-averse behavior (Theorem 5). In addition to being directly applicable in managerial settings, such as the sales agent model, this result also allows us to compare the categorization induced by monotonicity in optimal mixtures with Lang (2017)'s categorization based on first- versus second-order ambiguity aversion. It also allows us to provide an extension of our MEU result to the variational preferences (Maccheroni et al. 2006), a class that has seen important applications, especially the use of multiplier preferences (Hansen and Sargent 2001) to model concern for robustness in finance and macroeconomics.

### 6.1. Implications for c-Linearly Biseparable Preferences

Ghirardato and Marinacci (2001) define and axiomatize a broad class of preferences they call *c-linearly biseparable*. This class includes, among others, the well-known maxmin expected utility with nonunique prior (MEU) (Gilboa and Schmeidler 1989), Choquet expected utility (Schmeidler 1989), and $\alpha$-MEU (in which preference is represented by a convex combination of MEU and max-max EU) models. Axiomatically, such preferences are those that (1) are nontrivial preference relations; (2) admit a constant act equivalent for each act (i.e., for each act $f$, there is some lottery $x_f$ such that $f \sim x_f$); (3) satisfy state-by-state monotonicity; (4) satisfy a weak form of continuity in mixing weights; and (5) for binary acts (i.e., acts of the form $xEy$ for all $E \subseteq S$, $x, y \in X$), satisfy Gilboa and Schmeidler's (1989) certainty independence, which weakens the independence axiom by requiring it to hold only when the common act $h$ being mixed with is a constant act. Observe that (1)–(4) are mild conditions that, by themselves, do not impose more than is already entailed in the MBA preferences. The key requirement is (5), which applies only to binary acts. In terms of numerical representation for binary acts, these properties imply that there is a unique capacity $\rho$ and a nonconstant von Neumann–Morgenstern utility function $u$ such that

$$W(xEy) \equiv u(x)\rho(E) + u(y)(1 - \rho(E)) \tag{7}$$

represents $\succsim$ over acts of the form $xEy$ for all $E \subseteq S$, $x, y \in X$ with $x \succsim y$. The quantity $\rho(E) + \rho(E^c)$ is useful in classifying ambiguity attitude in regard to bets on or against an event $E$ (i.e., in regard to acts measurable with respect to $\{E, E^c\}$). Specifically, $\rho(E) + \rho(E^c) < 1$ corresponds to (strict) ambiguity aversion toward such bets, $\rho(E) + \rho(E^c) = 1$ to ambiguity neutrality, and $\rho(E) + \rho(E^c) > 1$ to (strict) ambiguity-loving behavior. Our next result shows that, for any c-linearly biseparable preferences, monotonicity in optimal mixtures restricted to such bets implies $\rho(E) + \rho(E^c) \geq 1$, and under a mild additional condition, the converse holds as well.

**Theorem 2.** *Fix any event $E$. If $\rho(E) + \rho(E^c) < 1$, then a c-linearly biseparable preference violates monotonicity in optimal mixtures restricted to acts measurable with respect to $\{E, E^c\}$. The converse holds as long as either* $\max\{\rho(E), \rho(E^c)\} < 1$ *or* $\min\{\rho(E), \rho(E^c)\} = 0$.

**Proof of Theorem 2.** We begin by showing the first direction. If $\rho(E) + \rho(E^c) < 1$, then (7) implies that $\succsim$ over acts $h$ measurable with respect to $\{E, E^c\}$ are represented by $\min\{\rho(E)u(h(E)) + (1-\rho(E))u(h(E^c)), (1-\rho(E^c))u(h(E)) + \rho(E^c)u(h(E^c))\}$ because $\rho(E) < 1 - \rho(E^c)$ and $\rho(E^c) < 1 - \rho(E)$ ensure that $\rho(E)u(h(E)) + (1-\rho(E))u(h(E^c)) < (1-\rho(E^c))u(h(E)) + \rho(E^c)u(h(E^c))$ when $h(E) \succsim h(E^c)$ and $(1-\rho(E^c))u(h(E)) + \rho(E^c)u(h(E^c)) < \rho(E)u(h(E)) + (1-\rho(E))u(h(E^c))$ when $h(E^c) \succsim h(E)$. Fix any acts $f = xEy$, $g = wEz$ such that $x \succ y$, $z \succ w$, $x \succ w$, and $z \succ y$.[6] Consider the choice of an optimal act from the set $\{\alpha f + (1-\alpha)g : \alpha \in [0, 1]\}$. Whenever the slope $-\frac{u(z)-u(y)}{u(x)-u(w)}$ of the line connecting $(u(f(E)), u(f(E^c))$ and $(u(g(E)), u(g(E^c))$ lies strictly between the slope $-\frac{\rho(E)}{1-\rho(E)}$ of the linear indifference curves corresponding to preferences represented by $\rho(E)u(h(E)) + (1-\rho(E))u(h(E^c))$ over acts $h$ measurable with respect to $\{E, E^c\}$ and the slope $-\frac{1-\rho(E^c)}{\rho(E^c)}$ of the linear indifference curves corresponding to preferences represented by $(1-\rho(E^c))u(h(E)) + \rho(E^c)u(h(E^c))$ over those acts, it follows that $\min\{\rho(E)u(\alpha f(E) + (1-\alpha)g(E)) + (1-\rho(E))u(\alpha f(E^c) + (1-\alpha)g(E^c)), (1-\rho(E^c))u(\alpha f(E) + (1-\alpha)g(E)) + \rho(E^c)u(\alpha f(E^c) + (1-\alpha)g(E^c))\}$ is uniquely maximized by the interior $\alpha$ that equates expected utility across $E$ and $E^c$.[7] Therefore, suppose $\frac{1-\rho(E^c)}{\rho(E^c)} > \frac{u(z)-u(y)}{u(x)-u(w)} > \frac{\rho(E)}{1-\rho(E)}$. Such $x, y, w, z \in X$ exist because $u$ is a nonconstant expected utility over lotteries and $\rho(E) < 1 - \rho(E^c)$. Calculation shows that the optimal mixture is $\alpha^* = 1/(1 + (u(x) - u(y))/(u(z) - u(w))) \in (0, 1)$.

Next, consider any $f' = x'Ey'$ with $x' \succ x$ and $y' \sim y$. Observe that $f'$ weakly dominates $f$. If $\frac{u(z)-u(y')}{u(x')-u(w)} > \frac{\rho(E)}{1-\rho(E)}$ (i.e., if $x'$ is not so good as to lead to all weight on $f'$ being an optimal mixture between $f'$ and $g$), then the unique optimal mixture between $f'$ and $g$ is again the $\alpha$ that equates $u$ across $E$ and $E^c$. Calculation shows that this now occurs at $\alpha' = 1/(1 + (u(x') - u(y'))/(u(z) - u(w))) \in (0,1)$. Because $\alpha' < \alpha^*$, this is a violation of monotonicity in optimal mixtures restricted to acts measurable with respect to $\{E, E^c\}$.

We next prove the converse direction: for any event $E$, given either $\max\{\rho(E), \rho(E^c)\} < 1$ or $\min\{\rho(E), \rho(E^c)\} = 0$, if $\rho(E) + \rho(E^c) \geq 1$, then monotonicity in optimal mixtures restricted to acts measurable with respect to $\{E, E^c\}$ is satisfied. If $\rho(E) + \rho(E^c) \geq 1$, then (7) implies that $\succsim$ over acts $h$ measurable with respect to $\{E, E^c\}$ are represented by $\max\{\rho(E)u(h(E)) + (1-\rho(E))u(h(E^c)), (1-\rho(E^c))u(h(E)) + \rho(E^c)u(h(E^c))\}$ because $\rho(E) \geq 1-\rho(E^c)$ and $\rho(E^c) \geq 1 - \rho(E)$ ensure that $\rho(E)u(h(E)) + (1-\rho(E))u(h(E^c)) \geq (1-\rho(E^c))u(h(E)) + \rho(E^c)u(h(E^c))$ when $h(E) \succsim h(E^c)$ and $(1-\rho(E^c))u(h(E)) + \rho(E^c)u(h(E^c)) \geq \rho(E)u(h(E)) + (1-\rho(E))u(h(E^c))$ when $h(E^c) \succsim h(E)$. By measurability with respect to $\{E, E^c\}$, we can write the acts appearing in the statement of monotonicity in optimal mixtures as $f = xEy$, $g = wEz$, and $f' = x'Ey'$ for some $x, x', y, y', w, z \in X$ with $x' \succsim x$ and $y' \succsim y$. Suppose $f' \succ g$. Then, $\max\{\rho(E)u(x') + (1-\rho(E))u(y'), (1-\rho(E^c))u(x') + \rho(E^c)u(y')\} > \max\{\rho(E)u(w) + (1-\rho(E))u(z), (1-\rho(E^c))u(w) + \rho(E^c)u(z)\}$. This implies that, for all $\alpha \in [0,1)$, $\max\{\rho(E)u(x') + (1-\rho(E))u(y'), (1-\rho(E^c))u(x') + \rho(E^c)u(y')\} > \max\{\rho(E)u(\alpha x' + (1-\alpha)w) + (1-\rho(E))u(\alpha y' + (1-\alpha)z), (1-\rho(E^c))u(\alpha x' + (1-\alpha)w) + \rho(E^c)u(\alpha y' + (1-\alpha)z)\}$. Therefore, $f' \succ \alpha f' + (1-\alpha)g$ for all $\alpha \in [0,1)$, implying $\alpha' = 1 \in \arg\max\{\alpha f' + (1-\alpha)g : \alpha \in [0,1]\}$ is the unique such maximizer $\alpha'$ and, thus, that monotonicity in optimal mixtures is satisfied for acts measurable with respect to $\{E, E^c\}$. Next, suppose instead that $g \succ f'$. Because $f' \succsim f$, it follows that $g \succ f$. Using, again, the max representation and the linearity in mixtures of each component within the max, this implies $g \succ \alpha f + (1-\alpha)g$ for all $\alpha \in (0,1]$, implying $\alpha^* = 0 \in \arg\max\{\alpha f + (1-\alpha)g : \alpha \in [0,1]\}$ is the unique such maximizer $\alpha^*$ and, thus, that monotonicity in optimal mixtures is satisfied for acts measurable with respect to $\{E, E^c\}$. Finally, suppose that $f' \sim g$. If $f' \succ f$, then $g \succ f$, and the argument proceeds exactly as in the case in which $g \succ f'$. The only remaining possibility is that $f' \sim f$ so that $f' \sim f \sim g$. This is the only place in the argument where the condition that either $\max\{\rho(E), \rho(E^c)\} < 1$ or $\min\{\rho(E), \rho(E^c)\} = 0$ is needed. First, suppose $\max\{\rho(E), \rho(E^c)\} < 1$. Because $\rho(E) + \rho(E^c) \geq 1$, it follows that $\rho(E), \rho(E^c) \in (0,1)$, and then, $f' \sim f$ implies $u(x') = u(x)$ and $u(y') = u(y)$, and the optimal mixture weights for $f$ and $g$ are identical to those of $f'$ and $g$, satisfying monotonicity in optimal mixtures. Second, suppose $\min\{\rho(E), \rho(E^c)\} = 0$. Then, because $\rho(E) + \rho(E^c) \geq 1$, $\rho(E) + \rho(E^c) = 1$ and preferences restricted to acts measurable with respect to $\{E, E^c\}$ are SEU and, therefore, satisfy monotonicity in optimal mixtures. □

**Remark 1.** When $\rho(E) + \rho(E^c) < 1$ so that preferences are ambiguity averse for acts measurable with respect to $\{E, E^c\}$, the proof of Theorem 2 shows more than just that monotonicity in optimal mixtures restricted to acts measurable with respect to $\{E, E^c\}$ is violated. It shows that, starting from any $\{E, E^c\}$-measurable acts $f$ and $g$ (without loss of generality, label the acts so that $u(f(E)) \geq u(g(E))$) such that an interior $\alpha$-mixture is strictly preferred to both $f$ and $g$, *all* $f'$ formed from $f$ by improving the utility on $E$ generate violations of monotonicity in optimal mixtures unless the improvement is so large as to induce $f' \succsim \alpha f' + (1-\alpha)g$ for all $\alpha \in [0,1]$. Thus, not only is monotonicity in optimal mixtures violated, but starting from *any* acts for which an interior $\alpha$ is strictly optimal, *all* utility improvements on $E$ that do not push the optimal weight on the improved act to one lead to reduced weight being placed on the improved act.

**Remark 2.** Here is an example showing that the auxiliary conditions are needed for the converse to hold. Specifically, in the example, $\max\{\rho(E), \rho(E^c)\} = 1$ and $\min\{\rho(E), \rho(E^c)\} > 0$ (and, thus, $\rho(E) + \rho(E^c) \geq 1$), yet monotonicity in optimal mixtures for acts measurable with respect to $\{E, E^c\}$ is violated. Let $\rho(E) = 1$, $\rho(E^c) = 0.5$, $u(x') = u(y') = u(x) = 1$, $u(y) = 0$, $u(w) = 0.5$, $u(z) = 1.5$, $f = xEy$, $g = wEz$, and $f' = x'Ey'$. Observe that, for all $\alpha \in [0,1]$, $\alpha f' + (1-\alpha)g$ is evaluated as $\max\{\alpha + (1-\alpha) * 0.5, 0.5 * (\alpha + (1-\alpha) * 0.5) + 0.5 * (\alpha + (1-\alpha) * 1.5)\} = 0.5 * (\alpha + (1-\alpha) * 0.5) + 0.5 * (\alpha + (1-\alpha) * 1.5) = 1$, implying that all mixtures between $f'$ and $g$ are optimal. Next, observe that any mixture $\alpha f + (1-\alpha)g$ is evaluated as $\max\{\alpha + (1-\alpha) * 0.5, 0.5 * (\alpha + (1-\alpha) * 0.5) + 0.5 * (1-\alpha) * 1.5\} = \max\{\frac{1+\alpha}{2}, 1 - \frac{\alpha}{2}\}$, which is maximized only at $\alpha = 0$ and $\alpha = 1$. To see that monotonicity in optimal mixtures is violated, note that $1 \in \arg\max\{\alpha f + (1-\alpha)g\}$ and $\frac{1}{2} \in \arg\max\{\alpha f' + (1-\alpha)g\}$, but $\min\{\frac{1}{2}, 1\} \notin \arg\max\{\alpha f + (1-\alpha)g\}$.

What is the meaning of the extra condition, that either $\max\{\rho(E), \rho(E^c)\} < 1$ or $\min\{\rho(E), \rho(E^c)\} = 0$, needed for the converse? A simple preference condition that implies this extra condition for any c-linearly biseparable preferences is a strict version of state-by-state montonicity, ensuring that every state matters.

**Axiom 3.** (Strict State-by-State Monotonicity). *For all acts $f', f$, if $f'(s) \succsim f(s)$ for all $s \in S$ and $f'(s) \succ f(s)$ for some $s \in S$, then $f' \succ f$.*

In fact, requiring this only for binary acts is enough to imply that $0 < \rho(E) < 1$ for all nonempty events $E$ that are strict subsets of $S$, which is stronger than the statement that the extra condition holds for all events $E$.[8]

An immediate implication of Theorem 2 is that ambiguity aversion toward bets about an event is incompatible with monotonicity in optimal mixtures for c-linearly biseparable preferences. As we show next, this leads to a particularly sharp result for MEU preferences.

**6.1.1. Implications for MEU.** Recall from Section 2.1 that MEU preferences have a representation as in (1). Notice that, when the set $C$ contains only one probability measure, preferences are SEU. In all other cases, MEU preferences depart from SEU. Which MEU preferences satisfy monotonicity in optimal mixtures?

**Proposition 1.** *Fix any event $E$. An MEU preference satisfies monotonicity in optimal mixtures restricted to acts measurable with respect to $\{E, E^c\}$ if and only if all measures in $C$ assign $E$ the same probability.*

**Proof of Proposition 1.** If all $p \in C$ assign the same $p(E)$, MEU preferences restricted to acts measurable with respect to $\{E, E^c\}$ are SEU preferences and, therefore, satisfy monotonicity in optimal mixtures restricted to acts measurable with respect to $\{E, E^c\}$. Suppose $p(E)$ varies across measures in $C$. The conclusion that monotonicity in optimal mixtures restricted to acts measurable with respect to $\{E, E^c\}$ is violated then follows as a special case of Theorem 2 because MEU preferences satisfy (7) with capacity $\rho(E) \equiv \min_{p\in C} p(E)$ for all $E \subseteq S$, and $p(E)$ varying then implies $\rho(E) + \rho(E^c) < 1$. □

Applying Proposition 1 across all events $E$ reveals that MEU preferences satisfy monotonicity in optimal mixtures if and only if they are SEU preferences. It follows that an alternative axiomatic characterization of SEU preferences in an Anscombe–Aumann setting is obtained by weakening the independence axiom by replacing it with the two novel axioms used by Gilboa and Schmeidler (1989) in axiomatizing MEU, namely certainty independence and uncertainty aversion, and then strengthening state-by-state monotonicity by replacing it with the monotonicity in optimal mixtures condition.

### 6.2. Implications for the Smooth Ambiguity Model

Are there ambiguity-averse preferences that can satisfy monotonicity in optimal mixtures? In this section, we show that the answer is yes. To do so, we consider the smooth ambiguity model (Klibanoff et al. 2005). Recall from Section 2.2 that such preferences have a representation as in (5). For results in this section, we further assume that $\phi$ is twice continuously differentiable on the interior of $u(X) = [0, \infty)$ with $\phi' > 0$ and $\phi'' \le 0$. Concavity of $\phi$ implies ambiguity aversion (see Klibanoff et al. 2005). We provide an upper bound on the coefficient of ambiguity aversion, $-\frac{\phi''(x)}{\phi'(x)}$, that is sufficient and, if $\mu$ is unrestricted, necessary for such preferences to satisfy monotonicity in optimal mixtures (Theorems 3 and 4). Thus, when applied to smooth ambiguity preferences, monotonicity in optimal mixtures is compatible with ambiguity aversion as long as the aversion is not too strong.

**Theorem 3.** *Preferences represented by the smooth ambiguity model as in (5) with $\phi$ twice continuously differentiable on the interior of $u(X) = [0, \infty)$, $\phi' > 0$, and $\phi'' \le 0$ satisfy monotonicity in optimal mixtures if $\phi$ is everywhere at most as concave as natural log, i.e.,*

$$-\frac{\phi''(a)}{\phi'(a)} \le \frac{1}{a}, \text{ for all } a > 0.$$

**Proof of Theorem 3.** For $\alpha \in [0, 1]$, $v \in u(X)^S$, and act $g \in \mathcal{F}$, define

$$W^g(\alpha, v) \equiv \int \phi\left(\sum_{\hat{s}} \big(\alpha v(\hat{s}) + (1-\alpha)u\big(g(\hat{s})\big)\big)p(\hat{s})\right) d\mu(p). \tag{8}$$

If each of the cross-partial derivatives with respect to $\alpha$ and the $s^{th}$ component of $v$, $W^g_{\alpha v(s)}(\alpha, v)$, are nonnegative for all $\alpha \in (0, 1)$ and strictly positive $v$, then by, for example, theorem 2 from Milgrom and Roberts (1990), $W^g$ is supermodular with respect to $\alpha$ and any component $s$ of $v$. This is sufficient for monotonicity in optimal mixtures because by, for example, theorem 5 from Milgrom and Roberts (1990), it implies that the increased state-by-state utility from improving $f$ to a weakly dominating $f'$ results in $\arg\max_{\alpha\in[0,1]} W^g(\alpha, u \circ f) \le_s \arg\max_{\alpha\in[0,1]} W^g(\alpha, u \circ f')$. We now show that, when $-\frac{\phi''(a)}{\phi'(a)} \le \frac{1}{a}$ for all a > 0, these cross-partials are indeed nonnegative. By differentiating, we obtain

$$W^g_\alpha(\alpha, v) = \int \left(\sum_{\hat{s}} \big(v(\hat{s}) - u\big(g(\hat{s})\big)\big)p(\hat{s})\right) \times \phi'\left(\sum_{\hat{s}} \big(\alpha v(\hat{s}) + (1-\alpha)u\big(g(\hat{s})\big)\big)p(\hat{s})\right) d\mu(p)$$

and

$$\begin{aligned} &W^g_{\alpha v(s)}(\alpha, v) \\ &\quad = \int \Bigg[ p(s)\phi'\left(\sum_{\hat{s}} (\alpha v(\hat{s}) + (1-\alpha). \times u\big(g(\hat{s})\big))p(\hat{s})\right) \\ &\qquad + \alpha p(s)\left(\sum_{\hat{s}} \big(v(\hat{s}) - u\big(g(\hat{s})\big)\big)p(\hat{s})\right) \\ &\qquad \times \phi''\left(\sum_{\hat{s}} \big(\alpha v(\hat{s}) + (1-\alpha)u\big(g(\hat{s})\big)\big)p(\hat{s})\right)\Bigg] d\mu(p). \end{aligned} \tag{9}$$

From $-\frac{\phi''(a)}{\phi'(a)} \leq \frac{1}{a}$ with $a = \sum_{\hat{s}}(\alpha v(\hat{s}) + (1-\alpha)u(g(\hat{s})))p(\hat{s}) > 0$, we obtain that, for all $p$,

$$\begin{aligned}\phi'\left(\sum_{\hat{s}}(\alpha v(\hat{s}) + (1-\alpha)u(g(\hat{s})))p(\hat{s})\right) \\ \geq -\phi''\left(\sum_{\hat{s}}(\alpha v(\hat{s}) + (1-\alpha)u(g(\hat{s})))p(\hat{s})\right) \\ \times\left(\sum_{\hat{s}}(\alpha v(\hat{s}) + (1-\alpha)u(g(\hat{s})))p(\hat{s})\right).\end{aligned} \tag{10}$$

Substituting (10) into (9) yields,

$$\begin{aligned}W^g_{\alpha v(s)}(\alpha, v) \geq \int\Bigg[p(s)\phi''\left(\sum_{\hat{s}}(\alpha v(\hat{s}) + (1-\alpha)u(g(\hat{s})))p(\hat{s})\right) \\ \times\left(-\sum_{\hat{s}}u(g(\hat{s}))p(\hat{s})\right)\Bigg]d\mu(p) \geq 0\end{aligned}$$

as claimed. □

**Theorem 4.** *For $\phi$ twice continuously differentiable on the interior of $u(X) = [0, \infty)$ with $\phi' > 0$ and $\phi'' \leq 0$, if $-\frac{\phi''(a)}{\phi'(a)} > \frac{1}{a}$ for some $a > 0$, then there exists a measure $\mu$ such that preferences represented by (5) violate monotonicity in optimal mixtures.*

What is the reason for the role of $\mu$ in Theorem 4? Intuitively, the extent to which higher ambiguity aversion manifests itself in behavior depends on how much ambiguity the individual perceives, with more ambiguity leading to stronger effects of any given ambiguity aversion. This suggests that a $\mu$ reflecting an extremely ambiguous event should be a good candidate for generating the required violation as soon as ambiguity aversion exceeds the tightest possible bound. The proof uses this strategy, constructing the violation using bets on an event assigned only probabilities one and zero by measures in the support of $\mu$ with at least some measures assigning each.[9]

**Proof of Theorem 4.** By assumption, $\exists a > 0$ s.t. $-\frac{\phi''(a)}{\phi'(a)} > \frac{1}{a}$, which, because $\phi'(a) > 0$, implies $\phi'(a) + a\phi''(a) < 0$. We construct a $\mu$ that generates a violation of monotonicity in optimal mixtures. Let $\mu(p_1) = \mu(p_2) = \frac{1}{2}$, where $p_1$ and $p_2$ are probability measures on $S$ such that $\exists E \subset S$ with $p_1(E) = 1$ and $p_2(E) = 0$. Consider the following acts $f$ and $g$: $f = xEz$ and $g = zEx$, where $u(x) = 2a$ and $u(z) = 0$. By concavity of $\phi$, $\phi(a) \geq \frac{1}{2}\phi(\alpha 2a) + \frac{1}{2}\phi((1-\alpha)2a)$, which implies that, for all $\alpha \in [0, 1]$,

$$\frac{1}{2}f + \frac{1}{2}g \succsim \alpha f + (1-\alpha)g. \tag{11}$$

Define $\overline{W}(\alpha, v) \equiv W^{zEx}(\alpha, vE0)$, where $W^{zEx}(\alpha, vE0)$ is defined as in (8) in the proof of Theorem 3 with $g = zEx$ and $vE0$ is the vector representing the state-by-state utility of the act $yEz$, where $u(y) = v > 0$. Substituting our constructed $\mu$, we get

$$\overline{W}(\alpha, v) = \frac{1}{2}[\phi(\alpha v) + \phi((1-\alpha)2a)].$$

Differentiating w.r.t. $\alpha$ yields

$$\overline{W}_\alpha(\alpha, v) = \frac{1}{2}[v\phi'(\alpha v) - 2a\phi'((1-\alpha)2a)]. \tag{12}$$

Further differentiating w.r.t. $v$, we get

$$\overline{W}_{\alpha v}(\alpha, v) = \frac{1}{2}[\phi'(\alpha v) + \alpha v\phi''(\alpha v)]. \tag{13}$$

Now, observe that $\overline{W}_{\alpha v}(\alpha, v)$ is negative when $\alpha = \frac{1}{2}$ and $v = 2a$:

$$\overline{W}_{\alpha v}\left(\frac{1}{2}, 2a\right) = \frac{1}{2}[\phi'(a) + a\phi''(a)] < 0. \tag{14}$$

From (12), $\overline{W}_\alpha(\frac{1}{2}, 2a) = 0$, and thus, by (14), there exists a $b > 2a$ such that

$$\overline{W}_\alpha\left(\frac{1}{2}, b\right) < 0. \tag{15}$$

Furthermore, $\overline{W}(\alpha, v)$ is globally weakly concave in $\alpha$ because

$$\overline{W}_{\alpha\alpha}(\alpha, v) = \frac{1}{2}[v^2\phi''(\alpha v) + 4a^2\phi''((1-\alpha)2a)] \leq 0.$$

Therefore, there exists an $\hat{\alpha} < \frac{1}{2}$ such that, for all $\alpha \geq \frac{1}{2}$,

$$\overline{W}(\alpha, b) < \overline{W}(\hat{\alpha}, b). \tag{16}$$

Letting $f' = yEz$ and setting $v = b$, we see that $f'$ weakly dominates $f$, and any mixture $\alpha f' + (1-\alpha)g$ is evaluated according to $\overline{W}(\alpha, b)$. Although, by (11), $\frac{1}{2}f + \frac{1}{2}g$ is an optimal mixture of $f$ and $g$, (16) implies that any optimal mixture of $f'$ and $g$ must place weight strictly below $\frac{1}{2}$ on $f'$, a violation of monotonicity in optimal mixtures. □

**Remark 3.** In Remark 1, we observed that, for c-linear biseparable preferences that are ambiguity averse when restricted to $\{E, E^c\}$-measurable acts, when starting from *any* acts for which an interior mixture is strictly optimal, *all* utility improvements on $E$ that do not push the optimal weight on the improved act to one result in nonmonotonicities. The same property need not hold for smooth ambiguity preferences even when they violate monotonicity in optimal mixtures; specifically, one can have some regions in which, starting from a strictly optimal interior mixture, small utility improvements on

$E$ result in nonmonotonicity, and for other regions, the reaction to such improvements is monotonic without all weight being pushed to the improved act. The right-hand graph in Figure 2 from the sales agent example gives a clear illustration of this. Intuitively, this can happen because ambiguity aversion (absolute or relative, i.e., $-\frac{\phi''(a)}{\phi'(a)}$ or $-a\frac{\phi''(a)}{\phi'(a)}$) may vary as expected utility levels change. In contrast, c-linear biseparable preferences, because they satisfy Gilboa and Schmeidler's (1989) certainty independence when restricted to $\{E, E^c\}$-measurable acts, effectively have a constant (both absolute and relative) ambiguity attitude for a given partition $\{E, E^c\}$. Analogously to risk attitudes in expected utility, smooth ambiguity preferences have both absolute and relative ambiguity aversion simultaneously constant only when they are ambiguity neutral ($\phi$ linear). Axiomatically, they satisfy certainty independence only when they are ambiguity neutral.

These results on monotonicity in optimal mixtures in the context of the smooth ambiguity model are closely related to work on comparative statics of portfolios of random variables (risky assets) under expected utility. A strand of that literature addresses the question of when any first-order stochastic dominant shift in the (conditional on any realization of the other assets) distribution of an asset results in a risk-averse expected utility investor increasing that asset's share in the optimal portfolio. The answer is when utility is everywhere at most as concave as natural log (equivalently, $xu'(x)$ increasing, absolute risk aversion at any $x \le \frac{1}{x}$ or relative risk aversion everywhere at most one). Fishburn and Porter (1976) show this for the special case of one risky and one safe asset. Hadar and Seo (1990) extend this result to any two assets with independently distributed returns. Meyer and Ormiston (1994) extend this to the case in which returns across assets may be dependent (in which case, the shift being to the conditional distribution becomes important). Finally, Mitchell and Douglas (1997) establish the result for the case of $n$ (possibly dependent) assets.

The condition that utility is at most as concave as natural log also appears in the literature on general equilibrium with additively separable utilities, in which it has been identified as leading demand for contingent goods to have the gross substitutes property and to the existence of a unique equilibrium (see Dana 2001 for a survey). Returning to the consumer theory analogy, for additively separable utilities over consumption goods, natural log utility leads income and substitution effects to just offset each other, further suggesting a connection with monotonicity in optimal mixtures.

It is also interesting to note that this bound on relative risk aversion (in our case, relative ambiguity aversion) is stronger than the conditions on relative risk aversion shown recently in Lanier (2020) to be sufficient and, with unrestricted beliefs, necessary for all assets to be ordinary goods (i.e., not Giffen goods). This is to be expected because, returning to the consumer theory analogy, though Giffen goods are not gross substitutes, ordinary goods may or may not be gross substitutes. For example, in Figure 3 used to illustrate monotonicity in optimal mixtures in Section 5, the condition that utility on event F is an ordinary good requires that an optimum from the larger budget set not lie in the part of the thickened segment lying below the lower dot, and monotonicity in optimal mixtures also requires that it not be below the upper dot.

Although this section has focused on the smooth ambiguity model, other specific models in the literature, for example the confidence preference model of Chateauneuf and Faro (2009), are also capable of displaying ambiguity aversion while satisfying monotonicity in optimal mixtures under some conditions. In the next section, we turn to implications of monotonicity in mixtures that apply to *all* MBA preferences.

### 6.3. More General Implications: Bounds on Slopes in Utility Space at Different Points

Here, we provide a result (Theorem 5) identifying a relatively simple condition on general MBA preferences, violation of which implies violation of monotonicity in optimal mixtures. Specifically, we show that monotonicity in optimal mixtures bounds how different the slope of a "local" support of the better-than set (i.e., a line supporting the better-than set within an appropriate rectangle) at one point in utility space can be from the slope of such a local support at another point. Monotonicity in optimal mixtures is necessarily violated whenever these slopes change "too fast." One immediate implication is that preferences with kinked boundaries of convex better-than sets, which have multiple supporting slopes at a given point, cannot satisfy monotonicity in optimal mixtures. This implication generalizes (by dramatically expanding the preferences to which the conclusion applies) our previous finding that c-linearly biseparable preferences for which there exists an event $E$ with $\rho(E) + \rho(E^c) < 1$ violate monotonicity in optimal mixtures. Theorem 5 is not limited to implications about such "kinked" MBA preferences, however, and also may be seen as a generalization to MBA preferences of our Theorem 4 on smooth ambiguity preferences, which identifies high relative ambiguity aversion as a source of violations of monotonicity in optimal mixtures.

The specific form of the bound (17) derived in Theorem 5 delivers useful insights. The bound is on the ratio of slopes and becomes tighter as the points are translated up by adding positive constants (i.e., moved parallel to the 45° line on the utility graph).

When utility is unbounded above so that these positive constants may be taken to infinity, this ratio bound converges to one. Satisfying this bound, therefore, implies that absolute ambiguity aversion eventually disappears in the sense that, for any bounded rectangle of points, translating this rectangle up by adding large enough positive constants makes any convexity of indifference curves within the translated rectangle arbitrarily uniformly small. In particular, if preferences are convex (i.e., satisfy Schmeidler's uncertainty aversion), then the bound implies that preferences eventually approach expected utility (and, thus, ambiguity neutrality) in any such region of large enough stakes. Thus, one implication of Theorem 5 is that, when utility is unbounded above, any variational preferences (Maccheroni et al. 2006), which are the constant absolute ambiguity-averse preferences satisfying uncertainty aversion (Grant and Polak 2013) that are not SEU, must violate monotonicity in optimal mixtures.[10] This includes the multiplier preferences introduced by Hansen and Sargent (2001) to model concern for robustness to misspecification (see Strzalecki 2011). For experimental evidence suggesting that many subjects display decreasing rather than constant absolute ambiguity aversion, see, for example, Baillon and Placido (2019) and Berger and Bosetti (2020). The sense in which the slope bound is also a bound on relative ambiguity aversion and the relation of this bound with Theorem 4 is explained in the paragraph just before the proof of Theorem 5.

From these and our earlier results, we see that monotonicity in optimal mixtures provides a different categorization of ambiguity-averse preferences than the first- versus second-order ambiguity aversion (very roughly, kinked versus smooth indifference curves) categorization explored in Lang (2017). An implication of Theorem 5 is that all first-order ambiguity-averse MBA preferences and some second-order ambiguity-averse MBA preferences violate monotonicity in optimal mixtures. We also know from, for example, Theorem 3, that some second-order ambiguity-averse MBA preferences do satisfy monotonicity in optimal mixtures.

Theorem 5 also directly relates to applications such as the sales agent model from Section 2. Given a sales agent with some MBA preferences (and, thus, having a representation $V((u(f(s)))_{s\in S}))$, Theorem 5, when combined with Theorem 1 from Section 5, tells us that violations of the slope bounds given in (17) applied to acts from the sales agent model generate nonmonotonicities in the agent's reaction to sales compensation (i.e., make the likelihood of the agent pursuing a given sales prospect sometimes decrease in the compensation for that prospect).

Our result is presented in terms of the geometry of preferences restricted to two-dimensional utility spaces. To state it, we first need to define some sets of acts generating such spaces.

**Definition 3.** For nonempty events $E, F \subseteq S$, $z \in X$ and act $h$, let $\mathcal{F}^{E,F,z,h}$ denote the set of acts $f$ such that $f(s) \sim f(t)$ for all $s, t \in E \setminus F$ and $f(s) \sim f(t)$ for all $s, t \in F \setminus E$ and $\frac{1}{2}f(s) + \frac{1}{2}z \sim \frac{1}{2}f(E \setminus F) + \frac{1}{2}f(F \setminus E)$ for all $s \in F \cap E$, and $f(s) \sim h(s)$ for all $s \in (E \cup F)^c$.

Motivated by our sales agent model, $\mathcal{F}^{E,F,z,h}$ may be viewed as generated from mixtures of acts that, restricted to $E \cup F$, either (i) give some arbitrary payoff if $E$ occurs and a default payoff, denoted by $z$, otherwise or (ii) give some arbitrary payoff if $F$ occurs and the same default payoff $z$ otherwise. Outside of $E \cup F$, outcomes are fixed according to some given act $h$. For example, in the sales agent model, $E$ is the event that sale 1 is successful if worked on, and $F$ is the event that sale 2 is successful if worked on, and both the default payoff and the payoff if neither sale is successful if worked on is zero. Formally, in that model, $E = \{s_1, s_3\}$, $F = \{s_2, s_3\}$, and $u(z) = 0 = u(h(s_4))$.

**Remark 4.** (Convexity of $\mathcal{F}^{E,F,z,h}$). Fixing $z, h$, observe that the two-dimensional utility space specifying $u(E \setminus F)$ and $u(F \setminus E)$ allows us to represent all utility profiles generated by acts in $\mathcal{F}^{E,F,z,h}$. In fact, the set of all such acts corresponds to the subset of this two-dimensional space such that $u(E \setminus F) + u(F \setminus E) - u(z) \in u(X)$. A convex combination in the two-dimensional space given by $u(E \setminus F)$ and $u(F \setminus E)$ corresponds to the convex combination of the two associated acts in $\mathcal{F}^{E,F,z,h}$ with the same weights.

**Remark 5** (Dominance in $\mathcal{F}^{E,F,z,h}$). If $f, g \in \mathcal{F}^{E,F,z,h}$ are such that $\forall s \in (E \setminus F) \cup (F \setminus E): f(s) \geq g(s)$, then the same is true for all $s \in S$.

We also need to define the associated better-than sets.

**Definition 4.** Let $\succsim$ be an MBA preference, thus represented by $V((u(f(s)))_{s\in S})$ as in (6). For any events $E, F \subseteq S$ such that $E \setminus F$ and $F \setminus E$ are nonempty, $z \in Z$, $h \in \mathcal{F}$, and any act $k \in \mathcal{F}^{E,F,z,h}$, define $G(k) \equiv \{(u(f(E \setminus F)), u(f(F \setminus E))) : f \in \mathcal{F}^{E,F,z,h} \text{ and } f \succsim k\}$.

Graphically, we can represent acts in $\mathcal{F}^{E,F,z,h}$ as a subset of the points in a two-dimensional Cartesian coordinate system with the vertical coordinate representing the utility level the act delivers in event $F \setminus E$ and the horizontal coordinate representing the utility level the act delivers in event $E \setminus F$. The definition of $\mathcal{F}^{E,F,z,h}$ implies that this subset of points is exactly $H(z) \equiv \{(a, b) \in u(X) \times u(X) : a + b - u(z) \in u(X)\}$. Monotonicity of $V$ and the definition of $\mathcal{F}^{E,F,z,h}$ imply that all points $(a, b) \in H(z)$ such that $a \geq u(h(E \setminus F))$ and $b \geq u(h(F \setminus E))$ lie in $G(h)$.

The statement of the theorem makes central use of the following definition, giving conditions for a rectangle and two support lines in two-dimensional utility space to form a *support configuration* (as illustrated in Figure 4).

**Definition 5.** Let $\succsim$ be represented by $V((u(f(s)))_{s\in S})$ as in (6). Fix any events $E, F \subseteq S$ such that $E \setminus F$ and $F \setminus E$ are nonempty, $z \in X$, and acts $h, k$ interior to $\mathcal{F}^{E,F,z,h}$ such that $k(E\setminus F) \succsim h(E\setminus F)$ and $h(F\setminus E) \succsim k(F\setminus E)$. Given a rectangle $R \subseteq H(z) \times H(z)$ containing $(u(h(E\setminus F)), u(h(F\setminus E)))$ and $(u(k(E\setminus F)), u(k(F\setminus E)))$ in its interior and two distinct lines $l_h$ and $l_k$ in $u(X)\times u(X)$, $(R, l_h, l_k)$ is a support configuration if

i. $l_h$ intersects $G(h)\cap R$ only at $(u(h(E\setminus F)), u(h(F\setminus E)))$ and has an intersection with the left edge of rectangle $R$, $\{(q(E\setminus F), q(F\setminus E)) \in R : q(E\setminus F) = \min_{r\in R} r(E\setminus F)\}$;

ii. $l_k$ intersects $G(k)\cap R$ only at $(u(k(E\setminus F)), u(k(F\setminus E)))$ and has an intersection with the bottom edge of rectangle $R$, $\{(q(E\setminus F), q(F\setminus E)) \in R : q(F\setminus E) = \min_{r\in R} r(F\setminus E)\}$; and

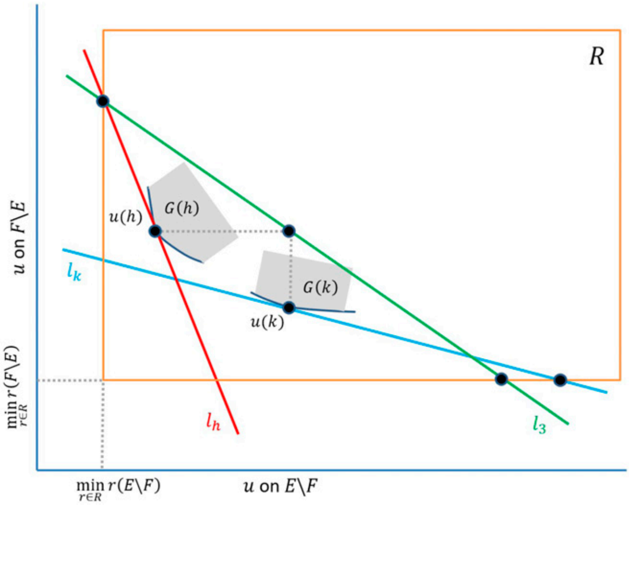
iii. the line $l_3$, passing through $(u(k(E\setminus F)), u(h(F\setminus E)))$ and the intersection of $l_h$ with the left edge of rectangle $R$, has an intersection with the bottom edge of rectangle $R$.

The definition requires that the rectangle $R$ is small enough so that it lies in $H(z)\times H(z)$ and so that the lines $l_h$ and $l_k$ intersect the upper sets $G(h)$ and $G(k)$ only at $u(h)$ and $u(k)$, respectively, and so that the lines $l_h$ ($l_k$ and $l_3$) intersect the left (bottom) side of $R$. The theorem gives a bound that monotonicity in optimal mixtures imposes on the ratio of the slopes of the lines $l_h$ and $l_k$ in any support configuration.

**Theorem 5.** *Let $\succsim$ be MBA preferences, thus represented by $V((u(f(s)))_{s\in S})$ as in (6).*

*Fix any events $E, F \subseteq S$ such that $E\setminus F$ and $F\setminus E$ are nonempty, $z \in X$, and acts $h, k$ interior to $\mathcal{F}^{E,F,z,h}$ such that $k(E\setminus F) \succsim h(E\setminus F)$ and $h(F\setminus E) \succsim k(F\setminus E)$.*

**Figure 4.** (Color online) Support Configuration




*If $\succsim$ satisfies monotonicity in optimal mixtures restricted to $\mathcal{F}^{E,F,z,h}$, then*

$$\frac{Slope(l_h)}{Slope(l_k)} \le \frac{u(k(E\setminus F)) - \min_{r\in R} r(E\setminus F)}{u(h(E\setminus F)) - \min_{r\in R} r(E\setminus F)} \times \frac{u(h(F\setminus E)) - \min_{r\in R} r(F\setminus E)}{u(k(F\setminus E)) - \min_{r\in R} r(F\setminus E)}, \tag{17}$$

*for all support configurations $(R, l_h, l_k)$.*

When $V$ is differentiable in the utility values, (17) may be written entirely in terms of $V$ and $u$ because

$$Slope(l_h) = -\frac{V_{E\setminus F}}{V_{F\setminus E}}\Big|_{u(h)} \tag{18}$$

and

$$Slope(l_k) = -\frac{V_{E\setminus F}}{V_{F\setminus E}}\Big|_{u(k)}, \tag{19}$$

where, for $A \subseteq S$, $V_A$ denotes the derivative of $V$ with respect to the utility attained on the event $A$.

It is worth noting that, if preferences satisfy Schmeidler's uncertainty aversion (and, thus, are convex in utility space, equivalently, $V$ is quasi-concave) when restricted to acts in $\mathcal{F}^{E,F,z,h}$ and $u(X) = [0,\infty)$, then support configurations always exist, and the only rectangles $R$ that need be considered are those with $\min_{r\in R} r(E\setminus F) + \min_{r\in R} r(F\setminus E) = u(z)$. When $u(z) = 0$, as in the sales agent model, this last condition yields $\min_{r\in R} r(E\setminus F) = \min_{r\in R} r(F\setminus E) = 0$, and thus, the right-hand side of the bound (17) is unchanged as utilities are multiplicatively scaled up. In this sense, monotonicity in optimal mixtures requires that such preferences not display too much relative ambiguity aversion anywhere. Note that the bound is satisfied with equality at all points for preferences that are Cobb–Douglas in utility space when restricted to $\mathcal{F}^{E,F,z,h}$ with $u(z) = 0$ (i.e., preferences representable on $\mathcal{F}^{E,F,z,h}$ by $u(f(E\setminus F))^{\gamma} u(f(F\setminus E))^{1-\gamma}$ or, equivalently, $\gamma \ln(u(f(E\setminus F))) + (1-\gamma)\ln(u(f(F\setminus E)))$ for some $\gamma \in (0,1)$). Thus, the bound can be viewed as saying that preferences (on these two-dimensional slices) must be everywhere at most as convex as Cobb–Douglas. This connection of monotonicity in optimal mixtures with Cobb–Douglas for uncertainty-averse preferences is a more general manifestation of the connection with natural log (and the corresponding connection with the gross substitutes property) we found for uncertainty-averse smooth ambiguity preferences in Section 6.2. In particular, with the $\mu$ used in the proof of Theorem 4, $\phi = \ln$ yields, for $u(z) = 0$, Cobb–Douglas restricted to $\mathcal{F}^{E,E^c,z,h}$, and the smooth ambiguity preferences restricted to $\mathcal{F}^{E,E^c,z,h}$ are at most as convex as Cobb–Douglas if and only if $\phi$ is at most as concave as natural log.

**Proof of Theorem 5.** Fix $V, u, E, F, z, h, k$ and a support configuration $(R, l_h, l_k)$ as in the statement of the theorem. Monotonicity of $V$ implies that, for any $f \in \mathcal{F}^{E,F,z,h}$, all points $(a, b) \in H(z) \times H(z)$ such that $a \geq u(f(E \setminus F))$ and $b \geq u(f(F \setminus E))$ lie in $G(f)$. Therefore, because $h$ and $k$ are interior and $l_h$ intersects $G(h) \cap R$ only at $(u(h(E \setminus F)), u(h(F \setminus E)))$ and $l_k$ intersects $G(k) \cap R$ only at $(u(k(E \setminus F)), u(k(F \setminus E)))$, it follows that both $l_h$ and $l_k$ must have negative and finite slopes as does $l_3$ by construction. Name the acts in $\mathcal{F}^{E,F,z,h}$ corresponding to the intersections of $l_h, l_k, l_3$ with the edges of $R$ as specified in (i), (ii), and (iii) of the definition of support configuration by $i, j$, and $m$, respectively (Figure 5).

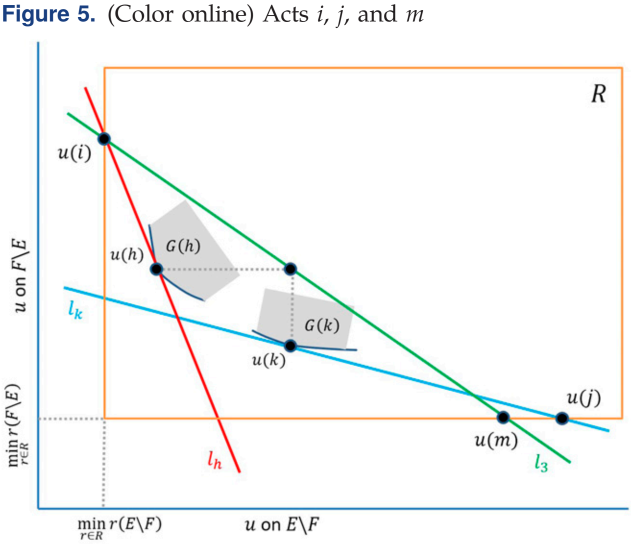
Observe that the slope of $l_3$ is the following:

$$\begin{aligned} Slope(l_3) &= -\frac{u(i(F \setminus E)) - u(h(F \setminus E))}{u(k(E \setminus F)) - u(i(E \setminus F))} \\ &= -\frac{u(i(F \setminus E)) - u(h(F \setminus E))}{u(h(E \setminus F)) - u(i(E \setminus F))} \\ &\quad \times \frac{u(h(E \setminus F)) - u(i(E \setminus F))}{u(k(E \setminus F)) - u(i(E \setminus F))} \\ &= Slope(l_h) \times \frac{u(h(E \setminus F)) - u(i(E \setminus F))}{u(k(E \setminus F)) - u(i(E \setminus F))}. \end{aligned} \tag{20}$$

Now, let $l_4$ be the line passing through $m$ and $k$. The slope of $l_4$ is (noting that $u(m(F \setminus E)) = u(j(F \setminus E))$)

$$\begin{aligned} Slope(l_4) &= -\frac{u(k(F \setminus E)) - u\big(j(F \setminus E)\big)}{u(m(E \setminus F)) - u(k(E \setminus F))} \\ &= -\frac{u(h(F \setminus E)) - u\big(j(F \setminus E)\big)}{u(m(E \setminus F)) - u(k(E \setminus F))} \\ &\quad \times \frac{u(k(F \setminus E)) - u\big(j(F \setminus E)\big)}{u(h(F \setminus E)) - u\big(j(F \setminus E)\big)} \\ &= Slope(l_3) \times \frac{u(k(F \setminus E)) - u\big(j(F \setminus E)\big)}{u(h(F \setminus E)) - u\big(j(F \setminus E)\big)}. \end{aligned} \tag{21}$$

**Figure 5.** (Color online) Acts $i$, $j$, and $m$




Combining (20) and (21) yields

$$\begin{aligned} Slope(l_4) = Slope(l_h) &\times \frac{u(h(E \setminus F)) - u(i(E \setminus F))}{u(k(E \setminus F)) - u(i(E \setminus F))} \\ &\times \frac{u(k(F \setminus E)) - u\big(j(F \setminus E)\big)}{u(h(F \setminus E)) - u\big(j(F \setminus E)\big)}. \end{aligned} \tag{22}$$

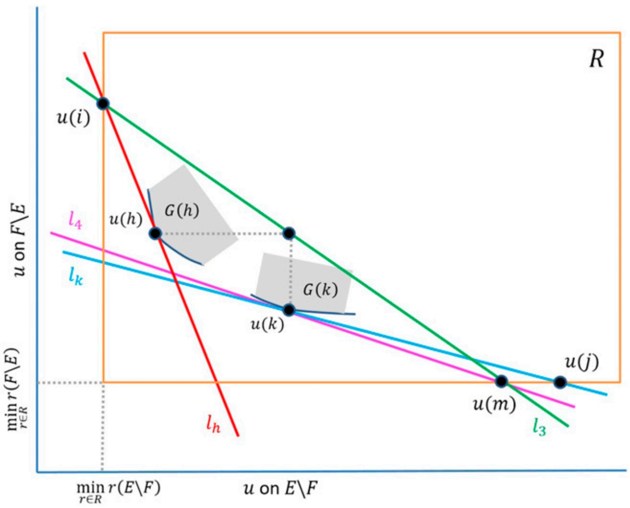
If $\succsim$ satisfies monotonicity in optimal mixtures restricted to $\mathcal{F}^{E,F,z,h}$, then, because by construction $h$ is the unique optimum on $l_h \cap R$, it must be that all optima on $l_3 \cap R$ have vertical coordinates weakly below $u(h(F \setminus E))$. Applying monotonicity in optimal mixtures restricted to $\mathcal{F}^{E,F,z,h}$ a second time yields that, because all optima on $l_3 \cap R$ have vertical coordinates weakly below $u(h(F \setminus E))$, all optima on $l_4 \cap R$ have vertical coordinates weakly below $u(k(F \setminus E))$. From this, we now show that, if $Slope(l_4) < Slope(l_k)$ (equivalently, $u(m)$ is to the left of u(j)), then monotonicity in optimal mixtures restricted to $\mathcal{F}^{E,F,z,h}$ must be violated (Figure 6). To see this, suppose to the contrary that $Slope(l_4) < Slope(l_k)$ and monotonicity in optimal mixtures restricted to $\mathcal{F}^{E,F,z,h}$ is satisfied. Because $k$ is uniquely optimal on $l_k \cap R$ (by the assumptions of the theorem), $V$ is monotonic and $Slope(l_4) < Slope(l_k)$, $k \succ f$ for all $(u(f(E \setminus F)), u(f(F \setminus E))) \in l_4 \cap R$ having vertical coordinates strictly below $u(k(F \setminus E))$ (as each such $(u(f(E \setminus F)), u(f(F \setminus E)))$ is weakly dominated by some point on $l_k \cap R$ that is strictly worse than $(u(k(E \setminus F)), u(k(F \setminus E)))$. Thus, because we earlier showed that monotonicity in optimal mixtures restricted to $\mathcal{F}^{E,F,z,h}$ implies that all optima on $l_4 \cap R$ have vertical coordinates weakly below $u(k(F \setminus E))$, it follows that $k$ is strictly optimal on $l_4 \cap R$. We next show that $k$ being strictly optimal on both $l_k \cap R$ and $l_4 \cap R$ with $Slope(l_4) < Slope(l_k)$ implies a violation of monotonicity in optimal mixtures restricted to $\mathcal{F}^{E,F,z,h}$, a contradiction. To this end, fix acts $f, g, f', g' \in \mathcal{F}^{E,F,z,h}$ such that $g = m$, $g' = j$, $(u(f(E \setminus F)), u(f(F \setminus E)))$ is the point at which $l_k$ intersects the left side of $R$, and $(u(f'(E \setminus F)), u(f'(F \setminus E)))$ is the point at which $l_4$ intersects the left side of $R$ (see Figure 6). Observe that the points in $l_4 \cap R$ correspond to the acts $\lambda f' + (1 - \lambda) g \in \mathcal{F}^{E,F,z,h}$ for $\lambda \in [0, 1]$, and the points in $l_k \cap R$ correspond to the acts $\lambda g' + (1 - \lambda) f \in \mathcal{F}^{E,F,z,h}$ for $\lambda \in [0, 1]$. Let $\lambda_1, \lambda_2 \in (0, 1)$ be the unique numbers such that $\lambda_1 f' + (1 - \lambda_1) g$ and $\lambda_2 g' + (1 - \lambda_2) f$ each correspond to $(u(k(E \setminus F)), u(k(F \setminus E)))$. It follows that $\lambda_1 u(f'(E \setminus F)) + (1 - \lambda_1) u(g(E \setminus F)) = \lambda_1 u(f(E \setminus F)) + (1 - \lambda_1) u(g(E \setminus F)) = \lambda_2 u(g'(E \setminus F)) + (1 - \lambda_2) u(f(E \setminus F))$, and thus,

$$\lambda_2 < 1 - \lambda_1. \tag{23}$$

Because $k$ is strictly optimal on both $l_k \cap R$ and $l_4 \cap R$,

$$\lambda_1 f' + (1 - \lambda_1) g \succ \lambda f' + (1 - \lambda) g \text{ for all } \lambda \neq \lambda_1 \tag{24}$$

**Figure 6.** (Color online) Violation of the Slope Bound (17)




and

$$\lambda_2 g' + (1-\lambda_2)f \succ \lambda g' + (1-\lambda)f \text{ for all } \lambda \neq \lambda_2. \quad (25)$$

Because $f'$ weakly dominates $f$, given (24), monotonicity in optimal mixtures restricted to $\mathcal{F}^{E,F,z,h}$ implies

$$\arg\max_{\lambda\in[0,1]} V\big(u\big(\lambda f + (1-\lambda)g\big)\big) \subseteq [0,\lambda_1]. \quad (26)$$

Because $g'$ weakly dominates $g$, given (25), monotonicity in optimal mixtures restricted to $\mathcal{F}^{E,F,z,h}$ implies

$$\arg\max_{\lambda\in[0,1]} V\big(u\big(\lambda f + (1-\lambda)g\big)\big) \subseteq [1-\lambda_2,1]. \quad (27)$$

By (23), $[0,\lambda_1]\cap[1-\lambda_2,1]=\emptyset$, a contradiction. Therefore, monotonicity in optimal mixtures restricted to $\mathcal{F}^{E,F,z,h}$ must be violated (and the violation occurs when applying the definition to either the acts $f',f,g \in \mathcal{F}^{E,F,z,h}$ or the acts $g',g,f \in \mathcal{F}^{E,F,z,h}$).

Thus, we have shown that monotonicity in optimal mixtures restricted to $\mathcal{F}^{E,F,z,h}$ implies $Slope(l_4) \geq Slope(l_k)$, which, because both are negative, is equivalent to

$$\frac{Slope(l_4)}{Slope(l_k)} \leq 1. \quad (28)$$

Applying (22) yields

$$\frac{Slope(l_4)}{Slope(l_k)} = \frac{Slope(l_h)}{Slope(l_k)} \times \frac{u(h(E\setminus F)) - u(i(E\setminus F))}{u(k(E\setminus F)) - u(i(E\setminus F))} \times \frac{u(k(F\setminus E)) - u\big(j(F\setminus E)\big)}{u(h(F\setminus E)) - u\big(j(F\setminus E)\big)}.$$

Therefore, monotonicity in optimal mixtures restricted to $\mathcal{F}^{E,F,z,h}$ implies

$$\frac{Slope(l_h)}{Slope(l_k)} \leq \frac{u(k(E\setminus F)) - u(i(E\setminus F))}{u(h(E\setminus F)) - u(i(E\setminus F))} \times \frac{u(h(F\setminus E)) - u\big(j(F\setminus E)\big)}{u(k(F\setminus E)) - u\big(j(F\setminus E)\big)}. \quad (29)$$

Because $u(i(E\setminus F)) = \min_{r\in R} r(E\setminus F)$ and $u(j(F\setminus E)) = \min_{r\in R} r(F\setminus E)$, (29) is the inequality (17). □

## 7. Conclusion

Our results demonstrate that moderately ambiguity-averse behavior is compatible with the monotone comparative static captured by monotonicity in optimal mixtures while also showing that nonmonotone reactions are a necessary feature when sufficiently strong ambiguity aversion is present. These are novel insights into the implications of models of ambiguity-sensitive behavior that are useful to keep in mind in managerial and other applications. They also suggest that investigating the prevalence and/or role of violations of monotonicity in optimal mixtures may be a fruitful avenue for future experimental or empirical analyses.

### Acknowledgments

The authors thank Gavriel Hirsch for excellent research assistance. They thank the referees and editors for helpful suggestions that improved the paper and Nemanja Antic, Sarah Auster, Rose-Anne Dana, Itzhak Gilboa, Mark Machina, Sujoy Mukerji, Chris Shannon, Marciano Siniscalchi, Jean-Marc Tallon, and several seminar audiences for helpful comments. Older versions of this paper were circulated under the title "Ellsberg Behavior: A Monotonicity Consideration and Its Implications."

## Appendix. Axioms for MBA preferences

To aid the reader in understanding the exact scope of MBA preferences, we recall the result from Cerreia-Vioglio et al. (2011) that the following four axioms characterize the MBA preference representation (6):

**Axiom A.1.** (Weak Order). $\succsim$ *are nontrivial, complete, and transitive.*

**Axiom A.2.** (State-by-State Monotonicity). *For all acts $f,g$, if $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.*

**Axiom A.3.** (Risk Independence). *For all lotteries $x,y,z \in X$ and $\alpha \in (0,1]$, if $x \succ y$, then $\alpha x + (1-\alpha)z \succ \alpha y + (1-\alpha)z$.*

**Axiom A.4.** (Archimedean Continuity). *For all acts $f,g,h$, if $f \succ g \succ h$, then there exist $\alpha,\beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h$.*

As previously mentioned, these axioms are quite weak, lending our theory a broad scope. The key limitation is Risk Independence, which implies expected utility treatment of "objective" risk (i.e., lotteries).

## Endnotes

[1] If $\underline{r} = \overline{r}$ and both of these equalities hold, then $q^*(w(v_1), w(v_2)) = [0, 1]$.

[2] If $(r(1-\delta)+\delta)u(w(v_1)) = ((1-r)(1-\delta)+\delta)u(w(v_2))$ for all $\delta, r \in [\underline{\delta}, \overline{\delta}] \times [\underline{r}, \overline{r}]$, which can happen only if the intervals for the parameters are degenerate or $u(w(v_1)) = u(w(v_2)) = 0$, then the agent is indifferent among all $q$, and $q^*(w(v_1), w(v_2)) = [0, 1]$.

[3] As shown in Theorem 3 in Section 6.2, this bound on ambiguity aversion in the smooth ambiguity model is sufficient to guarantee that monotonicity in optimal mixtures is satisfied not only in this example, but generally.

[4] Speaking in representational terms, it is uncontroversial that ambiguity-averse preferences can strictly value (at least some types of) smoothing of utility across states. There is some debate, however, about whether and when randomization by an individual over acts is evaluated as a state-by-state mixture of the utility profiles of the acts involved (see Ke and Zhang 2020 for discussion, references, and theory on this issue). The theory in this paper assumes that acts generating such utility mixtures are available for the individual to choose by some means. Whether this is via randomization or through other options available to the decision maker is not important for our results.

[5] See the appendix for a statement of the four axioms that they show characterize these preferences.

[6] With the convention that we label the acts so that $f(E) \succsim g(E)$, this describes all pairs of $\{E, E^c\}$-measurable acts that are not ordered by weak dominance and are not comonotonic. Given the form of preferences, these are the only candidates for pairs of $\{E, E^c\}$-measurable acts for which an interior mixture could be strictly optimal (see, e.g., Klibanoff 2001).

[7] Similarly, when the slope is not strictly between these bounds, at least one of the degenerate mixtures $\alpha = 0$ or $\alpha = 1$ is an optimum.

[8] Although it is easy to see the connection between $\max\{\rho(E), \rho(E^c)\} < 1$ and each event mattering, the role of $\min\{\rho(E), \rho(E^c)\} = 0$ is less immediate. Its role is to admit c-linearly biseparable preferences that, when restricted to $\{E, E^c\}$ measurable acts, are SEU with either $E$ or $E^c$ being given all weight. For example, when $E = S$ or $E = \emptyset$, preferences are always of this form. Such preferences both satisfy monotonicity in optimal mixtures restricted to such acts and have $\rho(E) + \rho(E^c) \geq 1$, thus satisfying the converse direction of Theorem 2.

[9] According to Jewitt and Mukerji (2017), no event is more ambiguous.

[10] Although some variational preferences have kinks, others do not. Furthermore, variational preferences need not be either c-linearly biseparable or smooth ambiguity preferences, so this finding does not follow from our earlier results.

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