Relating preference symmetry axioms

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Abstract

We show equivalence among various symmetry axioms in the literature.

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1 Setting and Notation

We adopt the setting and notation from Klibanoff, Mukerji and Seo [13]:

Let $S$ be a compact metric space and $\Omega = S^\infty$ the state space with generic element

$\omega = (\omega_1, \omega_2, \ldots)$. The state space $\Omega$ is also compact metric (Aliprantis and Border [1, Theorems 2.61 and 3.36]). Denote by $\Sigma_i$ the Borel $\sigma$-algebra on the $i$-th copy of $S$, and by $\Sigma$ the product $\sigma$-algebra on $S^\infty$. An act is a simple Anscombe-Aumann act, a measurable $f : S^\infty \to X$ having finite range (i.e., $f(S^\infty)$ is finite) where $X$ is the set of lotteries (i.e., finite support probability measures on an outcome space $Z$). The set of acts is denoted by $F$, and $\succeq$ is a binary relation on $F \times F$. As usual, we identify a constant act (an act yielding the same element of $X$ on all of $S^\infty$) with the element of $X$ it yields.

Denote by $\Pi$ the set of all finite permutations on $\{1,2,...\}$ i.e., all one-to-one and onto functions $\pi : \{1,2,...\} \to \{1,2,...\}$ such that $\pi(i) = i$ for all but finitely many $i \in \{1,2,...\}$. For $\pi \in \Pi$, let $\pi \omega = \left(\omega_{\pi(1)}, \omega_{\pi(2)}, \ldots\right)$ and $(\pi f)(\omega) = f(\pi \omega)$.

For any topological space $Y$, $\Delta(Y)$ denotes the set of (countably additive) Borel probability measures on $Y$. Unless stated otherwise, a measure is understood as a countably additive Borel measure. For later use, $ba(Y)$ is the set of finitely additive bounded real-valued set functions on $Y$, and $ba_+^1(Y)$ the set of nonnegative probability

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charges in $ba(Y)$. A measure $p \in \Delta(S^\infty)$ is called symmetric if the order doesn’t matter, i.e., $p(A) = p(\pi A)$ for all $\pi \in \Pi$, where $\pi A = \{\pi \omega : \omega \in A\}$. Denote by $\ell^\infty$ the i.i.d. measure with the marginal $\ell \in \Delta(S)$. Define $\int_{S^\infty} f dp \in X$ by $(\int_{S^\infty} f dp) (B) = (\int_{S^\infty} f (\omega) (B) dp (\omega))$. (Since $f$ is simple, this is well-defined.)

Fix $x_*, x^* \in X$ such that $x^* \succ x_*$. For any event $A \in \Sigma$, $1_A$ denotes the act giving $x^*$ on $A$ and $x_*$ otherwise. Informally, this is a bet on $A$. A finite cylinder event $A \in \Sigma$ is any event of the form $\{\omega : \omega_i \in A_i \text{ for } i = 1, \ldots, n\}$ for $A_i \in \Sigma_i$ and some finite $n$.

Endow $\Delta(S)$, $\Delta(\Delta(S))$ and $\Delta(S^\infty)$ with the relative weak* topology. To see what this is, consider, for example, $\Delta (\Delta(S))$.

The support of a probability measure $m \in \Delta(\Delta(S))$, denoted $\text{supp}(m)$, is a relative weak* closed set such that $m ((\text{supp}(m))^c) = 0$ and if $G \cap \text{supp}(m) \neq \emptyset$ for relative weak* open $G$, $m (G \cap \text{supp}(m)) > 0$. (See e.g., Aliprantis and Border [1, p.441].)

**Definition 1.1.** Let $\Psi_n (\omega) \in \Delta(S)$ denote the empirical frequency operator $\Psi_n (\omega) (A) = \frac{1}{n} \sum_{i=1}^n I (\omega_i \in A)$ for each event $A$ in $S$. Define the limiting frequency operator $\Psi$ by $\Psi (\omega) (A) = \lim_n \Psi_n (\omega) (A)$ if the limit exists and 0 otherwise. Also, to map given limiting frequencies or sets of limiting frequencies to events in $S^\infty$, we consider the natural inverses $\Psi^{-1} (\ell) = \{\omega : \Psi (\omega) = \ell\}$ and $\Psi^{-1} (L) = \{\omega : \Psi (\omega) \in L\}$ for $\ell \in \Delta(S)$ and $L \subseteq \Delta(S)$.

## 2 Relating Event Symmetry to the literature

We introduce conditions that will be assumed in our result. First, the following 5 axioms are standard in the literature.

**Axiom 1** (C-completeness). The restriction of $\succsim$ to $X$ is complete.

**Axiom 2** (Preorder). $\succsim$ is reflexive, transitive and the restriction of $\succsim$ to $X$ is complete.

Preorder allows $\succsim$ to be incomplete.

**Axiom 3** (Monotonicity). If $f (\omega) \succsim g (\omega)$ for all $\omega \in S^\infty$, $f \succsim g$.

**Axiom 4** (Risk Independence). For all $x, x', x'' \in X$ and $\alpha \in (0, 1)$, $x \succsim x'$ if and only if $\alpha x + (1 - \alpha) x'' \succsim \alpha x' + (1 - \alpha) x''$.

**Axiom 5** (Non-triviality). There exist $x, y \in X$ such that $x \succ y$. 

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To describe our remaining axioms, define the binary relation \( \succ^* \) derived from \( \succ \):

\[
f \succ^* g \text{ if } \alpha f + (1 - \alpha) h \succ \alpha g + (1 - \alpha) h \text{ for all } \alpha \in [0, 1] \text{ and } h \in \mathcal{F}.
\]

See Ghirardato, Maccheroni and Marinacci [10], Nehring ([14], [15], [16]).


**Axiom 6 (Event Symmetry).** For all finite cylinder events \( A \in \Sigma \) and finite permutations \( \pi \in \Pi \), \( A \sim^{*} \pi A \).

Some continuity axioms follow.\(^2\)

**Axiom 7 (Mixture Continuity of \( \succ \)).** For all \( f, g, h \in \mathcal{F} \), the sets \( \{ \lambda \in [0, 1] : \lambda f + (1 - \lambda) g \succ h \} \) and \( \{ \lambda \in [0, 1] : h \succ \lambda f + (1 - \lambda) g \} \) are closed in \([0, 1]\).

The following axiom is a counterpart of countably additive measures to preferences. See Arrow [2] and Ghirardato, Maccheroni and Marinacci [10].

**Axiom 8 (Monotone Continuity of \( \succ^* \)).** For all \( x, x', x'' \in X \), if \( A_n \downarrow \emptyset \) and \( x' \succ x'' \), then \( x' \succ^* x A_n x'' \) for some \( n \).

We now show that Event Symmetry relates quite closely to a variety of other conditions from the literature, including strengthenings of de Finetti [6]'s Exchangeability, Hewitt and Savage [11]'s Symmetry, of Seo [17]'s Dominance and of Klibanoff, Marinacci and Mukerji [12]'s Consistency. One of those conditions (condition (viii) below) requires some additional definitions.

**Definition 2.1.** For \( f \in \mathcal{F} \), \( f^\Psi \) is the (not necessarily simple) act uniquely defined as follows:

\[
f^\Psi (\omega) = \left\{ \begin{array}{ll} \int_S \alpha f d\ell^\infty \delta_x^* & \text{if } \ell = \Psi (\omega) \in \Delta (S) \\ \{ \omega : \Psi (\omega) \text{ is not defined} \} & \end{array} \right.
\]

Note this definition associates with each act \( f \) an act \( f^\Psi \) that, for each event \( \{ \omega : \Psi (\omega) = \ell \} \) corresponding to the limiting frequencies generated by \( \ell \), yields the lottery generated by \( f \) under the assumption that the i.i.d. process \( \ell^\infty \) governs the realization of the state.

Since \( f^\Psi \) need not be simple, but is an element of the space \( \hat{\mathcal{F}} \) of all bounded and measurable functions from \( \Omega \) to \( X \),\(^3\) it is necessary to consider extending \( \succ \) to \( \hat{\mathcal{F}} \). In particular, we consider extensions continuous in the following sense: \( \succ \) on \( \hat{\mathcal{F}} \) satisfies **Norm Continuity** if \( f \hat{\succ} g \) whenever \( f_k \hat{\succ} g_k \) for all \( k = 1, 2, \ldots \) and \( f_k \) and \( g_k \) norm-converge to \( f \) and \( g \) respectively.

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1. They state that Klaus Nehring is the first one to suggest using \( \succ^* \) of a given \( \succ \), in a 1996 talk
2. Mixture Continuity of \( \succ \) implies Mixture Continuity of \( \succ^* \).
3. More precisely, \( \hat{\mathcal{F}} \) is the collection of functions \( f : \Omega \to X \) that satisfy the following two properties:
   (i) for all \( x \in X \), \( \{ \omega : f(\omega) \succ x \} \in \Sigma \); and
   (ii) there exist \( x, y \in X \) such that \( x \hat{\succ} f(\omega) \hat{\succ} y \) for all \( \omega \in \Omega \).
Theorem 2.1. The following conditions are equivalent under Preorder and Mixture Continuity of $\succsim$:

(i) for every $f \in \mathcal{F}$ and $\pi \in \Pi$, $f \sim \frac{1}{2} f + \frac{1}{2} \pi f$,
(ii) for every $f \in \mathcal{F}$, $\pi \in \Pi$ and $\alpha \in [0, 1]$, $f \sim \alpha \pi f + (1 - \alpha) f$,
(iii) for every $f \in \mathcal{F}$ and $\pi_i \in \Pi$, $f \sim \frac{1}{n} \sum_{i=1}^{n} \pi_i f$,
(iv) for every $f \in \mathcal{F}$, $\pi_i \in \Pi$ and $\alpha_i \in [0, 1]$ with $\sum_{i=1}^{n} \alpha_i = 1$, $f \sim \sum_{i=1}^{n} \alpha_i \pi_i f$, and
(v) for every $f \in \mathcal{F}$ and $\pi \in \Pi$, $f \sim^* \pi f$.

Moreover, the above are equivalent to each of the following under C-completeness, Preorder, Mixture Continuity of $\succsim$, Monotonicity, Risk Independence, Non-triviality and Monotone Continuity of $\succsim^*$:

(vi) Event Symmetry,
(vii) for every $f, g \in \mathcal{F}$, if $\int f dp \succsim \int g dp$ for all symmetric $p \in \Delta (S^\infty)$, then $f \succsim g$.

Finally, if, in addition, there exists an extension $\succsim_*$ of $\succsim$ to $\hat{\mathcal{F}}$ satisfying Preorder and Norm Continuity, then the following is equivalent to all of the above:

(viii) for $f, g \in \mathcal{F}$, $f \succsim g$ if and only if $f^{\Psi} \succsim_\Psi g^{\Psi}$.

All of these conditions are strengthenings of Hewitt and Savage [11]’s symmetry: given $p \in \Delta (S^\infty)$, $p(A) = p(\pi A)$ for all finite cylinder events $A \in \Sigma$. In terms of preference, this translates into $1_A \sim 1_{\pi A}$ for all such events and all permutations $\pi \in \Pi$. Event Symmetry strengthens this by requiring the indifference to be preserved under mixture with any common third act. Under Preorder and Mixture Continuity of $\succsim$, conditions (i)-(v) each imply Event Symmetry.

Condition (ii) is closely related to Epstein and Seo [7]’s Strong Exchangeability, the first behavioral axiom in the literature that captures the idea that the agent views all experiments as identical, i.e., i.i.d. (See Epstein and Seo [7] for a behavioral interpretation of condition (ii). A similar interpretation applies to (i), (iii) and (iv).) Their axiom states that condition (ii) holds when $f$ depends only on a finite number of experiments. However, under their regularity axiom, their Strong Exchangeability extends to every act $f$ and hence is equivalent to condition (ii).

Epstein and Seo [7], under MEU, characterize two models of preferences where the decision maker is indifferent to permutations – one that restricts to identical experiments (no ambiguity on idiosyncratic factors) and permits ambiguity about parameters, and the other that allows ambiguity on idiosyncratic factors but rules out ambiguity about parameters. Only the former of the two models satisfies (ii). Epstein and Seo [9] go further and characterize a single model within MEU that reflects ambiguity on both at the same time. The unifying framework permits a behavioral distinction between idiosyncratic factors and parameters as two sources of ambiguity.

Condition (i) is a special case of condition (ii) when $\alpha = \frac{1}{2}$. Since Epstein and Seo [7] consider MEU models, they could restrict to the case $\alpha = \frac{1}{2}$. The above theorem shows that $\alpha = \frac{1}{2}$ is sufficient to capture the same idea in general as long as Preorder and Mixture Continuity of $\succsim$ hold.

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4Epstein and Seo [8] assume $f \sim \pi f$ for all permutations $\pi$ and acts $f$. They show that there is a modeling trade-off between this symmetry axiom, dynamic consistency and ambiguity.
De Castro and Al-Najjar ([3], [4]) use condition (iii) and its generalization to collections of transformations $\Gamma$ other than the finite permutations $\Pi$. They provide conditions on $\Gamma$ under which complete, transitive, monotonic, continuous and risk independent preferences satisfying (iii) with respect to $\Gamma$ are such that each act is indifferent to an associated act based on limiting frequencies (where the notion of limiting frequencies uses the given $\Gamma$) where the associated act is constructed much like $f^\Psi$ above with $\Gamma$-ergodic measures replacing the i.i.d. measures as parameters. They also prove a representation theorem for the utility function of the form

$$U(f) = \int_{\Delta(S)} \phi \left( \int_{S^\infty} u(f) \, d\ell^\infty \right) \, d\mu(\ell),$$

using an expected utility assumption on parameter-based acts added to the conditions on preferences and on $\Gamma$ mentioned in the previous sentence.

Condition (iv) strengthens condition (iii).

Condition (v) is stronger than Event Symmetry in that $f$ is not necessarily a binary act.

Condition (vii) is analogous to Seo [17]'s Dominance and Cerreia-Vioglio et. al.[5]'s Consistency, and condition (viii) to Klibanoff, Marinacci and Mukerji [12]'s Consistency. Seo’s Dominance is stated with lotteries over acts, objects that are not available in the domain of this paper, and condition (vii) restricts $p \in \Delta(S^\infty)$ to be symmetric while $p$ is unrestricted in Seo’s Dominance. Thus, the difference between the two is that Seo’s Dominance requires $f$ to induce a better lottery than $g$ under all processes, not just symmetric ones. The reason for the additional restriction here is to reflect the fact that the experiments are symmetric. We want to include as reflecting dominance, for example, the following case:

$$f(\omega) = x^* H_1 x^* \text{ and } g(\omega) = \left( \frac{1}{2} x^* + \frac{1}{2} x^* \right) H_2 x^*$$

where $H_i = \{ \omega \in S^\infty : \omega_i = H \}, S = \{ H, T \}$. The act $xAy$ gives $x$ on an event $A$ and $y$ otherwise. The act $f$ is a bet that the first coin comes up heads, and the act $g$ is a bet that the second coin comes up heads but with a less valuable reward for winning. Under symmetry of the experiments, it is intuitively clear that $f$ is better than $g$. Condition (vii) indeed implies that $f \succeq g$, while Seo’s Dominance would not – for example, when $p = \delta_{THTTT\ldots}, \int gdp \succ \int fdp$.

Cerreia-Vioglio et. al. [5]'s Consistency is similar to condition (vii), but instead of considering transformations (e.g. permutations) and/or all symmetric measures, they assume a family of objectively rational probability measures on a general state space and require $\int fdp \succeq \int gdp$ for all measures $p$ in that family. Taking the family to be all symmetric measures makes Cerreia-Vioglio et. al.’s Consistency exactly condition (vii). With their Consistency they provide representation theorems for Bewley preference, Choquet expected utility, variational preferences, uncertainty averse preferences and, as was mentioned earlier, the smooth ambiguity model. Also, they prove that the generalization of condition (v) to other collections of transformations $\Gamma$ together with
some (mild) axioms imply their Consistency when the family of objectively rational measures is the \( \Gamma \)-invariant measures.

The content of condition (viii) is that given the “knowledge” that everything is driven by some (as yet unknown) i.i.d. process, it seems reasonable that when evaluating an act, the individual would ultimately care only about the induced mapping from the space of i.i.d. processes to the lotteries generated under each process. Klibanoff, Marinacci and Mukerji [12]’s Consistency assumption says that when evaluating an act, an individual cares only about the induced mapping from probability measures on the state space to lotteries. Their assumption was stated in terms of acts and “second order acts” (maps from probability measures on the state space to outcomes). The latter objects do not appear as such in the present paper, but their role is played by the subset of acts measurable with respect to limiting frequency events, the acts \( f^{\Psi} \in \hat{F} \).

The identification (in terms of preference) of \( f \) with \( f^{\Psi} \) stated in condition (viii) is analogous to the identification of \( f \) with an “associated second order act” in Klibanoff, Marinacci and Mukerji [12]’s Consistency with the qualification that \( f^{\Psi} \) only induces the same mapping from probability measures to lotteries as \( f \) for i.i.d. probability measures. Thus, condition (viii) strengthens Klibanoff, Marinacci and Mukerji [12]’s Consistency to incorporate the known symmetry of the ordinates in the same way that condition (vii) strengthened Seo [17]’s Dominance. Theorem 2.1 says that, under our other axioms, each of these strengthenings is equivalent to Event Symmetry. One can view the equivalence of condition (iii) and condition (viii) as following from the i.i.d. case of the sufficient statistic result in de Castro and Al-Najjar [4].

3 Proof of Theorem 2.1

i)\( \Rightarrow \)v): Let \( f, h \in F \) and \( \pi \in \Pi \). Since \( \pi \) is a finite permutation, \( \pi^N \) is the identity for some \( N \). Thus, for any \( g \in F \),

\[ g^*(\omega) \equiv \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \pi_i^{-1} g(\omega) = \frac{1}{N} \sum_{i=1}^{N} \pi_i^{-1} g(\omega) \]

is well defined everywhere.

Step 1. \( g \sim g^* \).

By repeatedly applying i),

\[ g \sim \frac{1}{2} g + \frac{1}{2} \pi g \sim \frac{1}{2} \left( \frac{1}{2} g + \frac{1}{2} \pi g \right) + \frac{1}{2} \pi^2 \left( \frac{1}{2} g + \frac{1}{2} \pi g \right) \]

\[ = \frac{1}{2^2} \sum_{i=1}^{2^2} \pi_i^{-1} g \sim \frac{1}{2^k} \sum_{i=1}^{2^k} \pi_i^{-1} g, \]

for all \( k \).
For any positive integer \( n \), let \( q(n) \) be the quotient when we divide \( n \) by \( N \), and \( r(n) \) the remainder. (That is, \( n = q(n) \cdot N + r(n) \).) Then,

\[
\frac{1}{2^k} \sum_{i=1}^{2^k} \pi_i g = \frac{q(2^k)}{2^k} \sum_{i=1}^{N} \pi_i g + \frac{1}{2^k} \sum_{i=1}^{r(2^k)} \pi_i g
\]

\[
= \frac{q(2^k)}{2^k} N g^* + \frac{1}{2^k} \sum_{i=1}^{r(2^k)} \pi_i g.
\]

(Here \( \sum_{i=1}^0 a_i \) is understood to be 0.) Since \( r(2^k) \) can take at most finite number of integers, we can find a subsequence \( 2^{j(k)} \) of \( 2^k \) such that there is \( \bar{K} = r(2^{j(k)}) \) for all \( k \). Thus,

\[
g \sim \alpha_k g^* + (1 - \alpha_k) \left( \frac{1}{\bar{K}} \sum_{i=1}^{\bar{K}} \pi_i g \right) \text{ for all } k,
\]

where \( \alpha_k = \frac{q(2^{j(k)}) N}{2^k} \). (Note that \( 1 - \alpha_k = \frac{\bar{K}}{2^{j(k)}} \).) Since \( \alpha_k \to 1 \), Mixture Continuity implies \( g \sim g^* \).

Step 2. \( \alpha f + (1 - \alpha) h \sim \alpha \pi f + (1 - \alpha) h \).

By Step 1,

\[
\alpha f + (1 - \alpha) h \sim (\alpha f + (1 - \alpha) h)^* = \alpha f^* + (1 - \alpha) h^*.
\]

Similarly,

\[
\alpha \pi f + (1 - \alpha) h \sim \alpha (\pi f)^* + (1 - \alpha) h^* = \alpha f^* + (1 - \alpha) h^*.
\]

By Preorder, this step is proved.

v)\( \Rightarrow \)iv): By v),

\[
\alpha f + (1 - \alpha) h \sim \alpha \pi f + (1 - \alpha) h \tag{3.1}
\]

for all \( \alpha \in [0, 1] \) and \( h \in \mathcal{F} \). Note that \( f \sim \pi f \) when \( \alpha = 1 \).

Setting \( h = f \) gives,

\[
f \sim \alpha \pi f + (1 - \alpha) f \text{ for all } \alpha \in [0, 1]. \tag{3.2}
\]

Moreover, setting \( h = \beta f + (1 - \beta) \pi' f \) leads to

\[
f \sim (\alpha + (1 - \alpha) \beta) f + (1 - \alpha) (1 - \beta) \pi' f \quad \text{(by (3.2))}
\]

\[
= \alpha f + (1 - \alpha) (\beta f + (1 - \beta) \pi' f)
\]

\[
\sim \alpha \pi f + (1 - \alpha) (\beta f + (1 - \beta) \pi' f) \quad \text{(by (3.1))}
\]

\[
= (1 - \alpha) \beta f + \alpha \pi f + (1 - \alpha) (1 - \beta) \pi' f.
\]
Since $\alpha$ and $\beta$ can be taken arbitrarily, $f \sim \alpha f + \beta \pi f + (1 - \alpha - \beta) \pi' f$ for all $\alpha, \beta \in [0, 1]$. Since $f \sim \pi'' f$ for $\pi'' \in \Pi$,

$$f \sim \pi'' f \sim \alpha \pi'' f + \beta \pi'' f + (1 - \alpha - \beta) \pi'' f.$$  

For any $\pi_1, \pi_2, \pi_3 \in \Pi$, take $\pi = \pi_2 (\pi_1)^{-1}$, $\pi' = \pi_3 (\pi_1)^{-1}$ and $\pi'' = \pi_1$. Then, we get iv) for $n = 3$. Repeat the argument to show iv) for any $n$.

Clearly, iv) implies ii) and iii). Moreover, ii) implies i) and so does iii). Thus, we have shown that i)-v) are equivalent.

Assume the additional axioms to show equivalence of i)-vii).

v) $\iff$ vi): Clearly, v) implies vi). By Lemma 3.1 of Klibanoff, Mukerji and Seo [13], the converse holds.

vi) $\iff$ vii): vi) implies vii) by Lemma 3.1 of Klibanoff, Mukerji and Seo [13]. The converse holds since

$$\int (\lambda f + (1 - \lambda) h) \, dp = \int (\lambda \pi f + (1 - \lambda) h) \, dp \text{ for all symmetric } p \in \Delta(S^{\infty})$$

Assume the existence of an extension $\succsim$ satisfying Preorder and Norm Continuity to show the equivalence of i)-viii).

vi) $\iff$ viii): The easy direction is viii) implies vi): Note that

$$(\lambda f + (1 - \lambda) h)^\Psi = (\lambda \pi f + (1 - \lambda) h)^\Psi$$

for all $f, h \in \mathcal{F}$ and $\lambda \in [0, 1]$. Thus, viii) implies $\lambda f + (1 - \lambda) h \sim \lambda \pi f + (1 - \lambda) h$ for all $f, h \in \mathcal{F}$ and $\lambda \in [0, 1]$, which implies Event Symmetry.

We now turn to the other direction. Let $\succsim$ be any extension of $\succsim$ to $\hat{\mathcal{F}}$ satisfying Preorder and Norm Continuity. Define $\succsim^*$ on $\hat{\mathcal{F}}$ by

$$f \succsim^* g \text{ if } \alpha f + (1 - \alpha) h \succsim \alpha g + (1 - \alpha) h \text{ for all } \alpha \in [0, 1] \text{ and } h \in \hat{\mathcal{F}}.$$

(3.3)

Note that $\succsim$, $\succsim^*$ and $\succsim^*$ on $X$ all agree and can be represented by some non-constant mixture linear function $u$ on $X$.

Our argument proceeds by steps along the following lines: In steps 1-3, given an arbitrary $h' \in \mathcal{F}$, we construct sequences of simple acts that norm-converge to $h'$ from above and from below. In steps 4-7, we show that $\succsim^*$ is the unique extension of $\succsim^*$ satisfying C-complete Preorder, Norm Continuity, Monotonicity, Independence and Non-triviality. In step 8, we use the representation of $\succsim^*$ in Lemma 3.1 of Klibanoff, Mukerji and Seo [13] to construct a representation for $\succsim^*$, and use that representation to show vi) implies viii).

Step 1. For any $h' \in \mathcal{F}$, and $x, y \in X$ with $x \succ y$, there is $h \in \mathcal{F}$ such that $\frac{1}{2} h(\omega) + \frac{1}{2} y < \frac{1}{2} h'(\omega) + \frac{1}{2} x$ and $h(\omega) \succsim h'(\omega)$ for all $\omega \in \Omega$: Take $..., x_{-1}, x_0, x_1, ... \in X$ such that $x_0 = x$, $x_{-1} = y$ and $u(x_i) - u(x_{i-1}) = u(x) - u(y)$. For each $\omega \in \Omega$, set $h(\omega) = x_i$.
if \( x_{i-1} \prec h'(\omega) \preceq x_i \). Notice that \( h \in \mathcal{F} \) since \( h' \) is bounded above and below and \( u(x) - u(y) > 0 \). Then \( h \) does the job since \( h(\omega) \preceq h'(\omega) \) by construction and
\[
\frac{1}{2}u(h(\omega)) + \frac{1}{2}u(y) = \frac{1}{2}u(h(\omega)) - \frac{1}{2}u(x_i) + \frac{1}{2}u(x_{i-1}) + \frac{1}{2}u(x) < \frac{1}{2}u(h'(\omega)) + \frac{1}{2}u(x).
\]

Step 2. For any \( h' \in \hat{\mathcal{F}} \), there is \( h_k \in \mathcal{F} \) such that \( h_k \) norm-converges to \( h' \) and \( h_k(\omega) \succeq h'(\omega) \) for all \( \omega \in \Omega \): Take \( z_k, z \in X \) such that \( u(z_k) \triangleleft u(z) \). This is possible by Mixture Continuity. By Step 1, for each \( k \), since \( z_k \succ z \), there is \( h_k \in \mathcal{F} \) such that \( h_k(\omega) \succeq h'(\omega) \) and \( \frac{1}{2}h_k(\omega) + \frac{1}{2}z \prec \frac{1}{2}h'(\omega) + \frac{1}{2}z_k \) for all \( \omega \in \Omega \). Thus, \( \sup_{\omega \in \Omega} |u(h_k(\omega)) - u(h'(\omega))| \leq u(z_k) - u(z) \). For any \( x \succ y \), there exists \( K \) such that for all \( k > K \), \( u(z_k) - u(z) \leq u(x) - u(y) \). Thus, \( h_k \) norm-converges to \( h' \).

Step 3. For any \( h' \in \hat{\mathcal{F}} \), there is \( h_k \in \mathcal{F} \) such that \( h_k \) norm-converges to \( h' \) and \( h'(\omega) \succeq h_k(\omega) \) for all \( \omega \in \Omega \). Slightly change Steps 1 and 2.

Step 4. \( \succeq \) satisfies Monotonicity: Take \( f, g \in \hat{\mathcal{F}} \) such that \( f(\omega) \succeq g(\omega) \) for all \( \omega \in \Omega \). By Steps 2 and 3, there are \( f_k, g_k \in \mathcal{F} \) such that \( f_k, g_k \) norm-converge to \( f, g \) respectively, and \( f_k(\omega) \succeq f(\omega) \) and \( g(\omega) \succeq g_k(\omega) \) for all \( \omega \in \Omega \). Then, \( f_k(\omega) \succeq g_k(\omega) \) for all \( \omega \in \Omega \), and Monotonicity of \( \succeq \) implies \( f_k \succeq g_k \), hence \( f_k \succeq g_k \). Norm Continuity guarantees \( f \succeq g \).

Step 5. \( \preceq^* \) satisfies C-complete Preorder, Norm Continuity, Monotonicity, Independence and Non-triviality: \( \preceq^* \) inherits C-complete Preorder, Norm Continuity, Monotonicity and Non-triviality from the corresponding properties of \( \succeq \) and satisfies Independence by (3.3).

Step 6. \( \preceq^* \) extends \( \preceq^* \): Take \( f, g \in \mathcal{F} \) such that \( f \preceq^* g \), that is, \( \alpha f + (1 - \alpha) h \succeq \alpha g + (1 - \alpha) h \) for all \( \alpha \in [0, 1] \) and \( h \in \mathcal{F} \) (and thus \( \alpha f + (1 - \alpha) h \succeq \alpha g + (1 - \alpha) h \) for all \( \alpha \in [0, 1] \) and \( h \in \mathcal{F} \) since \( \succeq \) is an extension of \( \preceq \)). Now fix \( h' \in \hat{\mathcal{F}} \). By Step 2, there is \( h_k \in \mathcal{F} \) norm-converging to \( h' \). Moreover, by the mixture linearity of \( u, \alpha f + (1 - \alpha) h \), \( \alpha g + (1 - \alpha) h \), \( h_k \) norm-converge to \( f(1 - \alpha) h + \alpha g + (1 - \alpha) h' \) respectively. Since \( \alpha f + (1 - \alpha) h_k \succeq \alpha g + (1 - \alpha) h_k \) for all \( \alpha \in [0, 1] \) and \( k = 1, 2, \ldots \), Norm Continuity implies \( \alpha f + (1 - \alpha) h' \succeq \alpha g + (1 - \alpha) h' \) for all \( \alpha \in [0, 1] \). Since \( h' \) is arbitrary, \( f \succeq^* g \).

Step 7. All extensions of \( \succeq^* \) satisfying the axioms in Step 5 are the same: Assume \( \succeq^*_1, \succeq^*_2 \) on \( \hat{\mathcal{F}} \) are two such extensions. It is enough to show that \( f \succeq^*_1 g \) for \( f, g \in \hat{\mathcal{F}} \) implies \( f \succeq^*_2 g \) since the labeling of the extensions is arbitrary. By Steps 2 and 3, there are \( f_k, g_k \in \mathcal{F} \) such that \( f_k \) and \( g_k \) norm-converge to \( f \) and \( g \) respectively and \( f_k(\omega) \succeq^*_1 f(\omega), g(\omega) \succeq^*_1 g_k(\omega) \) for all \( \omega \in \Omega \). Thus, \( f_k \succeq^*_1 f, g \succeq^*_1 g_k \). Since \( \succeq^*_1, \succeq^*_2 \) coincide on \( \mathcal{F}, f_k \succeq^*_2 g_k \). Norm Continuity of \( \succeq^*_2 \) implies \( f \succeq^*_2 g \).

Step 8. vii) implies viii): By Lemma 3.1 of Klabinoff, Mukerji and Seo [13], there is \( M \subseteq \Delta(\Delta(S)) \) such that for all \( f, g \in \mathcal{F}, f \succeq^* g \) iff \( \int u(f)dp \geq \int u(g)dp \) for all \( p \in \{ \int \ell^\infty dm(\ell) : m \in M \} \). Define an extension \( \succeq^*_1 \) of \( \succeq^* \) to \( \hat{\mathcal{F}} \) by
\[
f \succeq^*_1 g \text{ iff } \int u(f)dp \geq \int u(g)dp \text{ for all } p \in \{ \int \ell^\infty dm(\ell) : m \in M \}.
\]
Then, one can check that $\succeq_1^*$ satisfies all the axioms in Step 5. By Step 7, $\succeq_1 = \succeq_1^*$. Therefore, $f \sim^* f^y$ and $g \sim^* g^y$ for any $f, g \in \mathcal{F}$ and hence $f \sim f^y$ and $g \sim g^y$. Transitivity of $\sim$ implies viii).

References


