Ellsberg behavior: a monotonicity consideration and its implications*

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Abstract

Consider the set of acts generated from all (Anscombe-Aumann) mixtures of two acts f and g, $\{\alpha f + (1-\alpha)g : \alpha \in [0,1]\}$, and think of choosing from this set. We propose a preference condition, *monotonicity in mixtures*, which says that clearly improving the act f (in the sense of weak dominance) makes putting more weight on f more desirable. We show that this property has strong implications for preferences exhibiting behavior as in the classic Ellsberg (1961) paradoxes. For example, we show that maxmin expected utility (MEU) preferences (Gilboa and Schmeidler 1989) satisfy monotonicity in mixtures if and only if the set of probability measures appearing in the preference representation is a singleton set. In other words, for MEU preferences, monotonicity in mixtures and Ellsberg behavior are incompatible. MEU is not the only class of preferences for which this incompatibility holds, and we explore

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several directions in which this stark finding may be extended, including to any preferences that rely on kinks to generate Ellsberg behavior. Moreover, we demonstrate that the incompatibility is not between monotonicity in mixtures and Ellsberg behavior (or even ambiguity aversion more generally) *per se* by showing compatibility between these properties for other models of preferences. For example, we show that smooth ambiguity preferences (Klibanoff, Marinacci and Mukerji 2005) can satisfy both properties simultaneously as long as they exhibit a coefficient of relative ambiguity aversion everywhere less than one. This last result is closely related to findings from earlier literature on comparative statics of risky portfolios under expected utility.

1 Introduction

This paper explores the tension between two aspects of preferences. Consider the set of acts generated from all (Anscombe-Aumann) mixtures of two acts f and g: $\{\alpha f + (1-\alpha)g : \alpha \in [0,1]\}$ and think of preferences over this set as inducing preferences over α. As one varies the acts f and g under consideration, the resulting preferences over α would be expected to change. One force we might expect to influence this change is the idea that making one of the acts (say, f) more attractive makes higher weights on f more attractive. A conservative notion of "more attractive" is the notion of state-by-state (weak) dominance. We propose a preference condition, *Monotonicity in Mixtures*, which says that clearly improving the act f (in the sense of weak dominance) makes putting more weight on f more desirable.

For preferences as in Ellsberg's (1961) classic paradoxes, there is another force that might influence preferences over α. Acts corresponding to intermediate weights α may have value as a hedge against ambiguity when f and g perform well under different distributions, as, for example, where f corresponds to winning a prize only if a red ball is drawn, g corresponds to winning only if a blue ball is drawn, the composition of red vs. blue balls is unknown, and $\frac{1}{2}f + \frac{1}{2}g$ corresponds to a sure 50% chance of winning a prize. We define *Ellsberg Behavior* as preferences that, at least for some event and some prizes, strictly value this hedging.

Is this hedging influence on preferences over α compatible with its role in responding to improvements? Specifically, what are the implications of Monotonicity in Mixtures for some leading models of preferences displaying Ellsberg behavior? We will show that the implications are stark under conditions applying to a broad class of models – in many cases (including MEU, Choquet Expected Utility (Schmeidler, 1989), α-MEU and others) Monotonicity in Mixtures and Ellsberg behavior are incompatible. They are not always incompatible, however, and we also provide some informative positive results on compatibility using the smooth ambiguity model.

An example of insight from our results can be seen in Auster (2014, 2015), who studies bilateral trade under ambiguity about quality. Optimal offer behavior on the part of an ambiguity averse buyer derived there involves hedging-motivated mixing between a pooling price and a price that will be accepted only by a low quality seller. One comparative static Auster examines is what happens to the mixing weight as the buyer's valuation of the high quality seller's good increases. This corresponds to an improvement in the payoff to the pooling price in the sense of weak dominance. When the buyer has MEU preferences, in line with our result (Theorem 2) on incompatibility with Monotonicity of Mixtures, there are many cases where the optimal response is to offer the pooling price less often. Our results on the smooth ambiguity model (Theorems 3 and 4) explain why such a result could occur only with sufficiently strong ambiguity aversion.

This paper is organized as follows. In section 2, we describe the formal setting and notation. In section 3, we introduce the basic axioms on preferences that we maintain throughout. In section 4, we define Ellsberg Behavior. In section 5, we define Monotonicity in Mixtures. The main results of the paper, describing implications of Ellsberg Behavior and Monotonicity in Mixtures, are in Section 6. The final section discusses some extensions.

2 Setting and Preliminaries

We operate within a standard Fishburn (1970)-style version of an Anscombe-Aumann (1963) setting. Let S be the finite set of states. An event E is a subset of S. Let Z be the set of prizes or outcomes. X is the set of all simple lotteries over prizes (i.e., the set of all finite-support probability distributions on Z). Observe that X is a convex set with respect to the following mixture operation: for $\alpha \in [0,1]$, and $x, y \in X$,

$\alpha x + (1-\alpha)y$ is the element of X defined, for all $z \in Z$, by

$$(\alpha x + (1-\alpha)y)(z) \equiv \alpha x(z) + (1-\alpha)y(z).$$

Acts are functions from S to X. Let $\mathcal{F}$ denote the set of all acts. Acts are the objects of choice. Preferences will be defined by a binary relation $\succsim$ over acts. The symmetric and asymmetric parts of $\succeq$ are denoted by $\sim$ and $\succ$, respectively. Mixtures over acts are defined through statewise mixing of the resulting lotteries: for $\alpha \in [0,1]$, and $f, g \in \mathcal{F}$, $\alpha f + (1-\alpha)g$ is the act defined, for all $s \in S$, by

$$(\alpha f + (1-\alpha)g)(s) \equiv \alpha f(s) + (1-\alpha)g(s).$$

For $x, y \in X$ and an event E, let xEy denote the act f s.t. $\forall s \in E : f(s) = x, \forall s \notin E : f(s) = y$. Constant acts are those that give the same lottery in all states (i.e., $f(s) = f(s'), \forall s, s' \in S$). In a standard abuse of notation, we sometimes use x to denote the constant act giving $x \in X$ in each state. An act f is an interior act if, for each state s, there exist $\overline{x}(s), \underline{x}(s) \in X$ such that $\overline{x}(s) \succ f(s) \succ \underline{x}(s)$.

A set-function $\rho : 2^S \to \mathbb{R}$ is a *capacity* if $\rho(\emptyset) = 0$, $\rho(S) = 1$, and, for all $A, B \subseteq S$ with $A \subseteq B$, $\rho(A) \leq \rho(B)$.

3 Preferences

Throughout, we will restrict attention to $\succsim$ satisfying a few standard axioms: Weak Order, State-by-state Monotonicity, Risk Independence and Archimedean Continuity. In our setting, these axioms define the MBA preferences of Cerreia-Vioglio et al. (2011) and are equivalent to assuming $\succsim$ can be represented by

$$V\left((u(f(s)))_{s \in S}\right) \tag{3.1}$$

where $u : X \to \mathbb{R}$ is a non-constant, affine utility function and $V : u(X)^S \to \mathbb{R}$ is normalized, monotonic and sup-norm continuous. (Note: $u(f(s)) \equiv \sum_z u(z)f(s)(z)$)

Axiom 1. ***Weak Order***: *$\succsim$ are non-trivial, complete and transitive.*

Axiom 2. ***State-by-state Monotonicity***: *For all acts f, g, if $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.*

Axiom 3. ***Risk Independence***: *For all lotteries $x, y, z \in X$ and $\alpha \in (0, 1]$, if $x \succ y$ then $\alpha x + (1-\alpha)z \succ \alpha y + (1-\alpha)z$.*

Axiom 4. ***Archimedean Continuity***: *For all acts f, g, h, if $f \succ g \succ h$ then there exist $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h$.*

Although all are standard, of these axioms Risk Independence is probably the most controversial. It rules out non-expected utility behavior over lotteries, and thus the departures from expected utility that are allowed by MBA preferences concern aggregation across states. In this sense, we restrict attention to preferences that may violate subjective expected utility but obey expected utility under "objective" risk. An advantage of doing so is that our analysis may be carried out in utility space, greatly facilitating our arguments.

4 Ellsberg Behavior

Motivated by Ellsberg's two-color experiment we formalize "Ellsberg behavior" as follows:

Axiom 5. ***Ellsberg Behavior***: *There exists an event $E \subseteq S$, $w, x, y \in X$ with $w \succ x \succsim y$ and an $\alpha \in (0, 1)$ such that*

$$\alpha wEy + (1-\alpha)xEw \succ wEy \sim xEw \tag{4.1}$$

In the two-color experiment, for example, if red and black balls are treated symmetrically, taking E as the event a red ball is drawn from the unknown urn, w as

\$100 for sure, $x = y$ as \$0 for sure, and $\alpha = \frac{1}{2}$ so that $\alpha wEy + (1-\alpha)xEw$ gives a 50% chance of \$100 and a 50% chance of \$0 no matter what color is drawn turns (4.1) into the typical Ellsberg pattern of preferring a 50-50 bet to a bet on either color from the unknown urn. Allowing $x \neq y$, $\alpha \neq 1/2$, and flexibility in the choice of E is designed to accommodate possibilities including that not all events may be perceived as ambiguous, asymmetries in the perception of E versus E^c, and stake- and event-dependence in ambiguity attitudes.

Subjective expected utility (SEU) preferences cannot satisfy Ellsberg Behavior. One way to see this is to observe that (4.1) is a direct violation of the Anscombe-Aumann Independence axiom: letting $f \equiv wEy$ and $g \equiv xEw$, if $f \sim g$, then Independence implies $\alpha f + (1-\alpha)g \sim f \sim g$.

Ellsberg Behavior is meant to be a fairly minimal and "local" condition. Under the assumption that Independence is violated somewhere, it is much weaker than common "global" properties appearing in the ambiguity literature such as Uncertainty Aversion (Schmeidler, 1989), Ambiguity Aversion (Epstein 1999, Ghirardato and Marinacci 2002) or Sure Expected Utility Diversification (Chateauneuf and Tallon, 2002). Its weakness makes our results showing that for a broad class of preferences there is a conflict between it and Monotonicity in Mixtures more powerful.

5 A Monotonicity Consideration: Monotonicity in Mixtures

The main novel property we introduce is the following:

Axiom 6. ***Monotonicity in Mixtures:*** *For all acts f, f', g such that $f'(s) \succsim f(s)$ for all $s \in S$, and all numbers $\alpha', \alpha \in [0,1]$ such that $\alpha' \geq \alpha$,*

$$\alpha' f + (1-\alpha')g \underset{(\succ)}{\succsim} \alpha f + (1-\alpha)g \Longrightarrow \alpha' f' + (1-\alpha')g \underset{(\succ)}{\succsim} \alpha f' + (1-\alpha)g$$

Monotonicity in Mixtures says that if increasing the weight on f from α to α' is good, then doing so for a dominating act f' is also good. Moving from α to α' is a shift away from g towards f (or f'). In these terms, Monotonicity in Mixtures says that replacing f by a weakly dominating f' can only expand the shifts away from g that are desirable. All subjective expected utility preferences satisfy Monotonicity in Mixtures.

6 Implications of Monotonicity in Mixtures

There is a potential conflict between Monotonicity in Mixtures and Ellsberg Behavior. For preferences exhibiting Ellsberg Behavior, intermediate weights α may have value as a hedge against ambiguity when f and g perform well under different distributions over states. On the other hand, Monotonicity in Mixtures says that improving f (in the sense of weak dominance) makes putting more weight on f more desirable. Is the hedging role of α under Ellsberg Behavior compatible with its role in responding to improvements? What are the implications of Monotonicity in Mixtures for some leading models of preferences displaying Ellsberg Behavior? These are questions to which we now turn.

6.1 Implications for MEU

We begin by considering a seminal model of ambiguity averse preferences: the Maxmin Expected Utility with Non-Unique Prior (MEU) model (Gilboa and Schmeidler, 1989). Each MEU preference can be represented by a functional of the following form:

$$\min_{p \in C} \sum_{s} u(f(s))p(s), \tag{6.1}$$

where u is a non-constant von Neumann-Morgenstern utility function and C is a non-empty, closed and convex set of probability measures over states.

Notice that when the set C contains only one probability measure, preferences are SEU. In all other cases, MEU preferences display Ellsberg Behavior. Formally:

Proposition 1. *An MEU preference displays Ellsberg Behavior if and only if the set of measures C is not a singleton.*

Proof of Proposition 1. Since Ellsberg Behavior is incompatible with SEU, it implies C is non-singleton. For the other direction, suppose C is non-singleton. Then there exists an event A s.t. $min_{p \in C} p(A) \neq max_{p \in C} p(A)$. Let $p_1 \equiv min_{p \in C} p(A)$ and $p_2 \equiv max_{p \in C} p(A)$ and note that $0 \leq p_1 < p_2 \leq 1$. By non-constancy of u, there are outcomes (i.e., degenerate lotteries) $\overline{x}, \underline{x}$ such that $u(\overline{x}) > u(\underline{x})$. If $\overline{x} \succ x \succsim \underline{x}$, then (6.1) evaluates $\overline{x}A\underline{x}$ as $p_1 u(\overline{x}) + (1 - p_1)u(\underline{x})$ and $xA\overline{x}$ as $p_2 u(x) + (1 - p_2)u(\overline{x})$. There are two cases to consider:

Case 1: $p_1 + p_2 \geq 1$. To show Ellsberg Behavior, let $E = A$, $w = \overline{x}$, $y = \underline{x}$, $x = \frac{p_1+p_2-1}{p_2} w + \frac{1-p_1}{p_2} y$ and $\alpha = \frac{1-p_1}{1-p_1+p_2}$. Then $w \succ x \succsim y$, and (6.1) yields $\alpha wEy + (1 - \alpha)xEw \succ wEy \sim xEw$ since

$$\begin{aligned}
&\frac{1-p_1}{1-p_1+p_2} u(\overline{x}) + \frac{p_2}{1-p_1+p_2} \left(\frac{p_1+p_2-1}{p_2} u(\overline{x}) + \frac{1-p_1}{p_2} u(\underline{x})\right) \\
=&\frac{p_2}{1-p_1+p_2} u(\overline{x}) + \frac{1-p_1}{1-p_1+p_2} u(\underline{x}) > p_1 u(\overline{x}) + (1-p_1)u(\underline{x}) \\
=&p_2 \left(\frac{p_1+p_2-1}{p_2} u(\overline{x}) + \frac{1-p_1}{p_2} u(\underline{x})\right) + (1-p_2)u(\overline{x}) \\
=&p_2 u(x) + (1-p_2)u(\overline{x}).
\end{aligned}$$

Case 2: $p_1 + p_2 \leq 1$. To show Ellsberg Behavior, let $E = A^c$, $w = \overline{x}$, $y = \underline{x}$, $x = \frac{1-p_1-p_2}{1-p_1} w + \frac{p_2}{1-p_1} y$ and $\alpha = \frac{p_2}{1-p_1+p_2}$. Then $w \succ x \succsim y$, and (6.1) yields

$\alpha wEy + (1-\alpha)xEw \succ wEy \sim xEw$ since

$$
\begin{aligned}
&\frac{p_2}{1-p_1+p_2}u(\overline{x}) + \frac{1-p_1}{1-p_1+p_2}\left(\frac{1-p_1-p_2}{1-p_1}u(\overline{x}) + \frac{p_2}{1-p_1}u(\underline{x})\right) \\
=&\frac{1-p_1}{1-p_1+p_2}u(\overline{x}) + \frac{p_2}{1-p_1+p_2}u(\underline{x}) > (1-p_2)u(\overline{x}) + p_2 u(\underline{x}) \\
=&(1-p_1)\left(\frac{1-p_1-p_2}{1-p_1}u(\overline{x}) + \frac{p_2}{1-p_1}u(\underline{x})\right) + p_1 u(\overline{x}) \\
=&(1-p_1)u(x) + p_1 u(\overline{x}).
\end{aligned}
$$

□

Which MEU preferences satisfy Monotonicity in Mixtures?

Proposition 2. *An MEU preference satisfies Monotonicity in Mixtures if and only if the set of measures C is a singleton.*

Proof of Proposition 2. If C is a singleton, MEU preferences are SEU preferences, and therefore satisfy Monotonicity in Mixtures. Suppose C is non-singleton. By Proposition 1, these preferences display Ellsberg Behavior. The conclusion that Monotonicity in Mixtures is violated then follows as a special case of Theorem 2 in Section 6.3, since MEU preferences satisfy (6.7) with capacity $\rho(A) \equiv min_{p \in C} p(A)$ for all $A \subseteq S$. □

These results reveal that for the MEU model, Monotonicity in Mixtures and Ellsberg Behavior are incompatible. In particular, MEU preferences satisfy Monotonicity in Mixtures if and only if they are SEU preferences.

Next we present a result showing that this incompatibility extends well beyond MEU. The proof of this result provides a constructive argument (with associated graphical intuition) showing how Monotonicity in Mixtures is violated in the presence of Ellsberg Behavior generated by kinks in preferences. As a further application of this result, we will see that the same stark incompatibility found under MEU applies to all *c-linearly biseparable* (Ghirardato and Marinacci, 2001) preferences, a large class that includes not only MEU, but Choquet Expected Utility (Schmeidler, 1989), α-Maxmin Expected Utility and more.

6.2 Implications for Ellsberg Behavior generated by kinks

Definition 1. For disjoint, non-empty events $A, B \subseteq S$ and act h, let $\mathcal{F}^{A,B,h}$ denote the set of acts f such that $f(s) \sim f(t)$ for all $s, t \in A$, $f(s) \sim f(t)$ for all $s, t \in B$ and $f(s) = h(s)$ for all $s \notin A \cup B$.

Definition 2. Let $\succsim$ be represented by $V\left((u(f(s)))_{s\in S}\right)$ as in (3.1). For any disjoint, non-empty events $A, B \subseteq S$ and any act $k \in \mathcal{F}^{A,B,h}$, define $G(k) \equiv \{(u(f(A)), u(f(B))) : f \in \mathcal{F}^{A,B,h}$ and $f \succsim k\}$.

Theorem 1. *Let $\succsim$ be represented by $V\left((u(f(s)))_{s\in S}\right)$ as in* (3.1).

Fix disjoint, non-empty events $A, B \subseteq S$ and interior act $h \in \mathcal{F}^{A,B,h}$.

If there exist a rectangle $R \subseteq u(X) \times u(X)$ containing $(u(h(A)), u(h(B)))$ in its interior and two distinct lines in $u(X) \times u(X)$ that intersect $G(h) \cap R$ only at $(u(h(A)), u(h(B)))$, then $\succsim$ violates Monotonicity in Mixtures.

Proof of Theorem 1. Fix V, u, A, B and h as in the statement of the theorem. Suppose that R is a rectangle in $u(X) \times u(X)$ containing $(u(h(A)), u(h(B)))$ in its interior, and l_1 and l_2 are distinct lines in $u(X) \times u(X)$ that intersect $G(h) \cap R$ only at $(u(h(A)), u(h(B)))$. Graphically, we can represent acts in $\mathcal{F}^{A,B,h}$ as points in a two-dimensional Cartesian coordinate system with the vertical coordinate representing the utility level the act delivers in event B, and the horizontal coordinate representing the utility level the act delivers in event A. Monotonicity of V implies that all points $(a, b) \in u(X) \times u(X)$ such that $a \geq u(h(A))$ and $b \geq u(h(B))$ lie in $G(h)$. Therefore, since l_1 and l_2 intersect $G(h) \cap R$ only at $(u(h(A)), u(h(B)))$, it follows that both l_1 and l_2 must have negative and finite slopes.

The main part of our argument constructing a violation of Monotonicity in Mixtures will assume that l_1 and l_2 intersect both the left and bottom sides of the rectangle R. However, for a given R as above, this need not hold (see Figure 6.1 for an illustration). Therefore, before turning to the main construction, we show that the existence of an R as in the theorem implies the existence of a (possibly smaller) rectangle $R' \subseteq R$ containing $(u(h(A)), u(h(B)))$ in its interior and such that l_1 and l_2 intersect both the left and bottom sides of R' and intersect $G(h) \cap R'$ only at $(u(h(A)), u(h(B)))$. Observe that, starting from the given R, by moving the bottom side upwards (but not all the way to $u(h(B))$) and/or the left side rightwards (but not all the way to $u(h(A))$), we can ensure that l_1 and l_2 intersect both these left and bottom sides. The resulting rectangle, R', still contains $(u(h(A)), u(h(B)))$ in its interior, and, since $G(h) \cap R' \subseteq G(h) \cap R$, l_1 and l_2 intersect $G(h) \cap R'$ only at $(u(h(A)), u(h(B)))$. Thus, it is without loss of generality to assume, as we do for the remainder of this proof, that l_1 and l_2 intersect both the left and bottom sides of the rectangle R.

We now construct a violation of Monotonicity in Mixtures. Since l_1 and l_2 are distinct and contain a common point, their slopes must differ. Without loss of generality, let l_1 have the steeper slope. Therefore, for points with horizontal coordinate below $u(h(A))$, l_1 lies above l_2, while for points with horizontal coordinate above $u(h(A))$, l_2 lies above l_1. Fix acts $f, g, f', g' \in \mathcal{F}^{A,B,h}$ such that $(u(f(A)), u(f(B)))$ is the point where l_2 intersects the left side of R, $(u(g(A)), u(g(B)))$ is the point where l_1 intersects the bottom side of R, $(u(f'(A)), u(f'(B)))$ is the point where l_1 intersects the left side of R and $(u(g'(A)), u(g'(B)))$ is the point where l_2 intersects the bottom side of R (See Figure 6.2). Observe that f' weakly dominates f (with strict dominance only on B), and g' weakly dominates g (with strict dominance only on A). Further observe that the points on l_1 contained in R correspond to the acts $\lambda f' + (1-\lambda)g$, $\lambda \in [0,1]$ and the points on l_2 contained in R correspond to the acts $\lambda g' + (1-\lambda)f$, $\lambda \in [0,1]$. Let $\lambda_1, \lambda_2 \in (0,1)$ be the unique numbers such that $\lambda_1 f' + (1-\lambda_1)g$ and $\lambda_2 g' + (1-\lambda_2)f$ each correspond to $(u(h(A)), u(h(B)))$. It follows that $\lambda_1 u(f'(A)) + (1-\lambda_1)u(g(A)) =$

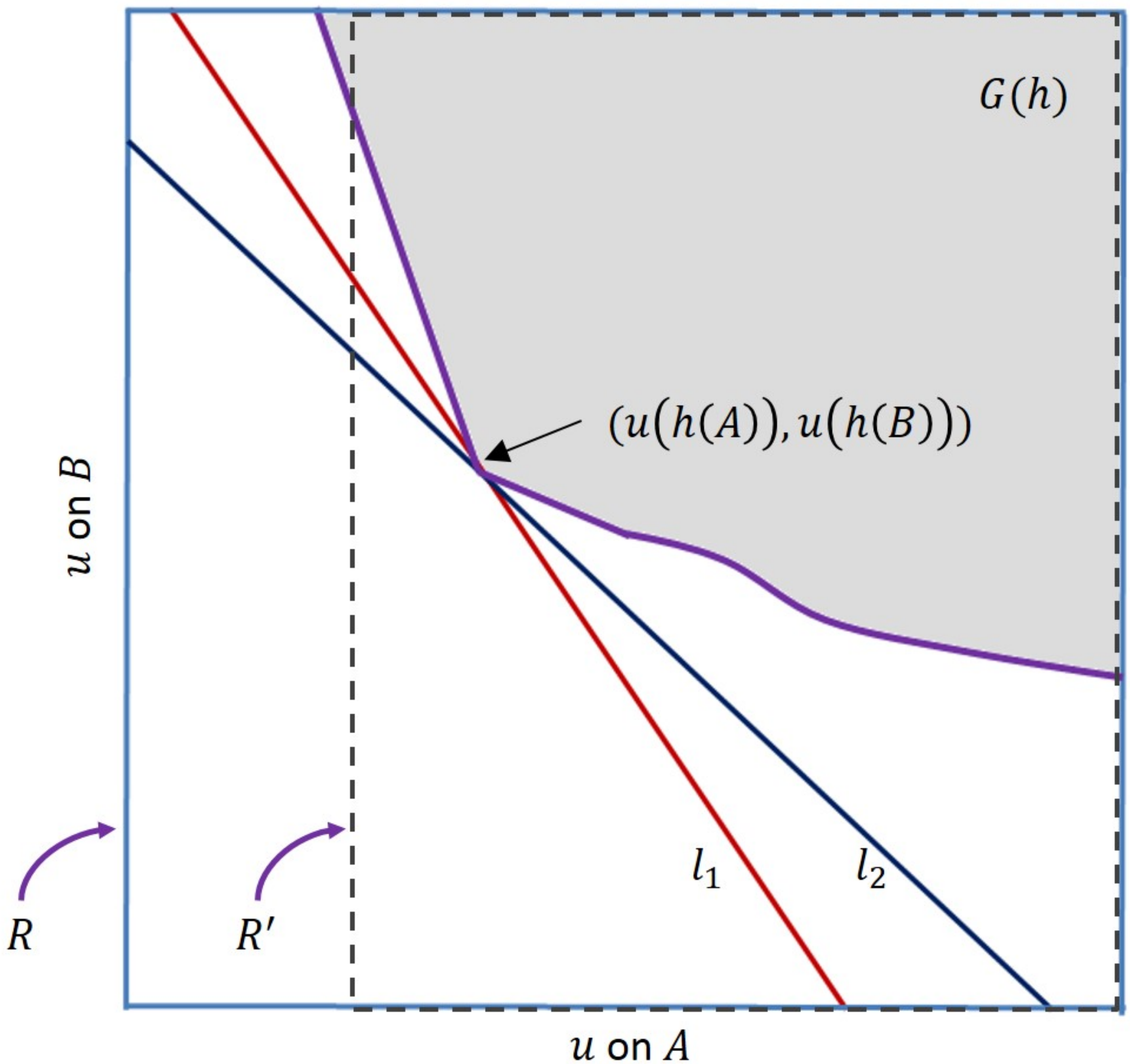

Figure 6.1: Shrinking rectangle R to get a smaller R' that lines l_1 and l_2 intersect on the left and bottom.

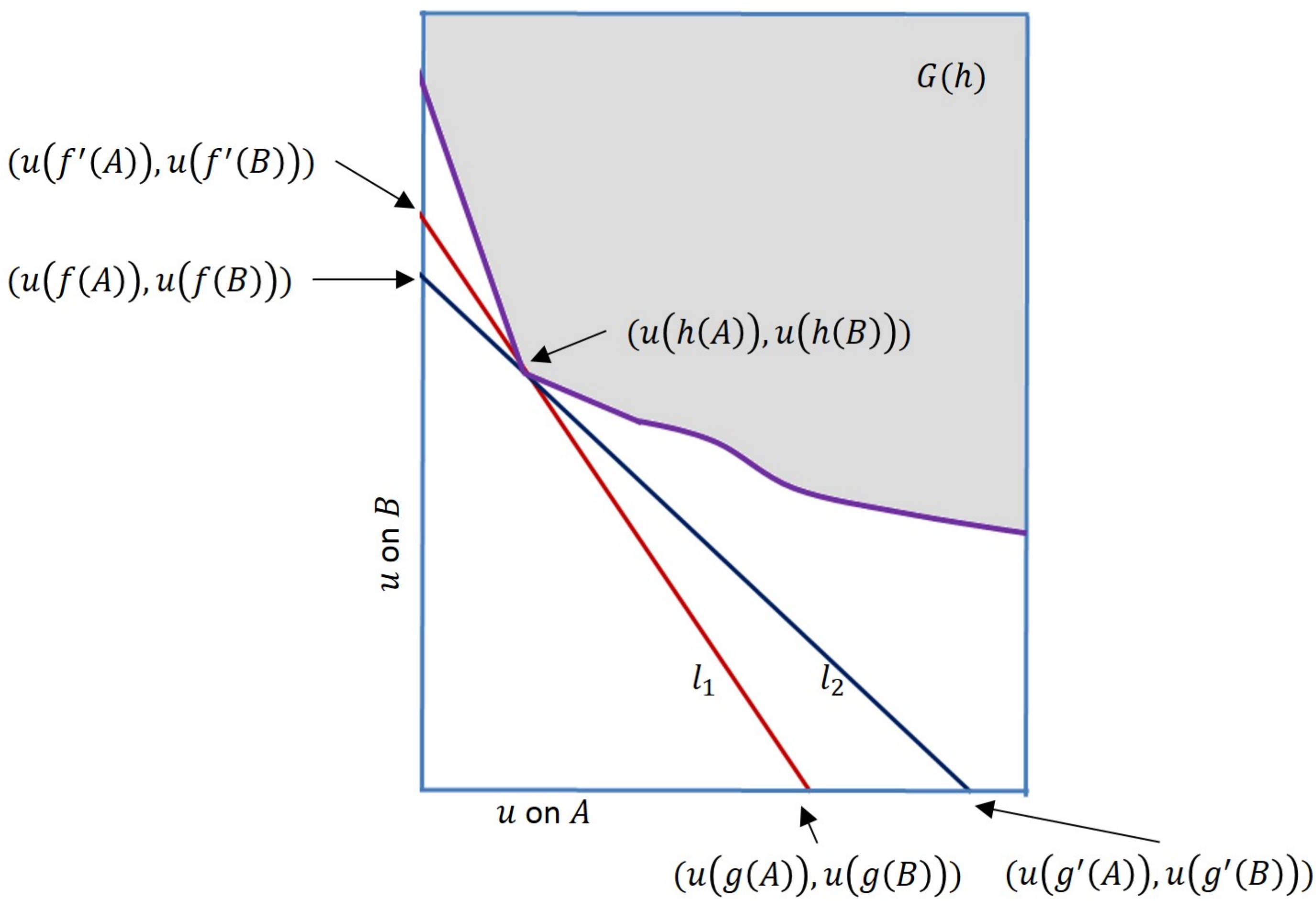

Figure 6.2: Choosing the acts f, g, f', g'

$\lambda_1 u(f(A)) + (1-\lambda_1)u(g(A)) = \lambda_2 u(g'(A)) + (1-\lambda_2)u(f(A))$, and thus

$$\lambda_2 < 1 - \lambda_1. \tag{6.2}$$

Since l_1 and l_2 intersect $G(h) \cap R$ only at $(u(h(A)), u(h(B)))$,

$$\lambda_1 f' + (1-\lambda_1)g \succ \lambda f' + (1-\lambda)g \text{ for all } \lambda \neq \lambda_1 \tag{6.3}$$

and

$$\lambda_2 g' + (1-\lambda_2)f \succ \lambda g' + (1-\lambda)f \text{ for all } \lambda \neq \lambda_2. \tag{6.4}$$

Because f' weakly dominates f, given (6.3), Monotonicity in Mixtures implies

$$\lambda_1 f + (1-\lambda_1)g \succ \lambda f + (1-\lambda)g \text{ for all } \lambda > \lambda_1. \tag{6.5}$$

Because g' weakly dominates g, given (6.5), Monotonicity in Mixtures implies

$$(1-\lambda_1)g' + \lambda_1 f \succ \lambda g' + (1-\lambda)f \text{ for all } \lambda < 1 - \lambda_1. \tag{6.6}$$

By (6.2), (6.6) contradicts (6.4). Therefore Monotonicity in Mixtures must be violated (and the violation occurs when applying the axiom to either the acts f', f, g or the acts g', g, f). (See also Figure 6.3, which illustrates this contradiction graphically. The yellow highlighted segments correspond to the acts on the right-hand sides of (6.5) and (6.6) respectively, which are strictly worse than the left-hand side acts corresponding to the lower endpoints of the highlighted segments). □

An implication of Theorem 1 is that any MBA preferences that use kinks (in at least some indifference curve in utility space) as their method of generating Ellsberg Behavior necessarily conflict with Monotonicity in Mixtures. This follows because such kinks allow there to exist the distinct lines on which the kink point is a "local" optimum (i.e., optimal among points on the line within the rectangle) that Theorem 1 relies on.

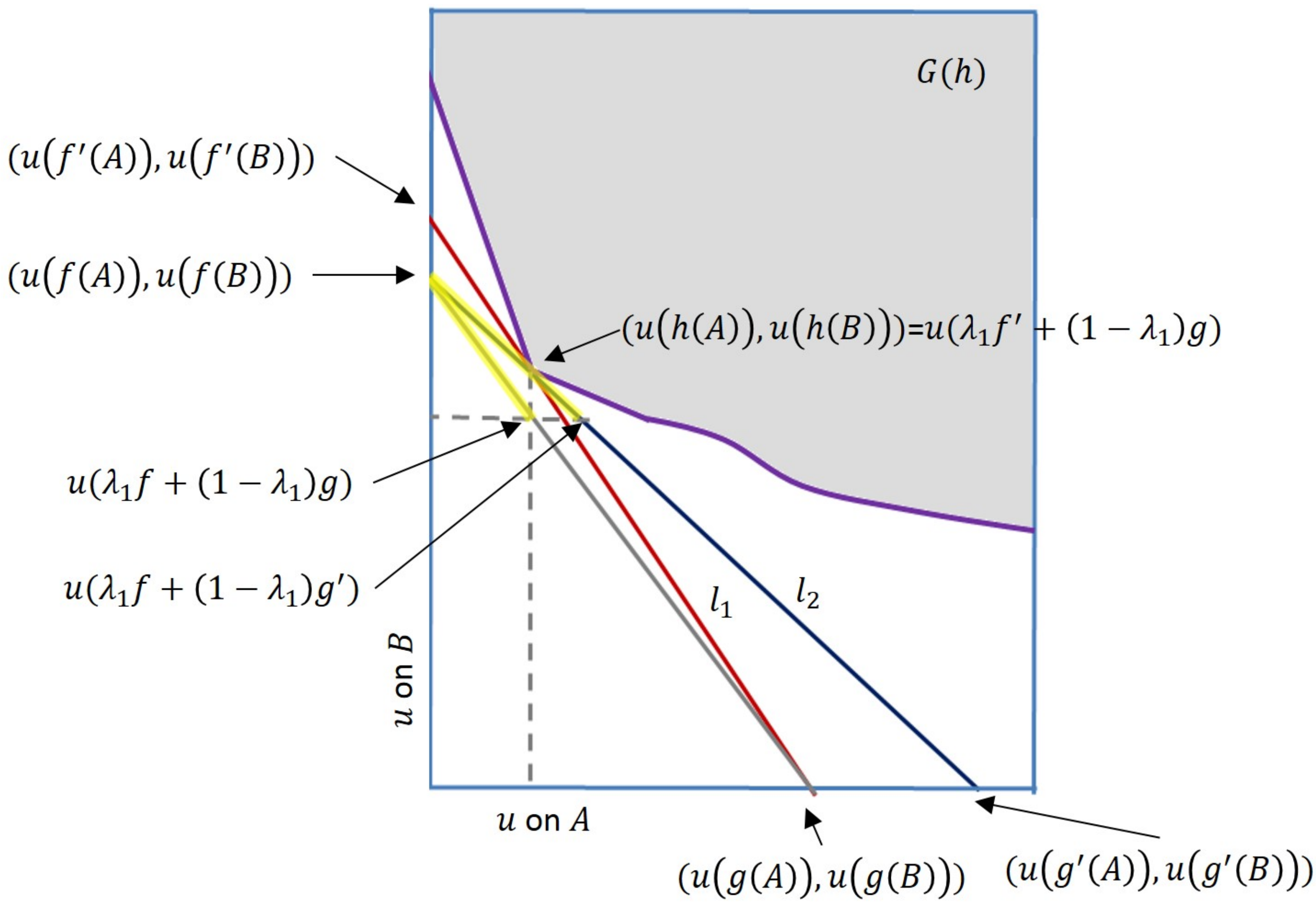

Figure 6.3: Violation of Monotonicity in Mixtures.

6.3 Implications for c-linearly biseparable preferences

Ghirardato and Marinacci (2001) define and axiomatize a broad class of preferences they call c-linearly biseparable. This class includes, among others, the well-known MEU, Choquet Expected Utility (Schmeidler, 1989), and α-MEU (where preference is represented by a convex combination of MEU and max-max EU) models. In terms of numerical representation, a key property satisfied by any c-linearly biseparable preference is that there is a unique capacity ρ such that

$$W(xEy) \equiv u(x)\rho(E) + u(y)(1 - \rho(E)) \tag{6.7}$$

represents $\succsim$ over acts of the form xEy for all $E \subseteq S$, $x, y \in X$ with $x \succsim y$.

Theorem 2. *All c-linearly biseparable preferences displaying Ellsberg Behavior must violate Monotonicity in Mixtures.*

Proof of Theorem 2: By Ellsberg Behavior, there exists an $E \subseteq S$, $w, x, y \in X$ with $w \succ x \succsim y$ and an $\alpha \in (0, 1)$ such that $\alpha wEy + (1 - \alpha)xEw \succ wEy \sim xEw$. By applying (6.7), we now show that, for this event E, $\rho(E) + \rho(E^c) < 1$. Since $wEy \sim xEw$, (6.7) implies

$$u(w)\rho(E) + u(y)(1 - \rho(E)) = u(w)\rho(E^c) + u(x)(1 - \rho(E^c)).$$

Thus,

$$\rho(E) + \rho(E^c) = \frac{2u(w) - u(x) - u(y)}{u(w) - u(x)}\rho(E) - \frac{u(x) - u(y)}{u(w) - u(x)}, \tag{6.8}$$

and, equivalently,

$$\rho(E) + \rho(E^c) = \frac{2u(w) - u(x) - u(y)}{u(w) - u(y)}\rho(E^c) + \frac{u(x) - u(y)}{u(w) - u(y)}. \tag{6.9}$$

There are two cases to consider:

Case 1: $\alpha w + (1 - \alpha)x \succsim \alpha y + (1 - \alpha)w$. By Ellsberg Behavior and (6.7), $(\alpha u(w) +$

$(1-\alpha)u(x))\rho(E) + (\alpha u(y) + (1-\alpha)u(w))(1-\rho(E)) > u(w)\rho(E) + u(y)(1-\rho(E))$. Thus,

$$\rho(E) < \frac{u(w) - u(y)}{2u(w) - u(x) - u(y)}.$$

Together with (6.8), this implies

$$\rho(E) + \rho(E^c) < \frac{u(w) - u(y)}{u(w) - u(x)} - \frac{u(x) - u(y)}{u(w) - u(x)} = 1,$$

as desired.

Case 2: $\alpha y + (1-\alpha)w \succ \alpha w + (1-\alpha)x$. By Ellsberg Behavior and (6.7), $(\alpha u(w) + (1-\alpha)u(x))(1-\rho(E^c)) + (\alpha u(y) + (1-\alpha)u(w))(\rho(E^c)) > u(w)\rho(E^c) + u(x)(1-\rho(E^c))$. Thus,

$$\rho(E^c) < \frac{u(w) - u(x)}{2u(w) - u(x) - u(y)}.$$

Together with (6.9), this again implies

$$\rho(E) + \rho(E^c) < \frac{u(w) - u(x)}{u(w) - u(y)} + \frac{u(x) - u(y)}{u(w) - u(y)} = 1,$$

as desired.

Since for this E, $\rho(E) + \rho(E^c) < 1$, we can now use Theorem 1: Apply Theorem 1, with $A = E$, $B = E^c$, $R = u(X) \times u(X)$, h an interior constant act, and lines through $(u(h(E)), u(h(E^c)))$ with slopes $\frac{1}{4}(-\frac{1-\rho(E)}{\rho(E)}) + \frac{3}{4}(-\frac{\rho(E^c)}{1-\rho(E^c)})$ and $\frac{3}{4}(-\frac{1-\rho(E)}{\rho(E)}) + \frac{1}{4}(-\frac{\rho(E^c)}{1-\rho(E^c)})$ respectively (if $\rho(E) = 0$, replace $-\frac{1-\rho(E)}{\rho(E)}$ by any finite number n such that $n < -\frac{\rho(E^c)}{1-\rho(E^c)}$) to conclude that Monotonicity in Mixtures is violated. $\square$

6.4 Implications for the Smooth Ambiguity Model

Are there preferences that can both satisfy Monotonicity in Mixtures and exhibit Ellsberg Behavior? In this section, we show that the answer is yes. To do so, we consider the smooth ambiguity model (Klibanoff, Marinacci and Mukerji, 2005). In our setting, each smooth ambiguity preference can be represented by a functional of the following form:

$$\int \phi \left(\sum_s u(f(s))p(s) \right) d\mu(p), \tag{6.10}$$

where u is a non-constant von Neumann-Morgenstern utility function, ϕ is a continuous and strictly increasing function and μ is a countably additive probability measure over probability measures over states. For some results in this section we further assume that ϕ is twice continuously differentiable with $\phi' > 0$ and $\phi'' < 0$ and $u(X) = [0, \infty)$. Such preferences exhibit Ellsberg Behavior if and only if μ has a non-singleton support (Proposition 3). We provide an upper bound on the coefficient of ambiguity aversion, $-\dfrac{\phi''(x)}{\phi'(x)}$, that is sufficient and, if one wants not to restrict μ, necessary for such preferences to satisfy Monotonicity in Mixtures (Theorems 3 and 4). Thus, when applied to smooth ambiguity preferences, Monotonicity in Mixtures is compatible with ambiguity aversion as long as the aversion isn't too strong.

Proposition 3. *Preferences represented by the smooth ambiguity model as in* (6.10) *with ϕ strictly increasing and strictly concave exhibit Ellsberg Behavior if and only if μ has a non-singleton support.*

Proof of Proposition 3: If μ has only one measure in its support, then (6.10) reduces to a strictly increasing transformation of an SEU preference and thus cannot exhibit Ellsberg Behavior. For the other direction, suppose that μ has a non-singleton support. Then there exists an event A s.t. $min_{p \in supp(\mu)} p(A) < \int p(A) d\mu(p) < max_{p \in supp(\mu)} p(A)$. By non-constancy of u, there are outcomes (i.e., degenerate lotteries) $\overline{x}, \underline{x}$ such that $u(\overline{x}) > u(\underline{x})$. There are two cases to consider:

Case 1: $\overline{x} A \underline{x} \succsim \underline{x} A \overline{x}$. To show Ellsberg Behavior, let $E = A$, $w = \overline{x}$, $y = \underline{x}$, $\alpha = \frac{1}{2}$, and x be the lottery $\lambda \overline{x} + (1-\lambda)\underline{x}$ such that $\overline{x} A \underline{x} \sim x A \overline{x}$. Then $w \succ x \succsim y$, and (6.10) yields $\alpha wEy + (1-\alpha)xEw \succ wEy \sim xEw$ since strict concavity of ϕ implies

$$\int \phi\left(\frac{1}{2}u(\overline{x})p(A) + \frac{1}{2}u(\underline{x})(1-p(A)) + \frac{1}{2}u(x)p(A) + \frac{1}{2}u(\overline{x})(1-p(A))\right) d\mu(p)$$
$$> \int \left(\frac{1}{2}\phi\left(u(\overline{x})p(A) + u(\underline{x})(1-p(A))\right) + \frac{1}{2}\phi\left(u(x)p(A) + u(\overline{x})(1-p(A))\right)\right) d\mu(p).$$

Case 2: $\overline{x}A^c\underline{x} \succsim \underline{x}A^c\overline{x}$. To show Ellsberg Behavior, let $E = A^c$, $w = \overline{x}$, $y = \underline{x}$, $\alpha = \frac{1}{2}$, and x be the lottery $\lambda\overline{x} + (1-\lambda)\underline{x}$ such that $\overline{x}A^c\underline{x} \sim xA^c\overline{x}$. Then $w \succ x \succsim y$, and (6.10) yields $\alpha wEy + (1-\alpha)xEw \succ wEy \sim xEw$ since strict concavity of ϕ implies

$$\int \phi\left(\frac{1}{2}u(\underline{x})p(A) + \frac{1}{2}u(\overline{x})(1-p(A)) + \frac{1}{2}u(\overline{x})p(A) + \frac{1}{2}u(x)(1-p(A))\right) d\mu(p)$$
$$> \int \left(\frac{1}{2}\phi\left(u(\underline{x})p(A) + u(\overline{x})(1-p(A))\right) + \frac{1}{2}\phi\left(u(\overline{x})p(A) + u(x)(1-p(A))\right)\right) d\mu(p).$$

□

Theorem 3. *Preferences represented by the smooth ambiguity model as in* (6.10) *with ϕ twice continuously differentiable, $\phi' > 0$, $\phi'' \leq 0$ and $u(X) = [0, \infty)$ satisfy Monotonicity in Mixtures if ϕ is everywhere at most as concave as natural log:*

$$-\frac{\phi''(a)}{\phi'(a)} \leq \frac{1}{a}, \text{ for all } a > 0.$$

Proof of Theorem 3: For $\alpha \in [0,1]$, $v \in u(X)^S$ and act $g \in \mathcal{F}$, define

$$W^g(\alpha, v) \equiv \int \phi\left(\sum_{\hat{s}}(\alpha v(\hat{s}) + (1-\alpha)u(g(\hat{s})))p(\hat{s})\right) d\mu(p). \qquad (6.11)$$

If each of the cross-partial derivatives with respect to α and the s^{th} component of v, $W^g_{\alpha v(s)}(\alpha, v)$, are always non-negative, this is sufficient for Monotonicity in Mixtures, since it implies that the increased state-by-state utility from improving f to a weakly dominating f' can only increase the desirability of increasing the mixing weight from

α to α'. We now show that when $-\frac{\phi''(a)}{\phi'(a)} \leq \frac{1}{a}$, for all $a > 0$ these cross-partials are indeed non-negative. By differentiating, we obtain

$$W^g_\alpha(\alpha, v) = \int \left(\sum_{\hat{s}} (v(\hat{s}) - u(g(\hat{s})))p(\hat{s}) \right) \phi' \left(\sum_{\hat{s}} (\alpha v(\hat{s}) + (1-\alpha)u(g(\hat{s})))p(\hat{s}) \right) d\mu(p)$$

and

$$\begin{aligned} & W^g_{\alpha v(s)}(\alpha, v) \\ = & \int \left[p(s)\phi' \left(\sum_{\hat{s}} (\alpha v(\hat{s}) + (1-\alpha)u(g(\hat{s})))p(\hat{s}) \right) \right. \\ & \left. + \alpha p(s) \left(\sum_{\hat{s}} (v(\hat{s}) - u(g(\hat{s})))p(\hat{s}) \right) \phi'' \left(\sum_{\hat{s}} (\alpha v(\hat{s}) + (1-\alpha)u(g(\hat{s})))p(\hat{s}) \right) \right] d\mu(p). \end{aligned} \tag{6.12}$$

From $-\frac{\phi''(a)}{\phi'(a)} \leq \frac{1}{a}$ with $a = \sum_{\hat{s}} (\alpha v(\hat{s}) + (1-\alpha)u(g(\hat{s})))p(\hat{s}))$, we obtain that, for all p,

$$\begin{aligned} & \phi' \left(\sum_{\hat{s}} (\alpha v(\hat{s}) + (1-\alpha)u(g(\hat{s})))p(\hat{s}) \right) \\ \geq & - \phi'' \left(\sum_{\hat{s}} (\alpha v(\hat{s}) + (1-\alpha)u(g(\hat{s})))p(\hat{s}) \right) \left(\sum_{\hat{s}} (\alpha v(\hat{s}) + (1-\alpha)u(g(\hat{s})))p(\hat{s}) \right). \end{aligned} \tag{6.13}$$

Substituting (6.13) into (6.12) yields,

$$W^g_{\alpha v(s)}(\alpha, v) \geq \int \left[p(s)\phi'' \left(\sum_{\hat{s}} (\alpha v(\hat{s}) + (1-\alpha)u(g(\hat{s})))p(\hat{s}) \right) \left(- \sum_{\hat{s}} u(g(\hat{s}))p(\hat{s}) \right) \right] d\mu(p) \geq 0$$

as claimed. □

Theorem 4. *For ϕ twice continuously differentiable, $\phi' > 0$, $\phi'' \leq 0$ and $u(X) = [0, \infty)$, if $-\frac{\phi''(a)}{\phi'(a)} > \frac{1}{a}$, for some $a > 0$, then there exists a measure μ such that preferences represented by* (6.10) *violate Monotonicity in Mixtures.*

Proof of Theorem 4: By assumption, $\exists a > 0$ s.t $-\frac{\phi''(a)}{\phi'(a)} > \dfrac{1}{a}$ which, since $\phi'(a) > 0$, implies $\phi'(a) + a\phi''(a) < 0$.

We construct a μ that will generate a violation of Monotonicity in Mixtures. Let

$\mu(p_1) = \mu(p_2) = \frac{1}{2}$, where p_1 and p_2 are probability measures on S such that $\exists E \subset S$ with $p_1(E) = 1$ and $p(E) = 0$.

Consider the following acts f and g: $f = xEz$ and $g = zEx$ where $u(x) = 2a$ and $u(z) = 0$.

By concavity of ϕ, $\phi(a) \geq \frac{1}{2}\phi(\alpha a) + \frac{1}{2}\phi((1-\alpha)a)$, which implies that for all $\alpha \in [0,1]$,

$$\frac{1}{2}f + \frac{1}{2}g \succsim \alpha f + (1-\alpha)g \tag{6.14}$$

Define $\overline{W}(\alpha, v) \equiv W^{zEx}(\alpha, vE0)$ where $W^{zEx}(\alpha, vE0)$ is defined as in (6.11) in the proof of Theorem 3 with $g = zEx$ and $vE0$ is the vector representing the state-by-state utility of the act yEz where $u(y) = v$. Substituting our constructed μ, we get

$$\overline{W}(\alpha, v) = \frac{1}{2}[\phi(\alpha v) + \phi((1-\alpha)2a)].$$

Differentiating w.r.t. α yields

$$\overline{W}_\alpha(\alpha, v) = \frac{1}{2}[v\phi'(\alpha v) - 2a\phi'((1-\alpha)2a)] \tag{6.15}$$

Further differentiating w.r.t. v, we get

$$\overline{W}_{\alpha v}(\alpha, v) = \frac{1}{2}[\phi'(\alpha v) + \alpha v\phi''(\alpha v)] \tag{6.16}$$

Now, observe that $\overline{W}_{\alpha v}(\alpha, v)$ is negative when $\alpha = \frac{1}{2}$ and $v = 2a$:

$$\overline{W}_{\alpha v}(\frac{1}{2}, 2a) = \frac{1}{2}[\phi'(a) + a\phi''(a)] < 0. \tag{6.17}$$

From (6.15), $\overline{W}_\alpha(\frac{1}{2}, 2a) = 0$, and thus, by (6.17), there exists a $b > 2a$ such that

$$\overline{W}_\alpha(\frac{1}{2}, b) < 0. \tag{6.18}$$

Therefore, there exists an $\hat{\alpha} < \frac{1}{2}$ such that

$$\overline{W}(\frac{1}{2}, b) < \overline{W}(\hat{\alpha}, b). \tag{6.19}$$

Letting $f' = yEz$ and setting $v = b$, we see that f' weakly dominates f. However, while from (6.14) $\frac{1}{2}f + \frac{1}{2}g \succsim \hat{\alpha}f + (1-\hat{\alpha})g$, (6.19) implies $\hat{\alpha}f' + (1-\hat{\alpha})g \succ \frac{1}{2}f' + \frac{1}{2}g$, violating Monotonicity in Mixtures. $\square$

These results on Monotonicity in Mixtures in the context of the smooth ambiguity model are closely related to work on comparative statics of portfolios of random variables (risky assets) under expected utility. A strand of that literature addresses the question of when any first-order stochastic dominant shift in the (conditional on any realization of the other assets) distribution of an asset will result in a risk-averse expected utility investor increasing that asset's share in the optimal portfolio. The answer is when utility is everywhere at most as concave as natural log (equivalently, $xu'(x)$ increasing, absoute risk aversion at any $x \leq \frac{1}{x}$, or relative risk aversion everywhere at most 1). See Mitchell and Douglas (1997), Meyer and Ormiston (1994), Hadar and Seo (1990), Fishburn and Burr Porter (1976).

The condition that utility is at most as concave as natural log also appears in the literature on general equilibrium with additively separable utilities, where it has been identified as leading demand for contingent goods to have the gross substitutes property and to existence of a unique equilibrium (see Dana (2001) for a survey).

7 Extensions

7.1 Bounds on slopes of indifference curves in utility space at different points

Our result (Theorem 1) ruling out Ellsberg behavior-generating kinks shows how multiple slopes at a given point on the boundary of the better than set generates a violation of Monotonicity in Mixtures. This idea can be extended to show that for smooth parts of boundaries, given two distinct points Monotonicity in Mixtures places a bound on the ratio of the slopes of the boundaries at those two points. Thus Monotonicity in Mixtures is necessarily violated whenever these slopes change "too fast".

Theorem 5. *Let* $\succsim$ *be represented by* $V\left((u(f(s)))_{s\in S}\right)$ *as in* (3.1).

Fix disjoint, non-empty events $A, B \subseteq S$ *and interior acts* $h, k \in \mathcal{F}^{A,B,h}$ *such that* $k(A) \succsim h(A)$ *and* $h(B) \succsim k(B)$.

If there exist a rectangle $R \subseteq u(X) \times u(X)$ *containing* $(u(h(A)), u(h(B)))$ *and* $(u(k(A)), u(k(B)))$ *in its interior and two distinct lines* l_h *and* l_k *in* $u(X) \times u(X)$ *such that*

(i) l_h intersects $G(h) \cap R$ only at $(u(h(A)), u(h(B)))$ and has an intersection with $\{(q(A), q(B)) \in R : q(A) = \min_{r\in R} r(A)\}$,

(ii) l_k intersects $G(k) \cap R$ only at $(u(k(A)), u(k(B)))$ and has an intersection with $\{(q(A), q(B)) \in R : q(B) = \min_{r\in R} r(B)\}$,

(iii) the line, l_3, passing through $(u(k(A)), u(h(B)))$ and the intersection of l_h with $\{(q(A), q(B)) \in R : q(A) = \min_{r\in R} r(A)\}$ has an intersection with $\{(q(A), q(B)) \in R : q(B) = \min_{r\in R} r(B)\}$, and

(iv) $\frac{Slope(l_h)}{Slope(l_k)} > \frac{u(k(A))-\min_{r\in R} r(A)}{u(h(A))-\min_{r\in R} r(A)} \times \frac{u(h(B))-\min_{r\in R} r(B)}{u(k(B))-\min_{r\in R} r(B)}$, then $\succsim$ violates Monotonicity in Mixtures.

Proof of Theorem 5: Fix V, u, A, B, h, k, R, l_h and l_k as in the statement of the theorem. Graphically, we can represent acts in $\mathcal{F}^{A,B,h}$ as points in a two-dimensional Cartesian coordinate system with the vertical coordinate representing the utility level the act delivers in event B, and the horizontal coordinate representing the utility level the act delivers in event A. Monotonicity of V implies that, for any $f \in \mathcal{F}^{A,B,h}$ all points $(a, b) \in u(X) \times u(X)$ such that $a \geq u(f(A))$ and $b \geq u(f(B))$ lie in $G(f)$. Therefore, since l_h intersects $G(h) \cap R$ only at $(u(h(A)), u(h(B)))$ and l_k intersects $G(k) \cap R$ only at $(u(k(A)), u(k(B)))$, it follows that both l_h and l_k must have negative and finite slopes, as does l_3 by construction. Name the acts in $\mathcal{F}^{A,B,h}$ corresponding to the intersections of l_h, l_k, l_3 with the edges of R as specified in (i),(ii) and (iii) of the theorem by g, f and f' respectively.

Observe that the slope of l_3 is the following:

$$\begin{aligned} Slope(l_3) &= -\frac{u(g(B)) - u(h(B))}{u(k(A)) - u(g(A))} = -\frac{u(g(B)) - u(h(B))}{u(h(A)) - u(g(A))} \times \frac{u(h(A)) - u(g(A))}{u(k(A)) - u(g(A))} \\ &= Slope(l_h) \times \frac{u(h(A)) - u(g(A))}{u(k(A)) - u(g(A))}. \end{aligned} \tag{7.1}$$

Now let l_4 be the line passing through f' and k. The slope of l_4 is:

$$\begin{aligned} Slope(l_4) &= -\frac{u(k(B)) - u(f(B))}{u(f'(A)) - u(k(A))} = -\frac{u(h(B)) - u(f(B))}{u(f'(A)) - u(k(A))} \times \frac{u(k(B)) - u(f(B))}{u(h(B)) - u(f(B))} \\ &= Slope(l_3) \times \frac{u(k(B)) - u(f(B))}{u(h(B)) - u(f(B))}. \end{aligned} \tag{7.2}$$

Combining (7.1) and (7.2) yields,

$$Slope(l_4) = Slope(l_h) \times \frac{u(h(A)) - u(g(A))}{u(k(A)) - u(g(A))} \times \frac{u(k(B)) - u(f(B))}{u(h(B)) - u(f(B))}. \tag{7.3}$$

If $\succsim$ satisfies Monotonicity in Mixtures, then, since by construction h is the unique optimum on $l_h \cap R$, it must be that all optima on $l_3 \cap R$ have vertical coordinates weakly below $u(h(B))$. Applying Monotonicity in Mixtures a second time yields that, since all optima on $l_3 \cap R$ have vertical coordinates weakly below $u(h(B))$, all optima on $l_4 \cap R$ have vertical coordinates weakly below $u(k(B))$. From this, we will now show that if $\mathit{Slope}(l_4) < \mathit{Slope}(l_k)$ then Monotonicity in Mixtures must be violated. To see this, suppose to the contrary that $\mathit{Slope}(l_4) < \mathit{Slope}(l_k)$ and Monotonicity in Mixtures is satisfied. Since k uniquely optimal on $l_k \cap R$ (by the assumptions of the Theorem), V is monotonic and $\mathit{Slope}(l_4) < \mathit{Slope}(l_k)$, $k \succ j$ for all $(u(j(A)), u(j(B))) \in l_4 \cap R$ having vertical coordinate strictly below $u(k(B))$ (as each such $(u(j(A)), u(j(B)))$ is weakly dominated by some point on $l_k \cap R$ that is strictly worse than $(u(k(A)), u(k(B)))$. Thus, k strictly optimal on $l_4 \cap R$. However, k, R, l_k and l_4 now satisfy the conditions of Theorem 1 and therefore imply Monotonicity in Mixtures is violated, a contradiction. Therefore, we have shown that

$$\frac{\mathit{Slope}(l_4)}{\mathit{Slope}(l_k)} > 1 \tag{7.4}$$

implies violation of Monotonicity in Mixtures.

Applying (7.3) yields

$$\frac{\mathit{Slope}(l_4)}{\mathit{Slope}(l_k)} = \frac{\mathit{Slope}(l_h)}{\mathit{Slope}(l_k)} \times \frac{u(h(A)) - u(g(A))}{u(k(A)) - u(g(A))} \times \frac{u(k(B)) - u(f(B))}{u(h(B)) - u(f(B))}.$$

Therefore,

$$\frac{\mathit{Slope}(l_h)}{\mathit{Slope}(l_k)} > \frac{u(k(A)) - u(g(A))}{u(h(A)) - u(g(A))} \times \frac{u(h(B)) - u(f(B))}{u(k(B)) - u(f(B))} \tag{7.5}$$

implies violation of Monotonicity in Mixtures. Since $u(g(A)) = \min_{r \in R} r(A)$ and $u(f(B)) = \min_{r \in R} r(B)$, (7.5) is the inequality in the statement of the theorem. $\square$

When V is differentiable in the utility values, (7.5) may be written entirely in terms of V and u, as

$$Slope(l_h) = -\frac{V_A}{V_B} \mid_{u(h)}$$

and

$$Slope(l_k) = -\frac{V_A}{V_B} \mid_{u(k)}$$

where for $E \subseteq S$, V_E denotes the derivative of V with respect to the utility attained on the event E.

One application of this generalization is to the class of Variational preferences (Maccheroni, Marinacci and Rustichini, 2006): These preferences need not have kinks to display Ellsberg Behavior. Nonetheless, when utility is unbounded above, all Variational preferences generating Ellsberg Behavior must violate Monotonicity in Mixtures because they can be shown to somewhere violate the slope bounds.

7.2 Monotonicity in Optimal Mixtures

The following is a weakening of Monotonicity in Mixtures to have a more directly comparative static flavor:

Axiom 7. ***Monotonicity in Optimal Mixtures****: For all acts f, f', g such that $f'(s) \succsim f(s)$ for all $s \in S$, $\alpha^* f + (1-\alpha^*)g$ optimal in $\{\alpha f + (1-\alpha)g : \alpha \in [0,1]\}$ implies there exists an $\alpha' \geq \alpha^*$ such that $\alpha' f' + (1-\alpha')g$ is optimal in $\{\alpha f' + (1-\alpha)g : \alpha \in [0,1]\}$.*

It is immediate that Monotonicity in Mixtures implies Monotonicity in Optimal Mixtures, so this is potentially a weakening of Monotonicity in Mixtures. However, by inspection of our earlier proofs, one can verify that all of our results in Section 6 providing conditions under which Monotonicity in Mixtures is violated also show violations of Monotonicity in Optimal Mixtures.

7.3 Further directions

- In addition to the smooth ambiguity model, there are other models of preferences capable of displaying Ellsberg Behavior in the literature while sometimes also satisfying Monotonicity in Mixtures. For example, the Confidence preference model of Chateauneuf and Faro (2009) can do so in some instances. As of yet, we have not been able to find nice conditions under which this occurs, but we think it would be interesting to do so.

- Ellsberg Behavior is, as was mentioned, much weaker than typical global ambiguity aversion/uncertainty aversion conditions in the literature. Mukerji and Tallon (2003) (see also Higashi et al., 2008) have an axiom A1 that is also meant to be a weak/local condition, and it may be interesting to explore the relationship between A1 and Ellsberg Behavior.

- Just as the compatibility results for the smooth ambiguity model and Monotonicity in Mixtures are related to compatibility results for expected utility and conditional FOSD shifts increasing optimal portfolio shares, our incompatibility results for preferences with kinks or slopes in utility space changing too rapidly along indifference curves could be applied to show that non-expected utility risk preferences that use kinks to generate risk aversion would not be compatible with conditional FOSD shifts always increasing optimal portfolio shares.

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