Module 17: Mechanism Design & Optimal Auctions

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Examples:

- Auctions
- Bilateral trade
- Production and distribution in society

General Setup

- N agents
- Each agent has private information θ_i; $\theta = \{\theta_i\}_{i=1}^N$.
- Outcomes $y \in Y$; often allocation plus transfers: $y = \{k, t_1, .., t_N\}$.
- Utility $u_i = u_i(y, \theta)$
  - Quasi-linear utility: $u_i = u_i^k(\theta) - t_i$.
- Mechanism designer's objective: "Implement" a choice rule $\psi : \Theta \to Y$ to maximize objective; *e.g.*,
  - *Efficiency:* maximize $\sum_i u_i^k(\theta)$
  - *Revenue:* maximize $\mathbb{E}_\theta \left[\sum_i t_i(\theta)\right]$

Definition. A choice rule $\psi : \Theta \to Y$ is incentive compatible with respect to an equilibrium concept "X" if each agent revealing his type truthfully (*i.e.*, reporting $\tilde{\theta}_i = \theta_i$) is an "X"-equilibrium.

Equilibrium Concepts

1. Dominant-strategy (strategy-proof) implementation: For all i, θ_i, $\tilde{\theta}_i$, θ_{-i} and $\tilde{\theta}_{-i}$

$$u_i\left(\psi\left(\theta_i,\tilde{\theta}_{-i}\right);\theta\right) \geq u_i\left(\psi\left(\tilde{\theta}_i,\tilde{\theta}_{-i}\right);\theta\right)$$

   - Reporting truthfully is an optimal strategy for each agent irrespective of the others' strategies.
   - Quite restrictive.

2. Bayesian Nash implementation:

   - There is a common prior π over θ, and the agents' beliefs $\pi_i(\cdot|\theta_i)$ over Θ_{-i} are given by Bayesian updating.
   - For all For all i, θ_i and $\tilde{\theta}_i$

$$\mathbb{E}_{\pi_i(\cdot|\theta_i)}u_i\left(\psi\left(\theta_i,\theta_{-i}\right);\theta\right) \geq \mathbb{E}_{\pi_i(\cdot|\theta_i)}u_i\left(\psi\left(\tilde{\theta}_i,\theta_{-i}\right);\theta\right)$$

   - Reporting truthfully is an optimal strategy on expectation, given beliefs $\pi_i(\cdot|\theta_i)$.

3. Ex-post implementation: For all i, θ_i, $\tilde{\theta}_i$ and θ_{-i}

$$u_i\left(\psi\left(\theta_i,\theta_{-i}\right);\theta\right) \geq u_i\left(\psi\left(\tilde{\theta}_i,\theta_{-i}\right);\theta\right)$$

   - Ea. agent finds it optimal to report truthfully given that others also report truthfully - after others' types are revealed ("no regret").
   - Advantage: Robust against different priors and higher order beliefs.

Revelation Principle

- Set of all mechanisms has little structure.
- Focus on a particular class of mechanism: Revelation mechanism $S_i = \Theta_i$; *i.e.*, strategy is to state a type $\tilde{\theta}$.

Theorem. *(Revelation Principle for Bayesian Nash implementation) A choice rule ψ is (partially) implementable by any mechanism if and only if it is incentive compatible.*

- *Proof:* Skipped.

- Very robust result.
  - Holds for all standard implementation concepts.
- If agents control actions a_i on top of common decision ψ, then one can replace any mechanism with a centralized mechanism where
  - Each agent reports his type $\tilde{\theta}_i$; and
  - the mechanism designer recommends actions $\tilde{a}_i$.
  - In equilibrium, the agents are truthful $\tilde{\theta}_i = \theta_i$ and obedient $(a_i = \tilde{a}_i)$.

  i.e., Moral hazard together with adverse selection *(Myerson, Ecta '82)*
- If agents can act sequentially and acquire further information, then one can replace any mechanism with a centralized mechanism where
  - Agents report everything they have learned so far ; and
  - the mechanism designer recommends actions $\tilde{a}_i$.
  - In equilibrium, the agents are truthful and obedient.
- Not robust to:
  - Communication costs
  - Bounded rationality.
- Full vs. Partial implementation:
  - *Partial:* $\psi(\theta)$ is an equilibrium.
  - *Full:* $\psi(\theta)$ is the only equilibrium.

Optimal Auctions

- N bidders.
- $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ with pdf f.
- Mechanism specifies:
  1. Allocation function $p_i : \left[\underline{\theta}, \overline{\theta}\right]^N \rightarrow [0,1]$ for each agent i such that $p_i \geq 0$ and $\sum_i p_i \leq 1$.
     - If the seller has n objects for sale, then $\sum_i p_i \leq n$.
  2. Transfer function $t_i : \left[\underline{\theta}, \overline{\theta}\right] \rightarrow \mathbb{R}$ for each agent i.
- Independent private values (IPV) model: $u_i(\theta_i) = \theta_i p_i - t_i$
- Revenue: $\sum_i t_i + (1 - \sum_i p_i)\,\theta_0$
  - θ_0: seller's value. Can be shown that the seller can disclose θ_0 wolog.

Examples of Auctions

1. First-Price Auction: $p_i(\theta) = 1$ if $\theta_i > \theta_{-i}$, and $t_i(\theta_i) = p_i(\theta)\, b(\theta_i)$.
   - $b(\theta_i)$ is the bid of type θ_i.
   - Under symmetry assumptions.
   - Otherwise: Maskin and Riley (REStud, 2000)
2. Second-Price Auction: $p_i(\theta) = 1$ if $\theta_i > \theta_{-i}$, and $t_i(\theta_i) = p_i(\theta)\, b\left(\theta_{(2)}\right)$.
   - $b\left(\theta_{(2)}\right)$ is the second-highest bid.
3. All-pay Auction: $p_i(\theta) = 1$ if $\theta_i > \theta_{-i}$, and $t_i(\theta_i) = b(\theta_i)$.
4. Raffle: $n(\theta_i) = \#$ of tickets, $p(\theta) = \frac{n(\theta_i)}{\sum_j n(\theta_j)}$, and $t_i(\theta_i) = c\, n(\theta_i)$.

Revenue Maximization

$$\begin{aligned} \max \quad & \mathbb{E}_{\theta}\left[\sum_i t_i\left(\theta_i\right)+\left[1-\sum_i p_i\left(\theta\right)\right]\theta_0\right] \\ \textbf{s.t.} \quad & u_i\left(\theta_i;\theta_i\right)\geq 0 \\ & u_i\left(\theta_i;\theta_i\right)\geq u_i\left(\theta_i;\tilde{\theta}_i\right) \end{aligned}$$

where $u_i\left(\theta_i;\tilde{\theta}_i\right)=\mathbb{E}_{\theta_{-i}}\left[p_i\left(\tilde{\theta}_i,\theta_{-i}\right)\theta_i - t\left(\tilde{\theta}_i,\theta_{-i}\right)\right]$.

Proposition. *is IC if and only if*

1. $u_i\left(\theta_i;\theta_i\right)=u_i\left(\underline{\theta};\underline{\theta}\right)+\int_{\underline{\theta}}^{\theta_i}\mathbb{E}_{\theta_{-i}}\left[p_i\left(s,\theta_{-i}\right)\right]ds$ *(IC-FOC)*
2. $\mathbb{E}_{\theta_{-i}}\left[p_i\left(\theta_i,\theta_{-i}\right)\right]$ increases in θ_i (Monotonicity)

(IR) can be replaced by $u\left(\underline{\theta};\underline{\theta}\right)=0$.

- *Proof:* Similar to the single-agent case.
- Re-write objective function:

$$\text{Revenue}=\mathbb{E}_{\theta}\left[\sum_i p_i\left(\theta\right)\theta_i+\left[1-\sum_i p_i\left(\theta\right)\right]\theta_0-\sum_i u_i\left(\theta_i;\theta_{-i}\right)\right]$$

- Calculate expected rent:

$$\begin{aligned} \mathbb{E}_{\theta_i}\left[u_i\left(\theta_i;\theta_{-i}\right)\right] &= \underbrace{u_i\left(\underline{\theta};\underline{\theta}\right)}_{=0\ \text{(IR)}}+\int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\theta}}^{\theta_i}\mathbb{E}_{\theta_{-i}}\left[p_i\left(s,\theta_{-i}\right)\right]ds\quad\underbrace{dF\left(\theta_i\right)}_{-\left[1-F\left(\theta_i\right)\right]'d\theta_i} \\ &= -\underbrace{\left[\mathbb{E}_{\theta_{-i}}\left[p\left(\theta_i,\theta_{-i}\right)\right]\left[1-F\left(\theta_i\right)\right]\right]_{\underline{\theta}}^{\bar{\theta}}}_{=0}+\int_{\underline{\theta}}^{\bar{\theta}}\mathbb{E}_{\theta_{-i}}\left[p_i\left(\theta_i,\theta_{-i}\right)\right]\left[1-F\left(\theta_i\right)\right]d\theta_i \\ &= \mathbb{E}_{\theta}\left[p_i\left(\theta\right)\frac{1-F\left(\theta_i\right)}{f\left(\theta_i\right)}\right] \end{aligned}$$

- Compile:

$$\text{Revenue}=\mathbb{E}_{\theta}\left[\sum_i p_i\left(\theta\right)\left[\underbrace{\theta_i-\frac{1-F\left(\theta_i\right)}{f\left(\theta_i\right)}}_{MR\left(\theta_i\right)}-\theta_0\right]\right]+\theta_0$$

Proposition. (Revenue Equivalence): *Any auction that has the same allocation function, generates the same revenue.*

Proof.

- Revenue depends on $p(\cdot)$, but not on $t(\cdot)$.

$\square$

- *Implication:* What matters is allocations; not "how you get there".
- Optimal Auction:
  - Award good to agent i if $MR(\theta_i) > \max\{\theta_0, MR(\theta_{-i})\}$.
  - If $MR(\theta)$ increases in θ, then (Monotonicity) is satisfied, and we have an optimal auction. Otherwise, we need to "iron it".

Implementation:

- First-price auction with reserve price $r = MR^{-1}(\theta_0)$.
- Second-price auction with entry fee $e = MR^{-1}(\theta_0) F^{N-1}(MR^{-1}(\theta_0))$.

Example:

- N bidders, $\theta_i \sim U[0,1]$, $\theta_0 = 0$.
- $MR(\theta) = 2\theta - 1$.
- Award good to agent with highest value if $\theta \geq \frac{1}{2}$; *i.e.*, reserve price $r = \frac{1}{2}$.
- Note: $r > \theta_0$. Why? (By increasing the reserve price, the seller can reduce information rents.)

Deriving bidding strategies:

- Assume that bidding functions are (i) monotone, and (ii) symmetric.
- First-price auction:

$$\begin{aligned} u_i(\theta_i, \theta_i) &= \mathbb{E}_{\theta_{-i}}\left[(\theta_i - b(\theta_i)) p_i(\theta)\right] = F^{N-1}(\theta_i)\left[\theta_i - b(\theta_i)\right] \\ u_i(\theta_i, \theta_i) &= \int_{\underline{\theta}}^{\theta_i} \mathbb{E}_{\theta_{-i}}\left[p(s, \theta_{-i})\right] ds = \int_{\underline{\theta}}^{\theta_i} F^{N-1}(s)\, ds \end{aligned}$$

- Equating the two expressions, we obtain

$$b(\theta) = \theta - \frac{\int_{\underline{\theta}}^{\theta} F^{N-1}(s)\, ds}{F^{N-1}(\theta)}$$

Asymmetries:

- Suppose $\theta_i \sim F_i(\cdot)$ (*i.e.*, valuations come from different distributions).
- Define: $MR_i(\theta_i) = \theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}$
- Revenue $= \mathbb{E}_\theta\left[\sum_i p_i(\theta)\left[MR(\theta_i) - \theta_0\right]\right] + \theta_0$
- If bidder j has ex-ante higher valuation than bidder i (*i.e.*, if $\frac{1-F_j(\theta)}{f_j(\theta)} > \frac{1-F_i(\theta)}{f_i(\theta)}$), then bias auction in favor of θ_i. (Formally, we say that $\theta_j >_{HRO} \theta_i$.)
  - If $\theta_i = \theta_j - \epsilon$, then still allocate good to bidder i.
  - Favor weak bidders to induce the stronger bidders to bid higher.

Welfare Maximization (First Best)

$$\max_{p_i(\cdot)} \left\{ \mathbb{E}_\theta \left[\sum_i p_i(\theta)\, \theta_i + \left[1 - \sum_i p_i(\theta)\right] \theta_0 \right] \right\}$$

- *Solution:* Allocate the good to the agent with the highest valuation (incl. seller)
  - $p_i(\theta) = 1$ if and only if $\theta_i > \theta_j$ for all $j \neq i$ (otherwise 0).
- Implementation:
  1. First-price auction with reserve price θ_0.
  2. Second-price auction with reserve price θ_0.
  3. All-pay auction with reserve price θ_0.

References

Bolton and Dewatripont, (2005), *Contract Theory*, MIT Press.

Ortner J., (2013), Lecture Notes.