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Definitions of Ambiguous Events and the Smooth Ambiguity Model

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Abstract We examine a variety of preference-based definitions of ambiguous events in the context of the smooth ambiguity model. We first consider the definition proposed in Klibanoff et al (2005) based on the classic Ellsberg two-urn paradox (Ellsberg (1961)), and show that it satisfies several desirable properties. We then compare this definition with those of Nehring (1999), Epstein and Zhang (2001), Zhang (2002) and Ghirardato and Marinacci (2002). Within the smooth ambiguity model, we show that Ghirardato and Marinacci (2002) would identify the same set of ambiguous and unambiguous events as our definition while Epstein and Zhang (2001) and Zhang (2002) would yield a different classification. Moreover, we discuss and formally identify two key sources of the differences compared to Epstein and Zhang (2001) and Zhang (2002). The more interesting source is that these two definitions can confound non-constant ambiguity attitude and the ambiguity of an event.

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1 Introduction

There are a variety of preference-based definitions of ambiguous events in the literature. In Klibanoff et al (2005), we proposed one such definition based on the classic Ellsberg two-urn paradox (Ellsberg (1961)). We showed, in the context of the smooth ambiguity model of decision making developed in Klibanoff et al (2005), that this definition is characterized by, roughly, disagreement in the probability assigned to an event by the various probability measures that are subjectively relevant. In the current paper we do two things: First, we show that this definition

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has many attractive properties in the context of the smooth ambiguity model. We show that the set of unambiguous events must be a λ-system – a collection of sets satisfying closure under complements and finite disjoint unions. The literature has argued that it is reasonable to expect the set of unambiguous events to be a λ-system. Furthermore, we show that an event being unambiguous is connected to the existence of a qualitative probability relation on a λ-system including that event. Second, we use the smooth ambiguity model to compare this definition with others in the literature, including those of Nehring (1999), Epstein and Zhang (2001), Zhang (2002), Ghirardato and Marinacci (2002) and Cerreia-Vioglio et al (2011). We show that Epstein and Zhang (2001) and Zhang (2002) do not generally identify the same set of ambiguous and unambiguous events as we do. Moreover, we discuss and formally identify two key sources of the differences compared to Epstein and Zhang (2001) and Zhang (2002). One source is that these two definitions can confound non-constant ambiguity attitude and the ambiguity of an event. The other source is that our definition takes advantage of the additional structure we place on the environment to determine that some events are ambiguous when these two definitions would not. In contrast, we show that Ghirardato and Marinacci (2002) identify the same set of ambiguous and unambiguous events as we do (as does the definition in Cerreia-Vioglio et al (2011), as that paper shows). A further result relevant to this discussion is that the only departures from expected utility that may arise in the smooth ambiguity model are also departures from probabilistic sophistication. Thus, differences in implicit assumptions about the ambiguity neutral benchmark cannot be a source of the different performance of the definitions in the smooth ambiguity model. Finally, as regards Nehring (1999), we show that all events we identify as unambiguous are so identified by his definition, however the converse remains an open question.

Other authors have previously critiqued aspects of some of these definitions. For example, Amarante and Filiz (2007) examine all of these definitions in the context of the maxmin expected utility model of Gilboa and Schmeidler (1989) and explore their properties in that setting. This leads them to some critiques of Epstein and Zhang (2001) and Zhang (2002) related to an incompatibility between a continuity requirement and these two definitions of unambiguous events. This point, along with other critiques, also appears in Nehring (2006). Parts of Kopylov (2007) might also be viewed as a critique and improvement upon aspects of Epstein and Zhang (2001). One result proved by Kopylov is that the collection of unambiguous events identified by Epstein and Zhang (2001) need not form a λ-system, but rather a weaker structure called a mosaic. None of these papers, nor, to our knowledge, any others, point out the challenge that non-constant ambiguity attitudes pose for attempts to identify ambiguous/unambiguous events. Working with the smooth ambiguity model both leads us to and allows us to examine this issue, since one of the main differences between the smooth ambiguity model and many of the other models in the literature is its use of ambiguity attitude non-constant in either an absolute or relative sense (or both), similar to non-constancy of risk attitudes in an absolute or relative sense that appears in expected utility theory. We show that certain of the definitions have trouble handling this non-constancy while others do not. In our final section, we discuss an example showing that other types of non-constant ambiguity attitude than those that can appear in the smooth ambiguity model may give our definition difficulty.

2 Ambiguity: ambiguous events and ambiguous acts

We begin by recalling the following setting, definitions and results from Klibanoff et al (2005):

Let $\mathcal{A}$ be the Borel σ-algebra of a separable metric space Ω, and $\mathcal{B}_1$ the Borel σ-algebra of $(0,1]$. Consider the state space $S = \Omega \times (0,1]$, endowed with the product σ-algebra $\Sigma \equiv \mathcal{A} \otimes \mathcal{B}_1$.

The space $(0,1]$ is introduced simply to model a rich set of lotteries as a set of Savage acts. For the remainder of this paper, all events will be assumed to belong to Σ unless stated otherwise.

We denote by $f : S \to \mathcal{C}$ a Savage act, where $\mathcal{C}$ is the set of consequences. We assume $\mathcal{C}$ to be an interval in $\mathbb{R}$ containing the interval $[-1,1]$. Given a preference $\succeq$ on the set of Savage acts, $\mathcal{F}$ denotes the set of all bounded Σ-measurable Savage acts; i.e., $f \in \mathcal{F}$ if $\{s \in S : f(s) \succeq x\} \in \Sigma$ for each $x \in \mathcal{C}$, and if there exist $x', x'' \in \mathcal{C}$ such that $x' \succeq f \succeq x''$.

Given the Lebesgue measure $\lambda : \mathcal{B}_1 \to [0,1]$, let $\pi : \Sigma \to [0,1]$ be a countably additive product probability such that $\pi(A \times B) = \pi(A \times (0,1])\lambda(B)$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}_1$. The set of all such probabilities π is denoted by Δ.

We call $\succsim$ a smooth ambiguity preference if it has a representation of the following form:

$$V(f) = \int_{\Delta} \phi \left[\int_S u\left(f\left(s\right)\right) d\pi \right] d\mu,$$

where $u : \mathcal{C} \to \mathbb{R}$ is a continuous, strictly increasing utility function, $\phi : \{u(x) : x \in \mathcal{C}\} \to \mathbb{R}$ is continuous and strictly increasing, and μ is a countably additive probability measure over Δ.

What makes an event ambiguous or unambiguous by our definition rests on a test of behavior, with respect to bets on the event, inspired by the Ellsberg two-urn experiment (Ellsberg (1961)). The role corresponding to bets on the draw from the urn with the known mixture of balls is played here by bets on events in $\{\Omega\} \times \mathcal{B}_1$. We say an event $E \in \Sigma$ is ambiguous if, analogous to the modal behavior observed in the Ellsberg experiment, betting on E is less desirable than betting on some event B in $\{\Omega\} \times \mathcal{B}_1$, and betting on E^c is *also* less desirable than betting on B^c. Similarly, we would also say E is ambiguous if both comparisons were reversed, or if one were indifference and the other were not.

Notation 2.1 *If $x, y \in \mathcal{C}$ and $A \in \Sigma$, xAy denotes the binary act which pays x if $s \in A$ and y otherwise.*

Definition 2.1 (KMM (2005)) *An event $E \in \Sigma$ is* unambiguous *if, for each event $B \in \{\Omega\} \times \mathcal{B}_1$, and for each $x, y \in \mathcal{C}$ such that $\delta_x \succ \delta_y$, either, $[xEy \succ xBy$ and $yEx \prec yBx]$ or, $[xEy \prec xBy$ and $yEx \succ yBx]$ or $[xEy \sim xBy$ and $yEx \sim yBx]$. An event is* ambiguous *if it is not unambiguous.*

The next proposition shows a shorter form of the definition that is equivalent to the original in the context of the smooth ambiguity model. Though this form lacks as immediate an identification with the Ellsberg experiment, it helps in understanding what makes an event unambiguous: an event is unambiguous if it is possible to calibrate the likelihood of the event with respect to events in $\{\Omega\} \times \mathcal{B}_1$.

Proposition 2.1 (KMM (2005)) *Assume $\succeq$ is a smooth ambiguity preference. An event $E \in \Sigma$ is unambiguous if and only if for each x and y with $\delta_x \succ \delta_y$,*

$$xEy \sim xBy \Longleftrightarrow yEx \sim yBx. \tag{1}$$

whenever $B \in \{\Omega\} \times \mathcal{B}_1$.

The next theorem relates ambiguity of an event to event probabilities in the smooth ambiguity representation.

Theorem 2.1 (KMM (2005)) *Assume $\succeq$ is a smooth ambiguity preference. If the event E is ambiguous according to Definition 2.1, then there exist μ-non-null sets Π', Π'' and $\gamma \in (0,1)$, such that $\pi(E) < \gamma$ for all $\pi \in \Pi'$ and $\pi(E) > \gamma$ for all $\pi \in \Pi''$. If the event E is unambiguous* *according to Definition 2.1, then, provided there is some nonempty open interval of utility values over which ϕ is strictly concave or strictly convex, there exists a $\gamma \in [0,1]$ such that $\pi(E) = \gamma$, μ-almost-everywhere.*[1]

Thus, in our model, if there is agreement about an event's probability then that event is unambiguous. Furthermore, if ϕ has *some* range over which it reflects either strict smooth ambiguity aversion or strict smooth ambiguity seeking then disagreement about an event's probability implies that the event is ambiguous. When the support Π of μ is finite, the meaning of disagreement about an event's probability in the theorem above simplifies to: there exist $\pi, \pi' \in \Pi$ such that $\pi(E) \neq \pi'(E)$.

We now turn to the substance of the current paper, which we begin by examining other properties of unambiguous events in the context of the smooth ambiguity model.

It has been widely argued in the literature that closure under finite intersection should not generally be expected from unambiguous events (e.g., Epstein and Zhang (2001), Ghirardato and Marinacci (2002), Zhang (2002)) and that, therefore, one should not generally expect the set of unambiguous events to form an algebra. A λ-system is a weaker mathematical structure dropping this intersection requirement of an algebra, and the cited papers argued that expecting the set of unambiguous events to be a λ-system is reasonable. Formally, say that a collection of events $\mathcal{A} \subseteq \Sigma$ is a λ-system if (i) $S \in \mathcal{A}$; (ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$; and (iii) $A_1, A_2 \in \mathcal{A}$ and $A_1 \cap A_2 = \varnothing \Rightarrow A_1 \cup A_2 \in \mathcal{A}$. This could be called a finite λ-system since (iii) requires only closure under finite disjoint unions rather than the closure under countable disjoint unions required in e.g., Billingsley (1986), p. 36. The following corollary to Theorem 2.1 observes that in the smooth ambiguity model, given our definition of unambiguous events, the collection of all unambiguous events always forms a λ-system. Examples such as in Zhang (2002) could be used to show it does not always form an algebra.

Corollary 2.1 *Suppose $\succeq$ is a smooth ambiguity preference and either ϕ is linear or there is some nonempty open interval of utility values over which ϕ is strictly concave or strictly convex. Let $\Lambda \subseteq \Sigma$ be the collection of all unambiguous events in Σ. Then Λ is a (finite) λ-system.*

Next, we identify an unambiguous act as an act which is measurable with respect to Λ, the collection of all unambiguous events in Σ.

Definition 2.2 An act $f \in \mathcal{F}$ is an **unambiguous act** if it is measurable with respect to Λ. Let $\mathcal{H}$ be the set of all unambiguous acts.

To gain further insight into our definition of an unambiguous event and its appropriateness in the context of our model, we introduce the following concept combining the standard notion of a qualitative probability with an additional necessary condition for the existence of a probability and likelihood revealed through bets on events. (Conditions (*i*) through (*iv*) in the definition define a qualitative probability, condition (*v*) is also necessary for the existence of a representing probability and condition (*vi*) connects this qualitative probability with $\succeq$ over bets.)

Definition 2.3 We call a binary relation $\succeq_q$ on a λ-system $\mathcal{A} \subseteq \Sigma$ a **qualitative probability relation for** $\succeq$ if it is (i) complete and transitive, (ii) $S \succ_q \emptyset$, (iii) $A \succeq_q \emptyset$ for $A \in \mathcal{A}$, (iv) for $A, B, C \in \mathcal{A}$, if $A \cap C = B \cap C = \emptyset$ then

$$A \succ_q B \iff A \cup C \succ_q B \cup C$$

[1] In the original, the condition on ϕ was replaced by equivalent assumptions on preferences. To make the current paper self-contained, in this and subsequent results, we state conditions directly on ϕ with the understanding that the interested reader can refer to Klibanoff et al (2005) for the equivalent assumptions on preferences. There it is shown that concavity (convexity) of ϕ corresponds to a behavioral notion of ambiguity aversion (seeking).

(v) for $A, B \in \mathcal{A}$,

$$A \succ_q B \iff B^c \succ_q A^c$$

and (vi) for $A, B \in \mathcal{A}$, $A \succeq_q B$ if there exist consequences x, y with $\delta_x \succ \delta_y$ such that

$$xAy \succeq xBy.$$

$\succeq$ is said to have **probabilistic beliefs** if there exists a qualitative probability relation for $\succeq$ on all of Σ.

If $\mathcal{A}$ is an algebra, conditions (i)-(iv) imply (v) (see e.g., Kreps (1988), p. 118). However, as observed by Zhang (1999), this is not true if $\mathcal{A}$ is merely a λ-system. Since (v) is a necessary condition for the existence of a probability representing $\succeq_q$ on $\mathcal{A}$, it makes sense to include it here. In any case, the result below on the existence of a qualitative probability relation for $\succeq$ is false without condition (v).

The next result uses this preference based notion of qualitative probability relation to give an alternative characterization of ambiguous events in our setting.

Corollary 2.2 *Assume $\succeq$ is a smooth ambiguity preference and either ϕ is linear or there is some nonempty open interval of utility values over which ϕ is strictly concave or strictly convex. Fix an event $E \in \Sigma$. Then E is unambiguous if and only if there exists a qualitative probability relation for $\succeq$ on some λ-system that is a superset of $\{E, E^c, \Omega \times \mathcal{B}\}$.*

Remark 2.1 The "if" part of the claim in Corollary 2.2 does not depend on $\succeq$ being a smooth ambiguity preference or the conditions on ϕ.

The existence of a qualitative probability relation for $\succeq$ on a λ-system containing $\{E, E^c, \Omega \times \mathcal{B}\}$ intuitively means that the event E can be compared in a consistent way with the rich set of events in the Borel σ-algebra $\mathcal{B}$.

The most important class of preferences exhibiting probabilistic beliefs are the probabilistically sophisticated preferences of Machina and Schmeidler (1992). Besides probabilistic beliefs, they also require some additional conditions which are superfluous for our purposes. Given the smooth ambiguity model, the only departure from expected utility that may arise is one due to ambiguity sensitive behavior, behavior that is not probabilistically sophisticated, as formally detailed in the following corollary.

Corollary 2.3 *Assume $\succeq$ is a smooth ambiguity preference. Consider the following four properties:*

(i) *$\succeq$ has probabilistic beliefs*
(ii) *$\succeq$ is probabilistically sophisticated*
(iii) *$\succeq$ has a subjective expected utility representation*
(iv) *For each event E, there exists a $\gamma \in [0, 1]$ such that $\pi(E) = \gamma$, μ-a.e.*
Then,

$$(iv) \Longrightarrow (iii) \Longrightarrow (ii) \Longrightarrow (i).$$

Moreover, all four properties are equivalent whenever there is some nonempty open interval of utility values over which ϕ is strictly concave or strictly convex, while the first three are equivalent (and true) whenever ϕ is linear.

3 Relating to other notions of ambiguity

In this section, we compare, in the context of the smooth ambiguity model, our behavioral definition of ambiguity with the behavioral definitions proposed in Epstein and Zhang (2001), Zhang (2002), Ghirardato and Marinacci (2002) and Nehring (1999). Throughout this section we assume that Ω is finite and that $\Sigma = 2^{\Omega} \otimes \mathcal{B}$.

We first relate our definition to the notion of ambiguity developed in Epstein and Zhang (2001). Their notion of ambiguity was designed to apply to a wide variety of models of preferences.

Definition 3.1 (Epstein-Zhang (2001)) An event T is **unambiguous** if: (a) for all disjoint subevents A, B of T^c, acts h, and outcomes x^*, x, z, z',

$$\begin{pmatrix} x^* & \text{if} & s \in A \\ x & \text{if} & s \in B \\ h(s) & \text{if} & s \in T^c \setminus (A \cup B) \\ z & \text{if} & s \in T \end{pmatrix} \succsim \begin{pmatrix} x & \text{if} & s \in A \\ x^* & \text{if} & s \in B \\ h(s) & \text{if} & s \in T^c \setminus (A \cup B) \\ z & \text{if} & s \in T \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x^* & \text{if} & s \in A \\ x & \text{if} & s \in B \\ h(s) & \text{if} & s \in T^c \setminus (A \cup B) \\ z' & \text{if} & s \in T \end{pmatrix} \succsim \begin{pmatrix} x & \text{if} & s \in A \\ x^* & \text{if} & s \in B \\ h(s) & \text{if} & s \in T^c \setminus (A \cup B) \\ z' & \text{if} & s \in T \end{pmatrix};$$

and (b) the condition obtained if T is everywhere replaced by T^c in (a) is also satisfied. Otherwise, T is **ambiguous** .

This definition works by looking for conditional likelihood reversals over events in the complement of the event being tested for ambiguity. Our definition instead looks for likelihood reversals involving the event being tested for ambiguity and events in $\mathcal{B}$. Note that such a strategy was not available to Epstein and Zhang since they wished their definition to have power in environments that might lack an appropriately rich set of unambiguous events.

Is our definition of ambiguity identical to that of Epstein and Zhang (2001) when applied to the smooth ambiguity model? The answer is no. How do they differ? It is of interest to note at the outset, given the many discussions comparing notions of ambiguity due to Epstein and Zhang (2001) and Ghirardato and Marinacci (2002), the reason for the difference cannot be that our definition confounds ambiguity with probabilistically sophisticated departures from expected utility. As we have shown in the previous subsection (Corollaries 2.2 and 2.3), whenever our definition identifies an event as ambiguous, behavior is not probabilistically sophisticated. Nevertheless, one might still expect, since we can take advantage of the rich structure of our model, that any difference would lie in the direction of our definition classifying more events as ambiguous than Epstein and Zhang. This is part of the story: in Example 3.1 we show that there may be some events that are ambiguous according to our definition that are not according to Epstein and Zhang.

Example 3.1 Let $\Omega = \{\omega_1, \ldots, \omega_n\}$. The measure μ assigns probability $1/2$ to both π_0 and π_1, where π_0 and π_1 yield marginals on Ω of

$$\pi_0(\omega_1) = \lambda_1,\ \pi_0(\omega_j) = \frac{1-\lambda_1}{n-1},\ j \neq 1$$

and

$$\pi_1(\omega_1) = \lambda_2,\ \pi_1(\omega_j) = \frac{1-\lambda_2}{n-1},\ j \neq 1,$$

respectively, with $\lambda_1 < \lambda_2$*. The utility function is* $u(x) = x$ *(risk neutrality). The function* ϕ *is* $\phi(x) = -e^{-\alpha x}$ *with* $\alpha > 0$*.*

Since $\pi_0(\omega_1) = \lambda_1 < \lambda_2 = \pi_1(\omega_1)$ *and* ϕ *is strictly concave, Theorem 2.1 implies that the event* $\omega_1 \times [0,1)$ *is ambiguous according to the definition in this paper. We now demonstrate that it is unambiguous according to the definition of Epstein and Zhang (2001) (Definition 3.1). To this end, consider* $T = \omega_1 \times [0,1)$ *in their definition. Notice that all events in* T^c *are assigned the same relative weights under* π_0 *as under* π_1*. Specifically, for any* $E \subseteq T^c$, $\pi_0(E) = \frac{1-\lambda_1}{1-\lambda_2}\pi_1(E)$*. Under the specified preferences, the top pair of acts in Definition 3.1 is evaluated according to*

$$-\frac{1}{2}e^{-\alpha(\lambda_1 z+\pi_0(A)x^*+\pi_0(B)x+\int_{s\in T^c\setminus(A\cup B)}h(s)\pi_0(s))}-\frac{1}{2}e^{-\alpha(\lambda_2 z+\pi_1(A)x^*+\pi_1(B)x+\int_{s\in T^c\setminus(A\cup B)}h(s)\pi_1(s))}, \tag{2}$$

and

$$-\frac{1}{2}e^{-\alpha(\lambda_1 z+\pi_0(A)x+\pi_0(B)x^*+\int_{s\in T^c\setminus(A\cup B)}h(s)\pi_0(s))}-\frac{1}{2}e^{-\alpha(\lambda_2 z+\pi_1(A)x+\pi_1(B)x^*+\int_{s\in T^c\setminus(A\cup B)}h(s)\pi_1(s))}, \tag{3}$$

respectively. Substituting $\pi_0(E) = \frac{1-\lambda_1}{1-\lambda_2}\pi_1(E)$ *and simplifying yields (2)* $\geq$ *(3) if and only if*

$$\pi_1(A)x^* + \pi_1(B)x \geq \pi_1(A)x + \pi_1(B)x^*. \tag{4}$$

The evaluation of the lower pair of acts in Definition 3.1 differs only by substituting z' *for* z*. It is not hard to show that preference in the lower pair is also determined by (4). Therefore condition (a) of the definition is satisfied. Setting* $T = (\omega_1 \times [0,1))^c$ *and noting that, again, all events in* T^c *will be assigned the same relative weights under any* π *leads to the conclusion that condition (b) of the definition holds as well. Therefore,* $\omega_1 \times [0,1)$ *is unambiguous according to Definition 3.1.* ■

The intuition behind this example is that Epstein and Zhang (2001) ambiguity requires a reversal in the relative likelihoods of two events lying in the complement of the candidate ambiguous event. The preferences in the example have the property that the ambiguity on the candidate event affects the likelihoods of all events in its complement in the same way – thus relative likelihoods in the complement are unchanged. In the formal result below, we show that such situations are in some sense rare in that by perturbing the beliefs of the decision maker even slightly one may have the ambiguity generate the differences in relative likelihoods needed for Definition 3.1.

This is not the whole story however. We show in Example 3.2 that the Epstein and Zhang definition may, more surprisingly, classify as ambiguous some events that are unambiguous according to our definition. This may appear somewhat strange, given that if E is unambiguous in our framework (under regularity conditions on ϕ) then preferences restricted to acts measurable with respect to the λ-system generated by $\{\emptyset, E, E^c, \Omega \times \mathcal{B}, S\}$ are probabilistically sophisticated (and, in fact, are expected utility).

Example 3.2 Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$*. The measure* μ *assigns probability* $1/2$ *to both* π_0 *and* π_1*, where* π_0 *and* π_1 *yield marginals on* Ω *of*

$$\pi_0(\omega_1) = 0.5, \pi_0(\omega_2) = 0.22, \pi_0(\omega_3) = 0.18, \pi_0(\omega_4) = 0.1,$$

and

$$\pi_1(\omega_1) = 0.5, \pi_1(\omega_2) = 0.02, \pi_1(\omega_3) = 0.18, \pi_1(\omega_4) = 0.3,$$

respectively. The utility function is $u(x) = x$ *(risk neutrality). The function* ϕ *is* $\phi(x) = ln(x)$*.*

Since $\pi_0(\omega_1) = \pi_1(\omega_1) = 0.5$, *the event* $\omega_1 \times [0,1)$ *is unambiguous according to the definition in this paper. We now demonstrate that it is ambiguous according to the definition of Epstein and Zhang (2001) (Definition 3.1). In their definition, let* $T = \omega_1 \times [0,1)$, $A = \omega_3 \times [0,1)$, $B = \omega_4 \times [0,1)$, $x^* = 1, x = 0, h(s) = 0, z = 0$, *and* $z' = 100$. *Calculation shows that for the top pair of acts in their definition the act on the left is strictly preferred to the one on the right, since* $ln(0.18) > 0.5\ln(0.1) + 0.5\ln(0.3)$. *However, for the bottom pair of acts the preference is reversed, as* $ln(50.18) < 0.5\ln(50.1) + 0.5\ln(50.3)$. ■

What is going on in this example? The function $\phi(x) = ln(x)$ does not reflect a constant ambiguity attitude (for a formal definition see Definition 6 in Klibanoff et al (2005)). In particular, as the acts under consideration get better and better in terms of expected utilities, the ambiguity aversion of such a decision maker diminishes. This is in close analogy to risk theory, since $ln(x)$ displays constant relative risk aversion but diminishing absolute risk aversion when used as a utility function. In our example, A is unambiguous, while B is ambiguous but has a higher average probability than A. When $z = 0$ the expected utilities of the acts are relatively low under all the measures and the portion of the $ln(x)$ function that is quite ambiguity averse is the relevant one. Here A gets favored over B. When considering $z' = 100$ however, the acts get considerably higher expected utilities and a portion of the $ln(x)$ function that is much less concave (ambiguity averse) is relevant. In the second case, ambiguity aversion has diminished enough that the decision maker is now willing to favor the ambiguous-but-higher-average-probability event, B over the unambiguous one, A. Examples of this kind may be constructed quite generally when ambiguity aversion is not constant. In such a case, changing payoffs on *any* event E, no matter what its ambiguity status, may lead to a conditional likelihood reversal between two events in E^c *if at least one of the two events in* E^c *is ambiguous.* This occurs because changing payoffs may change ambiguity attitude thus possibly affecting the decision maker's ranking of events. If all events in E^c are unambiguous, changing ambiguity attitude cannot affect the ranking of these events and thus a conditional likelihood reversal would not occur in this case.

This example suggests that when a rich set of events like $\mathcal{B}$ over which the decision maker has a probability is available, our approach allows one to distinguish between reversals due to the ambiguity of the event being tested and those due to changing ambiguity attitude.

As the next theorem shows, the differences identified in the above two examples are in some sense the only ones separating the two definitions of ambiguity in our setting. The second example is dealt with by the assumption of constant ambiguity aversion. The first example is dealt with by showing that perturbing the beliefs of the decision maker by as small amount as one wishes can eliminate this type of disagreement. We should note that the statement in part (b) of the theorem below (concerning differences in the direction of the first example) does not go as far as one might hope. In particular, the perturbation argument we develop works for events in $2^{\Omega} \times [0,1)$ rather than general events in Σ, assumes that μ has a finite support and, even though the perturbations required are arbitrarily small, we have not been able to rule out that they might change the ambiguity classification of some compound events (i.e., events not of the form $\omega \times [0,1)$). This contrasts with part (a) of the theorem (concerning differences in the direction of the second example), which applies to all events and whose proof does not make use of the finite support assumption. Part (a) shows quite strongly that differences in the direction of the second example indeed stem from non-constant ambiguity attitude.

Definition 3.2 An event $E \subseteq \Omega$ is **null** if, for all $x, y, z \in \mathcal{C}$,

$$\begin{pmatrix} x \text{ if } s \in E \times [0,1) \\ y \text{ if } s \in E^c \times [0,1) \end{pmatrix} \sim \begin{pmatrix} z \text{ if } s \in E \times [0,1) \\ y \text{ if } s \in E^c \times [0,1) \end{pmatrix}.$$

Proposition 3.1 *An event* $E \subseteq \Omega$ *is* ***null*** *if and only if* $\pi(E \times [0,1)) = 0$, μ*-a.e.*

Theorem 3.1 *Assume that Ω contains at least three non-null states and that $\phi(x) = -e^{-\alpha x}$, $\alpha > 0$ (i.e., preferences display constant ambiguity aversion (see Proposition 2 of Klibanoff et al (2005))).*

(a) If an event $E \in \Sigma$ is unambiguous according to Definition 2.1, then it is unambiguous according to the definition in Epstein-Zhang (2001).

(b) If an event $E \times [0,1)$ is ambiguous according to Definition 2.1, then there exists a sequence of perturbations of Π ($\equiv$ support of μ), denoted $\Pi(\epsilon_n)$, with $\lim_{n\to\infty} \Pi(\epsilon_n) = \Pi$ and $\lim_{n\to\infty} \epsilon_n = 0$, such that for all n, given $\Pi(\epsilon_n)$:

(i) The event $E \times [0,1)$ is ambiguous according to the definition in Epstein-Zhang (2001), and,

(ii) for each $\omega \in \Omega$, $\omega \times [0,1)$ is ambiguous according to Definition 2.1 if and only if $\omega \times [0,1)$ was ambiguous according to Definition 2.1 given Π.

An analogous result is true with $\alpha < 0$ and $\phi(x) = e^{-\alpha x}$ (constant ambiguity seeking).

We next turn to the definition of unambiguous event proposed by Zhang (2002). This is a strengthening of Epstein and Zhang (2001)'s definition. Instead of requiring the acts on T^c to have a specific structure, Zhang (2002) allows them to be arbitrary. Thus, an event is unambiguous according to Zhang (2002) if changing the outcome on the event never leads to reversals in preference on the complementary event. Formally:

Definition 3.3 (Zhang (2002)) An event T is **unambiguous** if: (a) for all acts f, g, and outcomes z, z',

$$\begin{pmatrix} f(s) \text{ if } & s \in T^c \\ z \text{ if } & s \in T \end{pmatrix} \succsim \begin{pmatrix} g(s) \text{ if } & s \in T^c \\ z \text{ if } & s \in T \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} f(s) \text{ if } & s \in T^c \\ z' \text{ if } & s \in T \end{pmatrix} \succsim \begin{pmatrix} g(s) \text{ if } & s \in T^c \\ z' \text{ if } & s \in T \end{pmatrix};$$

and (b) the condition obtained if T is everywhere replaced by T^c in (a) is also satisfied. Otherwise, T is **ambiguous** .

Since this definition classifies fewer events as unambiguous than Epstein and Zhang (2001), it is immediate that Example 3.2 applies equally to Zhang's definition, and thus it also shares with that definition the feature of allowing non-constant ambiguity aversion to interfere with its ability to identify which events are ambiguous. Although not as immediate, it is also true that Example 3.1 generates the same phenomenon under the Zhang (2002) definition as it did for Epstein and Zhang (2001). The uniformity of the probabilities on T^c in that example serves equally well to eliminate the reversals checked for in the Zhang (2002) definition as it did to eliminate the subset of those reversals checked for in Epstein and Zhang (2001). Therefore, in the context of the smooth ambiguity model, both of the key sources of difference we identified when discussing Epstein and Zhang (2001) also apply to Zhang's (2002) definition.

Next we relate our definition of ambiguity to the definition proposed by Ghirardato and Marinacci (2002). The definition they propose applies to a very general class of preferences which includes smooth ambiguity preferences. In their definition, they invoke a notion of a benchmark preference for $\succeq$ which is any subjective expected utility preference that is less ambiguity averse than $\succeq$ (in the sense that whenever an act f is ranked weakly above a lottery l by $\succeq$ then this is also true for the benchmark preference).

Definition 3.4 (Ghirardato and Marinacci (2002)) For a given benchmark preference $\geqslant$ for $\succeq$ the set of $\geqslant$ **−unambiguous acts**, denoted $\mathcal{H}^{gm}_{\geqslant}$, is the largest subset of $\mathcal{F}$ satisfying the following two conditions:

(A) For every $f \in \mathcal{H}^{gm}_{\geqslant}$ and every $x \in \mathcal{C}$,

$$x \geqslant f \iff x \succeq f \text{ and } x \leqslant f \iff x \preceq f;$$

(B) For every $f \in \mathcal{H}^{gm}_{\geqslant}$ and every $g \in \mathcal{F}$, if $\left\{g^{-1}(x) : x \in \mathcal{C}\right\} \subseteq \left\{f^{-1}(x) : x \in \mathcal{C}\right\}$, then $g \in \mathcal{H}^{gm}_{\geqslant}$.

Correspondingly, they define the set of unambiguous events $\Lambda^{gm}_{\geqslant}$ as the collection of all the 'upper pre-image' sets of the unambiguous acts $f \in \mathcal{H}^{gm}_{\geqslant}$. (An upper pre-image set of an act $f : S \to \mathcal{C}$ is a set of the form $\{s : f(s) \succeq x\}$ for $x \in \mathcal{C}$.)

Given any smooth ambiguity preference, there would seem to be a natural subjective expected utility benchmark: the preference generated by replacing ϕ with the identity (or any other strictly increasing linear function). Note, however, that this "benchmark" will not necessarily satisfy the requirement from Ghirardato and Marinacci (2002) for a benchmark, as it need not be less ambiguity averse than $\succeq$. Their definition could, nevertheless, in principle, be applied with any subjective expected utility preference sharing the same risk preferences as $\succeq$ as the benchmark, even if it were not less ambiguity averse than $\succeq$. It is in this expanded sense that we use their definition here.

In the following theorem, we show that for smooth ambiguity preferences (with a regularity condition on ϕ), the collection of unambiguous events (Λ) and the collection of unambiguous acts ($\mathcal{H}$) identified by the definitions based on Klibanoff et al (2005) (Definitions 2.1 and 2.2 in this paper, respectively) are identical to the collections identified by the corresponding definitions in Ghirardato and Marinacci (2002) with the "benchmark" $\geqslant$ taken to be the expected utility preference generated from the smooth ambiguity preference by taking ϕ linear.

Theorem 3.2 *Suppose $\succeq$ is a smooth ambiguity preference and either ϕ is linear or there is some nonempty open interval of utility values over which ϕ is strictly concave or strictly convex. Let $\geqslant$ be the subjective expected utility preference obtained from $\succeq$ by replacing ϕ with a linear function. Then, $\mathcal{H} = \mathcal{H}^{gm}_{\geqslant}$ and $\Lambda = \Lambda^{gm}_{\geqslant}$.*

If the assumptions of Theorem 3.2 were strengthened to require that ϕ were globally either linear, strictly concave or strictly convex, then the "benchmark" used in the theorem would indeed be a benchmark according to Ghirardato and Marinacci (2002) as well. In this case, Theorem 4 of the working paper version of Ghirardato and Marinacci (2002),[2] may be used to show that $\mathcal{H}^{gm}_{\geqslant}$ is actually independent of the particular benchmark $\geqslant$ and hence we may drop the subscript and, letting $\mathcal{H}^{gm}$ denote the set of acts identified as unambiguous in the Ghirardato-Marinacci definition and Λ^{gm} denote the unambiguous events, strengthen the conclusion of Theorem 3.2 to $\mathcal{H} = \mathcal{H}^{gm}$ and $\Lambda = \Lambda^{gm}$.

While Ghirardato and Marinacci (2002) provide a simple characterization of ambiguous acts and events for biseparable preferences they do not have a corresponding result for general preferences. In particular, smooth ambiguity preferences (except for expected utility) are not contained in the biseparable class. Hence, the formal relationship described above between our definitions and the Ghirardato-Marinacci definitions of ambiguous acts and events in the context of $\succeq$, does not follow from the characterization obtained in Ghirardato and Marinacci (2002). What explains the above result that the two definitions of ambiguous acts, ours and Ghirardato-Marinacci's, coincide in the smooth ambiguity model? The explanation lies in two key properties shared by the definitions. One is that according to both definitions, if two acts are measurable with respect to the same collection of events, then if one of the acts is unambiguous so is the other; the actual payoffs do not matter, only the partition generated by the acts. The other key property is that

[2] For convenience, we reproduce the relevant section of the theorem in the Appendix as Theorem A.1. It appears just before the statement of the proof of our Theorem 3.2.

the set of unambiguous events, according to Ghirardato-Marinacci, is a partition such that the ranking between acts measurable with respect to that partition is the same, whether given by the benchmark preference relation or whether given by the ambiguity sensitive preference relation: behavior with respect to these events is unaffected by ambiguity attitude. In our model there is a natural benchmark corresponding to any given preference, namely the subjective expected utility relation obtained by making ϕ linear without changing any other aspect of the given preference. Our definition of ambiguity identifies those events as unambiguous for which there is no disagreement about probabilities. In our model, ranking between acts measurable with respect to such events according to the given preference must coincide with that corresponding to the benchmark expected utility preference.

In the context of the smooth ambiguity model, under the same regularity assumptions on ϕ as in Theorem 3.2, it is also true that our definition identifies the same sets of unambiguous acts and events as the definition recently proposed in Cerreia-Vioglio et al (2011) (see their paper for a proof).

Finally, we turn to the definition proposed in (Nehring 1999, Definition 4). He formally states the definition only in the context of real-valued acts and risk neutrality. So as to make the comparison clearest, let us temporarily adopt these restrictions as well. Under these restrictions, we show that any event identified as by our Definition 2.1 as unambiguous is also unambiguous according to (Nehring 1999, Definition 4). It is an open question whether the converse implication always holds. We next state his definition. To do so, note that adding a real number to an act should be interpreted as generating another act by, state-by-state, adding the real number to the outcome of the original act.

Definition 3.5 (Nehring 1999) An event E is *unambiguous* if for all acts f and all $a, b \in \mathbb{R}$,

$$f \succeq (f-a)\, E\, (f-b) \Leftrightarrow aEb \succeq 0.$$

If ϕ is linear, then Definitions 2.1 and 3.5 both classify all events as unambiguous. When ϕ is nonlinear, the next result shows that any event unambiguous according to Definition 2.1 is also unambiguous according to Definition 3.5.

Theorem 3.3 *Suppose $\succeq$ is a smooth ambiguity preference, with $u(x) = x$ for $x \in \mathbb{R}$ and ϕ such that there is a nonempty open interval over which ϕ is strictly concave or strictly convex. An event $E \in \Sigma$ is unambiguous according to Definition 3.5 if it is unambiguous according to Definition 2.1.*

4 Beyond the smooth ambiguity model

We have argued that the definition of ambiguous events we used in Klibanoff et al (2005) performs well and has desirable properties in the context of the smooth ambiguity model. Is it true more generally, beyond the smooth ambiguity model, that this definition captures all that one might mean by ambiguous event? The following example, adapted from Nehring (1999), suggests this may not be the case.

Example 4.1 Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and let u be the utility function. Suppose that $\succsim$ over acts f are represented by $u(f(\omega_m))$ where ω_m is a state such that f yields a middle-ranked outcome (e.g. if $u(f(\omega_2)) \leq u(f(\omega_1)) \leq u(f(\omega_3))$ then $u(f(\omega_m)) = u(f(\omega_1))$).

Under our definition, all events are unambiguous according to this preference. To see this, observe that any bet on an event containing one or fewer states is treated as a bet on an event given zero probability and any bet on an event containing at least two states is treated as a bet on *an event given probability one. Thus bets on an event and on its complement may be calibrated to bets on (and against) an unambiguous zero probability event, and no reversals as in the two-color Ellsberg paradox (or the three color Ellsberg paradox) can occur.*

Nonetheless, preferences fail to be probabilistically sophisticated and might be motivated in terms of ambiguity and ambiguity attitude in the following way: suppose the individual considers all probability distributions over Ω possible, is infinitely ambiguity averse when evaluating "small" events (those with one or fewer states), and is infinitely ambiguity loving when evaluating "large" events (those with two or more states). Since the event of being equal to or above a middle-ranked outcome is always a "large event" while the events of being strictly below or of being strictly above a middle-ranked outcome are always "small" events, the preferences would be as in the example. Such an individual might be said to consider all non-trivial events ambiguous, but because their ambiguity attitude varies dramatically with the size of the events, something not permitted in the smooth ambiguity model, our definition fails to pick this up. ■

5 Conclusion

We have shown that a definition of unambiguous event motivated by Ellsberg's two-urn paradox and stated in Klibanoff et al (2005) works well in the context of the smooth ambiguity model. In that context, we also showed that the alternative definition proposed by Ghirardato and Marinacci (2002) reduces to our definition. In contrast, we show that the definitions proposed by Epstein and Zhang (2001) and Zhang (2002), when applied to the smooth ambiguity model, do not always agree with our definition. Moreover, we identify the sources of this difference, the most interesting of which is the fact that the Epstein and Zhang (2001) and Zhang (2002) definitions may classify an event as ambiguous when the preference reversal they use to do so is generated by ambiguity attitude changing with payoffs rather than ambiguity of the event. The fact that the literature prior to the smooth ambiguity model focused on models displaying constant ambiguity attitude as payoffs change (in particular any preference satisfying the C-Independence axiom of Gilboa and Schmeidler (1989) or its weakening in Maccheroni et al (2006) will display such constancy) perhaps explains why this aspect of these definitions remained unnoticed previously. Seeing how different definitions perform in the context of the smooth ambiguity model has, in this sense, proved quite valuable.

A Appendix: Proofs and Related Material

A.1 Corollary 2.1

Let $\succeq$ be a smooth ambiguity preference and either ϕ is linear or there is some nonempty open interval of utility values over which ϕ is strictly concave or strictly convex. First consider the case where ϕ is linear. In this case, preferences are expected utility and hence Λ coincides with Σ, a σ-algebra and therefore, a λ-system. Next consider the case where there is some nonempty open interval of utility values over which ϕ is strictly concave or strictly convex and let $E, F \in \Sigma$ *s.t.* $E \cap F = \varnothing$. By Theorem 2.1, E unambiguous implies there exists a $\gamma \in [0,1]$ such that $\pi(E) = \gamma$ for μ almost all π. Also, $\pi(E^c) = 1 - \gamma$ for μ almost all π. Similarly, there exists a $\gamma' \in [0,1]$ such that $\pi(F) = \gamma'$ and $\pi(F^c) = 1 - \gamma'$and for μ almost all π. Since π is a probability measure and $E \cap F = \varnothing$, it follows that $\pi(E \cup F) = \gamma + \gamma'$ for μ almost all π. Hence, by Theorem 2.1, $E \cup F$ is unambiguous. Finally, note that it follows directly from Definition 2.1 that S is unambiguous and that if E is unambiguous then so is E^c. ■

A.2 Corollary 2.2

"If ": Fix an event E. Suppose that $\succeq_q$ is a qualitative probability relation for $\succeq$ on a λ-system $\mathcal{A} \supseteq \mathcal{E} = \{E, E^c, \Omega \times \mathcal{B}\}$. By properties (i), (v), and (vi) in the definition of a qualitative probability relation, E is unambiguous.

"Only if": Let Λ denote the set of all unambiguous events in Σ. Suppose that E, and therefore all elements of $\mathcal{E}$, is unambiguous. By Corollary 2.1, Λ is a λ-system. By our hypothesis, $\mathcal{E} \subseteq \Lambda$. First, suppose there is some nonempty open interval of utility values over which ϕ is strictly concave or strictly convex. By Theorem 2.1, for every $A \in \Lambda$ there exists a $\gamma(A) \in [0,1]$ such that $\pi(A) = \gamma(A)$ for μ almost all π. So, a probability γ representing the likelihood relation on λ-system $\Lambda \supseteq \mathcal{E}$ exists and therefore a qualitative probability relation for $\succeq$ exists on the same λ-system. Alternatively, suppose ϕ is linear. In this case, preferences are expected utility and hence a probability γ representing the likelihood relation on λ-system $\Lambda \supseteq \mathcal{E}$ exists and therefore a qualitative probability relation for $\succeq$ exists on the same λ-system. Therefore E unambiguous implies the existence of a qualitative probability relation for $\succeq$ on a λ-system containing $\mathcal{E}$. ■

A.3 Corollary 2.3

By the representation $(iv) \implies (iii)$. The implications $(iii) \implies (ii) \implies (i)$ are obvious. By Theorem 2.1, $(i) \implies (iv)$ when there is some nonempty open interval of utility values over which ϕ is strictly concave or strictly convex. To see this, observe that if $\succeq_q$ is a qualitative probability relation for $\succeq$ on Σ, then by Corollary 2.2 all events $E \in \Sigma$ are unambiguous. By Theorem 2.1, provided there is some nonempty open interval of utility values over which ϕ is strictly concave or strictly convex, this implies agreement about each event's probability. As to the case where ϕ is linear, recall that $\succeq$ has an expected utility representation. ■

A.4 Results of Section 3

For the following proofs we need a piece of notation. As we assumed Ω to be finite in this section, we can use $[0,1]$ as an index set for Δ, that is, $\Delta = \{\pi_i : i \in [0,1]\}$. In this case, μ can be viewed as defined on the Borel σ-algebra of $[0,1]$, with support I.

A.4.1 Proposition 3.1

Suppose $\pi_i(E \times [0,1)) = 0$, for μ almost all i. By the smooth ambiguity representation,

$$f = \binom{x \text{ if } s \in E \times [0,1)}{y \text{ if } s \in E^c \times [0,1)} \sim \binom{z \text{ if } s \in E \times [0,1)}{y \text{ if } s \in E^c \times [0,1)} = g,$$

iff

$$\begin{aligned}
&\int_I \phi\left[\sum_{\omega\in E}\int_{[0,1)} u(x)\,\pi_i(\{\omega\times[0,1)\})\,dr + \sum_{\omega\in E^c}\int_{[0,1)} u(y)\,\pi_i(\{\omega\times[0,1)\})\,dr\right] d\mu \\
=& \int_I \phi\left[\sum_{\omega\in E}\int_{[0,1)} u(z)\,\pi_i(\{\omega\times[0,1)\})\,dr + \sum_{\omega\in E^c}\int_{[0,1)} u(y)\,\pi_i(\{\omega\times[0,1)\})\,dr\right] d\mu \\
\Leftrightarrow& \int_I \phi\left[\sum_{\omega\in E^c}\int_{[0,1)} u(y)\,\pi_i(\{\omega\times[0,1)\})\,dr\right] d\mu \\
=& \int_I \phi\left[\sum_{\omega\in E^c}\int_{[0,1)} u(y)\,\pi_i(\{\omega\times[0,1)\})\,dr\right] d\mu.
\end{aligned}$$

Therefore E is null. To prove the other direction, suppose E is null. To generate a contradiction assume $\pi_i(E \times [0,1)) > 0$ for $i \in H$ where $\mu(H) = c > 0$. Fix x and z such that $\delta_x \succ \delta_z$. Set $y = 0$. By the smooth ambiguity representation,

$$f = \binom{x \text{ if } s \in E \times [0,1)}{y \text{ if } s \in E^c \times [0,1)} \sim \binom{z \text{ if } s \in E \times [0,1)}{y \text{ if } s \in E^c \times [0,1)} = g,$$

iff

$$\int_I \phi\left[\sum_{\omega\in E}\int_{[0,1)} u(x)\,\pi_i(\{\omega\times[0,1)\})\,dr + \sum_{\omega\in E^c}\int_{[0,1)} u(0)\,\pi_i(\{\omega\times[0,1)\})\,dr\right]d\mu$$
$$= \int_I \phi\left[\sum_{\omega\in E}\int_{[0,1)} u(z)\,\pi_i(\{\omega\times[0,1)\})\,dr + \sum_{\omega\in E^c}\int_{[0,1)} u(0)\,\pi_i(\{\omega\times[0,1)\})\,dr\right]d\mu$$
$$\Leftrightarrow \int_I \phi\left[\sum_{\omega\in E}\int_{\{r\in[0,1)\mid(\omega,r)\in E\}} u(x)\,\pi_i(\{\omega\times[0,1)\})\,dr\right]d\mu$$
$$= \int_I \phi\left[\sum_{\omega\in E}\int_{\{r\in[0,1)\mid(\omega,r)\in E\}} u(z)\,\pi_i(\{\omega\times[0,1)\})\,dr\right]d\mu$$
$$\Leftrightarrow u(x) = u(z),$$

a contradiction. ■

A.4.2 Theorem 3.1

(a) Fix any event E that is unambiguous according to Definition 2.1. Suppose,

$$\begin{pmatrix} x^* & if & s\in A \\ x & if & s\in B \\ h(s) & if & s\in C\equiv E^c\setminus(A\cup B) \\ z & if & s\in E \end{pmatrix} \succeq \begin{pmatrix} x & if & s\in A \\ x^* & if & s\in B \\ h(s) & if & s\in C\equiv E^c\setminus(A\cup B) \\ z & if & s\in E \end{pmatrix}.$$

Then according to our representation,

$$\int_I -e^{-\alpha\left[u(z)\pi_i(E)+u\left(x^*\right)\pi_i(A)+u(x)\pi_i(B)+\int_C u(h(s))d\pi_i\right]}d\mu \geq \int_I -e^{-\alpha\left[u(z)\pi_i(E)+u(x)\pi_i(A)+u\left(x^*\right)\pi_i(B)+\int_C u(h(s))d\pi_i\right]}d\mu. \tag{5}$$

Since E is unambiguous according to Definition 2.1 and ϕ is strictly concave, Theorem 2.1 implies $\pi_i(E)$ is almost everywhere constant in i and hence the $u(z)$ term can be taken outside the integral on both sides and canceled. This can clearly be done, in exactly the same way, if z is replaced by some z' in the acts above. Therefore for any $z'\in\mathcal{C}$,

$$\begin{pmatrix} x^* & if & s\in A \\ x & if & s\in B \\ h(s) & if & s\in C\equiv E^c\setminus(A\cup B) \\ z' & if & s\in E \end{pmatrix} \succeq \begin{pmatrix} x & if & s\in A \\ x^* & if & s\in B \\ h(s) & if & s\in C\equiv E^c\setminus(A\cup B) \\ z' & if & s\in E \end{pmatrix}.$$

Since, under Definition 2.1, E unambiguous implies that E^c is unambiguous, a similar preference implication can be derived replacing E by E^c. Thus E is unambiguous according to the definition in Epstein and Zhang (2001).

(b)[3] Fix any $E\subseteq\Omega$ such that $E\times[0,1)$ is ambiguous according to definition 2.1. The strategy for showing that this event is ambiguous according to Epstein and Zhang (2001) after a suitable perturbation of the measures in Π is as follows: We eventually select a particular pair of acts for which we will show that there is the type of conditional likelihood reversal required by Epstein and Zhang (2001). Using our representation, we are able to find equations (6, 7) that are necessary for no reversal to occur. It turns out that there are essentially only two ways these two equations can hold simultaneously given that $\pi_i(E\times[0,1))$ is not constant. One way is for the relative probabilities of the events in the complement of $E\times[0,1)$ to remain unchanged, as happened in Example 3.1. A second, related way is that although the relative probabilities of the events in the complement do change, they do so in a way which happens to exactly "cancel out" on level sets of $\pi_i(E\times[0,1))$. We construct the perturbation of Π precisely to prevent these two circumstances from happening (and also to maintain the ambiguity/unambiguity of all events of the form $\omega\times[0,1)$ as required by part $b(ii)$ of the theorem). We then are able to conclude that the needed reversal in preference does occur and $E\times[0,1)$ is indeed ambiguous according to Epstein and Zhang (2001).

[3] We are grateful to Rakesh Vohra for his help in constructing this proof.

We begin by constructing the needed perturbation to prevent both the "cancelling out" on level sets problem (by making sure all such sets are singletons) and the constancy of relative probabilities problem (by direct perturbation). Since $E \times [0,1)$ is ambiguous and unambiguous events are closed under disjoint unions, there must exist an $\omega_1 \in E$ such that $\omega_1 \times [0,1)$ is ambiguous. Fix such an ω_1. Choose ω_2, $\omega_3 \in \Omega$ as follows. By Theorem 2.1, there exist $i, j \in I$ such that $\pi_i(E \times [0,1)) \neq \pi_j(E \times [0,1))$. Given such i and j, since probabilities sum to 1, there must be an $\omega' \in \Omega \backslash E$ such that $\omega' \times [0,1)$ is ambiguous. Set ω_2 equal to some such ω'. Choose ω_3 to be any element of $\Omega \backslash (E \cup \{\omega_2\})$ such that $\omega_3 \times [0,1)$ is non-null. Without loss of generality we may assume that such a ω_3 exists.[4]

Given Π, construct $\Pi(\epsilon_n)$ as follows:

Step 1: Recall that $I \subseteq [0,1]$. Divide I into sets of the form

$$I_\kappa \equiv \{i \in I \mid \pi_i(E \times [0,1)) = \kappa\}.$$

Let J' denote the collection of sets I_κ including exactly each such set that is not a singleton, has $\kappa \neq 0$ and $\kappa \neq 1$, and has $\pi_i(\omega_2 \times [0,1)) > 0$ for all $i \in I_\kappa$. Let $J = \{i \mid i \in I_\kappa \text{ for some } I_\kappa \in J'\}$. Let H' denote the collection of sets I_κ including exactly those that are not in J', and have $\kappa \neq 0$. Let $H = \{i \mid i \in I_\kappa \text{ for some } I_\kappa \in H'\}$. Choose $\epsilon_n > 0$ no larger than the minimum of {half the minimum distance between the κ values associated with the I_κ sets, half of $\min_{i \in J}(1 - \pi_i(E \times [0,1)))$, half of $\min_{i \in J} \pi_i(\omega_2 \times [0,1))$, half of $\min_{i \in H}(1 - \pi_i(\omega_2 \times [0,1)))$, half of $\min_{i \in H} \pi_i(E \times [0,1))$, half of

$$\max_{i,j \in I} |\pi_i(\omega_2 \times [0,1)) - \pi_j(\omega_2 \times [0,1))|,$$

and half of

$$\min_{i,j \in I, \omega \in E \text{ s.t. } \pi_i(\omega \times [0,1)) \neq \pi_j(\omega \times [0,1))} |\pi_i(\omega \times [0,1)) - \pi_j(\omega \times [0,1))|\}.$$

If there are $\omega \in E$ such that $\omega \times [0,1)$ is unambiguous, then transfer a total mass of ϵ_n to ω_1 taken evenly from these unambiguous states for each i.

For each $i \in J$, let

$$\pi_i^1(\epsilon_n)(\omega_1 \times [0,1)) = \pi_i(\omega_1 \times [0,1)) + i\epsilon_n$$

and

$$\pi_i^1(\epsilon_n)(\omega_2 \times [0,1)) = \pi_i(\omega_2 \times [0,1)) - i\epsilon_n.$$

Note that the way we have chosen ϵ_n and the fact that $I \subseteq [0,1]$ ensures that $E \times [0,1)$, $\omega_1 \times [0,1)$ and $\omega_2 \times [0,1)$ remain ambiguous according to definition 2.1 and that no new "bunchings" of $\pi_i(E \times [0,1))$ have been created.

Step 2: For each $i \in H$, let (if there was mass transferred to ω_1 from the unambiguous states in E in the previous step)

$$\pi_i^1(\epsilon_n)(\omega_1 \times [0,1)) = \pi_i(\omega_1 \times [0,1)) - i\epsilon_n,$$

otherwise

subtract the $i\epsilon_n$ from the ambiguous states in E in any way that doesn't push any below 0

and

$$\pi_i^1(\epsilon_n)(\omega_2 \times [0,1)) = \pi_i(\omega_2 \times [0,1)) + i\epsilon_n.$$

Note that the way we have chosen ϵ_n and the fact that $I \subseteq [0,1]$ ensures that $E \times [0,1)$, $\omega_1 \times [0,1)$ and $\omega_2 \times [0,1)$ and any other ambiguous $\omega \in E$ remain ambiguous according to Definition 2.1 and that no new "bunchings" of $\pi_i(E \times [0,1))$ have been created.

Step 3: For all points and for all $i \in I$ at which $\pi_i^1(\epsilon_n)$ has yet to be defined, let

$$\pi_i^1(\epsilon_n)(.) = \pi_i(.).$$

The end result of Steps 1 through 3 is a set of probabilities $\Pi^1(\epsilon_n) \equiv \{\pi_i^1(\epsilon_n)\}_{i \in I}$ that has the same set of ambiguous events of the form $\omega \times [0,1)$ as Π but, except (possibly) where $\pi_i^1(\epsilon_n)(E \times [0,1)) = 0$, $\pi_i^1(\epsilon_n)(E \times [0,1))$ takes on distinct values for each $i \in I$ and $\pi_i^1(\epsilon_n)(\omega_2 \times [0,1)) > 0$ for all $i \in I$.

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[4] If no such ω_3 exists, since there were assumed to be at least three $\omega \in \Omega$ such that $\omega \times [0,1)$ is non-null, it must be that at least two of these lie in E. If $E \times [0,1)$ above is replaced by $(\Omega \backslash E) \times [0,1)$, then $(\Omega \backslash E) \times [0,1)$ will be ambiguous according to Definition 2.1 and $\omega_2, \omega_3 \in E$ with the desired properties will exist. The proof may then be carried through for $(\Omega \backslash E) \times [0,1)$. This will then imply the result for $E \times [0,1)$ since both definitions of ambiguous event are closed under complementation.

Step 4: This step should only be undertaken if the ratio $\frac{\pi_i^1(\epsilon_n)(\omega_3\times[0,1))}{\pi_i^1(\epsilon_n)(\omega_2\times[0,1))}$ is constant across all $i \in I$ such that $\pi_i^1(\epsilon_n)(E \times [0,1)) > 0$. Notice that since $\omega_2 \times [0,1)$ is ambiguous, this can only occur if $\omega_3 \times [0,1)$ is ambiguous as well. Select an $m \in I$ such that

$$\pi_m^1(\epsilon_n)(E \times [0,1)) > 0.$$

Set

$$\pi_m^2(\epsilon_n)(\omega_3 \times [0,1)) = \pi_m^1(\epsilon_n)(\omega_3 \times [0,1)) + \frac{m}{2}\epsilon_n$$

and

$$\pi_m^2(\epsilon_n)(\omega_2 \times [0,1)) = \pi_m^1(\epsilon_n)(\omega_2 \times [0,1)) - \frac{m}{2}\epsilon_n.$$

Step 5: For all points and for all $i \in I$ at which $\pi_i^2(\epsilon_n)$ has yet to be defined, let

$$\pi_i^2(\epsilon_n)(.) = \pi_i^1(.).$$

Then define $\Pi(\epsilon_n) = \Pi^2(\epsilon_n)$. Notice, because of Step 4, that the ratio $\frac{\pi_i(\epsilon_n)(\omega_3\times[0,1))}{\pi_i(\epsilon_n)(\omega_2\times[0,1))}$ cannot be constant across all $i \in I$ such that $\pi_i(\epsilon_n)(E \times [0,1)) > 0$. This fact will be used in the proof below.

Above, ϵ_n was allowed to be real number strictly between zero and an upper bound that was the minimum of several terms. To form a sequence, select any sequence of numbers $\{\epsilon_n\}_{n=1}^{\infty}$ in this range such that the $\lim_{n\to\infty}\epsilon_n = 0$. Observe, that given our construction, $\lim_{n\to\infty}\Pi(\epsilon_n) = \Pi$ (specifically, $\lim_{n\to\infty}\pi_i(\epsilon_n)(E) = \pi_i(E)$ for all $i \in I$ and all $E \in \Sigma$). Now we proceed to show that, given any such $\Pi(\epsilon_n)$,the event $E \times [0,1)$ is ambiguous according to the definition in Epstein and Zhang (2001). Consider the pair of acts depicted below (where E, ω_2 and ω_3 are as in the construction of $\Pi(\epsilon_n)$ and $C \equiv (E \times [0,1))^c \setminus ((\omega_3 \times [0,t)) \cup (\omega_2 \times [t,1)))$).

$$\begin{pmatrix} 1 \; if \;\; s \in \omega_3 \times [0,t) \\ 0 \; if \;\; s \in \omega_2 \times [t,1) \\ 0 \; if \qquad s \in C \\ 0 \; if \;\; s \in E \times [0,1) \end{pmatrix}, \begin{pmatrix} 0 \; if \;\; s \in \omega_3 \times [0,t) \\ 1 \; if \;\; s \in \omega_2 \times [t,1) \\ 0 \; if \qquad s \in C \\ 0 \; if \;\; s \in E \times [0,1) \end{pmatrix}.$$

Given our representation of preferences, the decision maker is indifferent between these two acts iff

$$\sum_{i=1}^{k} -e^{-\alpha t \pi_i(\epsilon_n)(\omega_3\times[0,1))}\mu(i) = \sum_{i=1}^{k} -e^{-\alpha(1-t)\pi_i(\epsilon_n)(\omega_2\times[0,1))}\mu(i). \tag{6}$$

Let t^* be the unique t that solves equation (6). Such a t^*exists since the left-hand side is continuously increasing in t and is equal to -1 at $t = 0$ while the right-hand side is continuously decreasing in t and is equal to -1 at $t = 1$. If $E \times [0,1)$ were unambiguous according to Epstein and Zhang (2001), it would be the case that

$$\begin{pmatrix} 1 & if \;\; s \in \omega_3 \times [0,t^*) \\ 0 & if \;\; s \in \omega_2 \times [t^*,1) \\ 0 & if \qquad s \in C \\ u^{-1}(c) & if \;\; s \in E \times [0,1) \end{pmatrix} \sim \begin{pmatrix} 0 & if \;\; s \in \omega_3 \times [0,t^*) \\ 1 & if \;\; s \in \omega_2 \times [t^*,1) \\ 0 & if \qquad s \in C \\ u^{-1}(c) & if \;\; s \in E \times [0,1) \end{pmatrix}$$

for all $c \in \mathcal{U}$. In terms of our representation, this means

$$\begin{aligned} &\sum_{i=1}^{k} -e^{-\alpha\left[t^*\pi_i(\epsilon_n)(\omega_3\times[0,1))+c\pi_i(\epsilon_n)(E\times[0,1))\right]}\mu(i) \\ &= \sum_{i=1}^{k} -e^{-\alpha\left[(1-t^*)\pi_i(\epsilon_n)(\omega_2\times[0,1))+c\pi_i(\epsilon_n)(E\times[0,1))\right]}\mu(i) \;\text{ for all } c \in \mathcal{U}. \end{aligned} \tag{7}$$

Taking the derivative of both sides of equation (7) with respect to c and evaluating at $c = 0$,

$$\begin{aligned} &\sum_{i=1}^{k} -e^{-\alpha t^*\pi_i(\epsilon_n)(\omega_3\times[0,1))}\mu(i)\,\pi_i(\epsilon_n)(E \times [0,1)) \\ &= \sum_{i=1}^{k} -e^{-\alpha(1-t^*)\pi_i(\epsilon_n)(\omega_2\times[0,1))}\mu(i)\,\pi_i(\epsilon_n)(E \times [0,1)). \end{aligned} \tag{8}$$

In fact, since equation (7) is an identity in c, we can differentiate both sides as many times as we wish while maintaining equality. Calculating this out, we see

$$\begin{aligned}&\sum_{i=1}^{k} -e^{-\alpha t^* \pi_i(\epsilon_n)(\omega_3\times[0,1))}\mu(i)\left[\pi_i(\epsilon_n)(E\times[0,1))\right]^m \\ &= \sum_{i=1}^{k} -e^{-\alpha(1-t^*)\pi_i(\epsilon_n)(\omega_2\times[0,1))}\mu(i)\left[\pi_i(\epsilon_n)(E\times[0,1))\right]^m \qquad (9)\end{aligned}$$

for all $m = 0,1,2,...$. Since by construction of $\{\pi_i(\epsilon_n)\}_{i=1}^{k}$, either $\pi_i(\epsilon_n)(E\times[0,1)) = 0$, in which case those terms in the system of equations (9) get no weight, or, for the remaining i's we can strictly order the $\pi_i(\epsilon_n)$'s by the weight they give to $E\times[0,1)$. Let i^* be the index of the $\pi_i(\epsilon_n)$ that gives the largest such weight. Divide both sides of the each equation in the system of equations above by $[\pi_{i^*}(\epsilon_n)(E\times[0,1))]^m$. We can then rewrite the system as, for all $m = 0,1,2,\ldots,$

$$\begin{aligned}&-e^{-\alpha t^*\pi_{i^*}(\epsilon_n)(\omega_3\times[0,1))}\mu(i^*) \\ &\quad+\sum_{i\neq i^*}\frac{-e^{-\alpha t^*\pi_i(\epsilon_n)(\omega_3\times[0,1))}\mu(i)\left[\pi_i(\epsilon_n)(E\times[0,1))\right]^m}{\left[\pi_{i^*}(\epsilon_n)(E\times[0,1))\right]^m} \\ &= -e^{-\alpha(1-t^*)\pi_{i^*}(\epsilon_n)(\omega_2\times[0,1))}\mu(i^*) \\ &\quad+\sum_{i\neq i^*}\frac{-e^{-\alpha(1-t^*)\pi_i(\epsilon_n)(\omega_2\times[0,1))}\mu(i)\left[\pi_i(\epsilon_n)(E\times[0,1))\right]^m}{\left[\pi_{i^*}(\epsilon_n)(E\times[0,1))\right]^m}. \qquad (10)\end{aligned}$$

A necessary condition for the above system of equations to hold is that

$$-e^{-\alpha t^*\pi_{i^*}(\epsilon_n)(\omega_3\times[0,1))}\mu(i^*) = -e^{-\alpha(1-t^*)\pi_{i^*}(\epsilon_n)(\omega_2\times[0,1))}\mu(i^*). \qquad (11)$$

To see this, observe that for any $\delta > 0$ there exists an M such that for all $m > M$,

$$\begin{aligned}\delta >\ &\sum_{i\neq i^*}\frac{-e^{-\alpha t^*\pi_i(\epsilon_n)(\omega_3\times[0,1))}\mu(i)\left[\pi_i(\epsilon_n)(E\times[0,1))\right]^m}{\pi_{i^*}^n(\epsilon_n)(E\times[0,1))} \\ &-\sum_{i\neq i^*}\frac{-e^{-\alpha(1-t^*)\pi_i(\epsilon_n)(\omega_2\times[0,1))}\mu(i)\left[\pi_i^n(\epsilon_n)(E\times[0,1))\right]^m}{\pi_{i^*}^n(\epsilon_n)(E\times[0,1))} > -\delta.\end{aligned}$$

This is true because $0 \leq \pi_i(\epsilon_n)(E\times[0,1)) < \pi_{i^*}(\epsilon_n)(\omega_1\times[0,1)) \leq 1$ and all the other terms are bounded.

Given equation (11), we can cancel the i^* terms from both sides of the equations in the system (9). This gives a new system of equations. For this new system find the i such that $\pi_i(\epsilon_n)(E\times[0,1))$ gives the largest weight and repeat the above steps to show that

$$-e^{-\alpha t^*\pi_i(\epsilon_n)(\omega_3\times[0,1))}\mu(i) = -e^{-\alpha(1-t^*)\pi_i(\epsilon_n)(\omega_2\times[0,1))}\mu(i),$$

for that i. Canceling and repeating $k-2$ more times (or until the largest remaining $\pi_i(\epsilon_n)(E\times[0,1)) = 0$), we find

$$-e^{-\alpha t^*\pi_i(\epsilon_n)(\omega_3\times[0,1))} = -e^{-\alpha(1-t^*)\pi_i(\epsilon_n)(\omega_2\times[0,1))},$$

for all $i \in I$ such that $\pi_i(\epsilon_n)(\omega_1\times[0,1)) > 0$. This is only possible if

$$\frac{\pi_i(\omega_3\times[0,1))}{\pi_i(\omega_2\times[0,1))} = \frac{1-t^*}{t^*}$$

for all $i \in I$ such that $\pi_i(\epsilon_n)(\omega_1\times[0,1)) > 0$. As noticed above in the construction of $\Pi(\epsilon_n)$, this cannot be true. Therefore we have a contradiction and it cannot be that $E\times[0,1)$ unambiguous according to the definition in Epstein and Zhang (2001). ■

A.4.3 Theorem 3.2

We first report a result from the working paper version of Ghirardato and Marinacci (2002), where it is stated as Theorem 4.

Theorem A.1 *Let $\succeq$ be an ambiguity averse monotonic preference relation. For all $\geqslant_1$, $\geqslant_2$ belonging to the set of benchmark preferences corresponding to $\succeq$, we have*

$$\mathcal{H}^{gm}_{\geqslant_1} = \mathcal{H}^{gm}_{\geqslant_2} \equiv \mathcal{H}^{gm} \text{ and } \Lambda^{gm}_{\geqslant_1} = \Lambda^{gm}_{\geqslant_2} \equiv \Lambda^{gm}.$$

We now prove Theorem 3.2.

Proof of Theorem 3.2. Let $\geqslant$, the "benchmark", be the special case of $\succeq$ where ϕ is linear. That is, $\geqslant$ is represented by $V_{\geqslant}(f) = \sum_\omega \left[\int_{[0,1)} u(f(\omega,r))\,dr\right]\nu(\omega)$, where $\nu(\omega) = \int_I \pi_i(\{\omega \times [0,1)\})\,d\mu$ for all $\omega \in \Omega$.

Suppose ϕ is linear. Then $\mathcal{H}^{gm}_{\geqslant} = \mathcal{F}$. On the other hand, by Corollaries 2.2 and 2.3, ϕ linear means that all events are unambiguous. Hence, $\mathcal{H} = \mathcal{F}$ as well. For the rest of the proof we suppose ϕ is strictly concave or strictly convex over some open interval $J \subseteq \mathcal{U}$.

First we prove that $\mathcal{H} \subseteq \mathcal{H}^{gm}_{\geqslant}$. By Theorem 2.1, $\pi_i = \pi_j \equiv \bar{\pi}$ on Λ. Hence, we have, for $f \in \mathcal{H}$,

$$V(f) = \phi\left(\sum_\omega \left[\int_{[0,1)} u(f(\omega,r))\,dr\right]\bar{\pi}(\omega)\right). \tag{12}$$

Since ϕ is strictly increasing, the way $\succeq$ ranks acts belonging to $\mathcal{H}$ is equivalently given by the functional,

$$\sum_\omega \left[\int_{[0,1)} u(f(\omega,r))\,dr\right]\bar{\pi}(\omega).$$

Notice, $V_{\geqslant}(f)$ also reduces to the functional (12). Hence, $\succeq$ and $\geqslant$ agree on $\mathcal{H}$. Since $\mathcal{H}$ contains all the constant acts this proves that part (A) of Definition 3.4 holds for $\mathcal{H}$. On the other hand, it is immediate to see that part (B) holds as well. Thus, since $\mathcal{H}^{gm}_{\geqslant}$ is the largest subset satisfying (A) and (B), it follows that $\mathcal{H} \subseteq \mathcal{H}^{gm}_{\geqslant}$.

As to the converse inclusion, we consider the case of ϕ strictly concave on J (the strictly convex on J case being similar). We take an act $f \notin \mathcal{H}$ and go on to show that $f \notin \mathcal{H}^{gm}_{\geqslant}$ either. It is enough to consider acts f such that the expected utility of f according to each π in the support of μ lies in J. To see why this is the case, assume f is such that some of these expected utilities do not lie in J. Let $\{x_1, ..., x_n\}$ be the range of the finite-valued act f. Since u is strictly increasing on the interval $\mathcal{C}$, it is differentiable on $\mathcal{C}$ except on at most a countable subset M of $\mathcal{C}$. The function u is therefore locally Lipschitz on $\mathcal{C} - M$. Since J is an open interval and u is strictly increasing and continuous, $u^{-1}(J)$ is an open interval. Hence, $u^{-1}(J) \cap (\mathcal{C} - M) \neq \emptyset$, and so there exists $c \in u^{-1}(J)$ at which u is locally Lipschitz. Let $(c-\varepsilon, c+\varepsilon)$ be a neighborhood of c over which u is locally Lipschitz. Since J is an open interval, by taking ε small enough, we can assume that $[u(c)-\varepsilon, u(c)+\varepsilon] \subseteq u^{-1}(J)$. It is easy to check that there exist $\alpha \neq 0$ and $\beta \in \mathbb{R}$ such that $|\alpha x_i + \beta - c| \leq \varepsilon$ for each $i = 1, ...n$. Hence, by the local Lipschitz property, $|u(\alpha x_i + \beta) - u(c)| \leq \varepsilon$, and so

$$\left|\int u(\alpha f + \beta)\,d\pi - u(c)\right| \leq \varepsilon$$

for all probabilities $\pi \in \Delta$. Hence,

$$\left[\inf_{\pi \in \Delta} \int u(\alpha f + \beta)\,d\pi, \sup_{\pi \in \Delta} \int u(\alpha f + \beta)\,d\pi\right] \subseteq [u(c)-\varepsilon, u(c)+\varepsilon] \subseteq J.$$

Set $g = \alpha f + \beta$. As $g(s) = g(s')$ if and only if $f(s) = f(s')$ for all $s, s' \in S$, we have $\{f^{-1}(x) : x \in \mathcal{C}\} = \{g^{-1}(x) : x \in \mathcal{C}\}$. Hence, by (B) of Definition 3.4, $f \in \mathcal{H}^{gm}_{\geqslant}$ if and only if $g \in \mathcal{H}^{gm}_{\geqslant}$ and by Definition 2.2, $f \in \mathcal{H}$ if and only if $g \in \mathcal{H}$. All this shows that in what follows we can assume that the expected utility of f according to each π in the support of μ lies in J.

We denote the constant act valued at the μ-average expected utility of f by $\delta_{u^{-1}(e(\mu_f))}$. As $f \notin \mathcal{H}$, either $\delta_{u^{-1}(e(\mu_f))} \succ f$ or $\delta_{u^{-1}(e(\mu_f))} \sim f$. Suppose, first, the preference is strict. Since for the benchmark preference $\geqslant$, $\delta_{u^{-1}(e(\mu_f))}$ and f are indifferent, part (A) of Definition 3.4 is violated for f.

Next, suppose $\delta_{u^{-1}(e(\mu_f))} \sim f$. By the strict concavity of ϕ (on J) this implies that the expected utility of f under each π in the support of μ is the same. For, if not, then by strict concavity of ϕ,

$$V(f) = \int \phi\left[\int_S u(f(s))\,d\pi\right]d\mu < \phi\left(\int\int_S u(f(s))\,d\pi d\mu\right) = V\left(\delta_{u^{-1}(e(\mu_f))}\right), \tag{13}$$

and so $\delta_{u^{-1}(e(\mu_f))} \succ f$. Hence it remains to consider the case $f \notin \mathcal{H}$ with constant expected utility of f under each π in the support of μ. We show, next, that in this case condition (B) of Definition 3.4 is violated for f.

Since $f \notin \mathcal{H}$ there exists an $x \in \mathcal{C}$ such that the event $f^{-1}(x)$ is ambiguous. Call that event $A(x)$. Define $g : S \longrightarrow \mathcal{C}$ as follows,

$$g(s) = f(s) \text{ if } s \notin A(x),$$
$$g(s) = x' \text{ if } s \in A(x),$$

with $x' \notin \{f(s) : s \in S\}$ (recall that all our acts are finite valued). Since $A(x)$ is ambiguous there exist μ-non-null sets $I' \subseteq I$ and $I'' \subseteq I$ such that for any $i \in I'$ and $j \in I''$, $\pi_i(A(x)) \neq \pi_j(A(x))$. Since, there is constant expected utility of f under each π in the support of μ, for μ-almost-all i, j, $E_{\pi_i}(u \circ f) = E_{\pi_j}(u \circ f)$. Hence, for μ-almost-all $i \in I'$ and $j \in I''$, $E_{\pi_i}(u \circ f) = E_{\pi_j}(u \circ f)$. For all such i, j pairs,

$$\begin{aligned} E_{\pi_i} u(g) - E_{\pi_j} u(g) &= E_{\pi_i} u(f) - \pi_i(A(x))\left[u(x) - u(x')\right] \\ &\quad - E_{\pi_j} u(f) + \pi_j(A(x))\left[u(x) - u(x')\right] \\ &= \left[u(x) - u(x')\right]\left[\pi_j(A(x)) - \pi_i(A(x))\right] \\ &\neq 0. \end{aligned}$$

Therefore, it is not true that $E_{\pi_i}(u \circ g) = E_{\pi_j}(u \circ g)$ for μ-almost-all i, j. By strict concavity of ϕ (see Eq. 13) we then conclude that $\delta_{u^{-1}(e(\mu_g))} \succ g$. This violates part (B) of the Definition 3.4.

Hence, summing up, we may conclude $f \notin \mathcal{H}^{gm}_{\succcurlyeq}$. This completes the proof that $\mathcal{H} = \mathcal{H}^{gm}_{\succcurlyeq}$.

Since all acts are finite-valued, the λ-system Λ can be viewed as the set of all pre-image sets of the acts in $\mathcal{H}$. As $\Lambda^{gm}_{\succcurlyeq}$ is defined as the collection of all pre-image sets of acts in $\mathcal{H}^{gm}_{\succcurlyeq}$, it follows that $\Lambda = \Lambda^{gm}_{\succcurlyeq}$. ■

A.4.4 Theorem 3.3

According to the smooth ambiguity model, $f \succeq (f-a)E(f-b)$ if and only if

$$\int \phi\left[\int f(s)d\pi\right]d\mu \geq \int \phi\left[\int f(s)d\pi - \int_E a d\pi - \int_{E^c} b d\pi\right]d\mu. \tag{14}$$

Similarly, $aEb \succeq 0$ if and only if

$$\int \phi\left[\int_E a d\pi + \int_{E^c} b d\pi\right]d\mu \geq \int \phi[0]\, d\mu. \tag{15}$$

Let $E \in \Sigma$ be unambiguous according to Definition 2.1. By Theorem 2.1, there exists a $\gamma \in [0,1]$ such that $\pi(E) = \gamma$, μ-a.e.. In this case, (15) becomes

$$a\gamma + b(1-\gamma) \geq 0,$$

while (14) becomes

$$\int \phi\left[\int f(s)d\pi\right]d\mu \geq \int \phi\left[\int f(s)d\pi - (a\gamma + b(1-\gamma))\right]d\mu.$$

Since ϕ is strictly increasing, these are equivalent. Thus E is unambiguous according to Definition 3.5.■

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