**Risk sharing example: Pooling Investments**

A friend and I have initial wealth $W_a$ and $W_b$; $r_a$ and $r_b$ are our coefficients of absolute risk aversion.

We own portfolios which have a expected returns of $I_a$ and $I_b$, and variances $\text{Var}(I_a)$ and $\text{Var}(I_b)$, respectively.

Let $\beta$ be a cash transfer from me (indexed as a) to my friend (indexed as b).

Q: How do we efficiently split these streams of income?

To be determined:

How do we split the returns from each investment?  
Will there be side payments between us?  What will they be?

Let $a$ be my share of $I_a$, let $\beta$ be my share of $I_b$.

Let $\beta$ be a cash transfer from me (indexed as a) to my friend (indexed as b).

My utility/CEQ: 

\[
W_a - \beta + a I_a + \beta I_b + \frac{1}{2} r_a \text{var}(aI_a) + \frac{1}{2} r_a \text{var} (\beta I_b) \\
= W_a - \beta + a I_a + \beta I_b + \frac{1}{2} r_a \left[ a^2 \text{var}(I_a) + \beta^2 \text{var} (I_b) \right]
\]

Friend’s utility/CEQ: 

\[
W_b + \beta + (1-a) I_a + (1-\beta) I_b + \frac{1}{2} r_b \left[ (1-a)^2 \text{var}(I_a) + (1-\beta)^2 \text{var}(I_b) \right]
\]

Total value: $W_a + W_b + I_a + I_b + \text{Prem}_a + \text{Prem}_b$, where

\[
\text{Prem}_a = \frac{1}{2} \text{var}(I_a) \left[ r_a a^2 \% r_b (1-\alpha)^2 \right] \\
\text{Prem}_b = \frac{1}{2} \text{var}(I_b) \left[ r_a \beta^2 \% r_b (1-\beta)^2 \right]
\]

Note:

$C$  \quad \beta$ drops out: transfer does not affect efficiency

$C$  \quad a, \beta$ only enter through risk premia
Choose \( a, \beta \) to maximize total value:

\[
\frac{MV}{M} \cdot \left( r_a \var(I_a) \& r_b(1 \& a) \var(I_a) \right) \cdot 0
\]

\[
\frac{a}{1 \& a} \cdot \frac{r_b}{r_a}
\]

\[
\frac{MV}{M} \cdot 0 \bullet \frac{\beta}{1 \& \beta} \cdot \frac{r_b}{r_a}
\]

Therefore,

\[
a^\prime \beta^\prime \cdot \frac{r_b/r_a}{1 \%r_b/r_a}
\]

Efficiency dictates:

C Higher \( r_b/r_a \) (more risk-tolerant I am relative to my friend), more of the share I should take.

C Share does not depend on the riskiness of the stream.

C Does not depend on who initially owns which stream.

C Risk-neutral person bears entire risk.

C *Efficiency gains arise because pooling allows us to allocate risk so that it imposes least costs.*

Individual rationality dictates:

C ? must be chosen such that each of us does at least as well as we would if we did not pool our investments.

C range of possible values that satisfy individual rationality.