Module 15: Screening

Information Economics (Ec 515) · George Georgiadis

Non-Linear Pricing

- Quasi-linear model
- Consumer (agent)
  - Type: θ (taste)
  - Utility $u = \theta q - t$ (taste * quality - price)
- Firm's Profit: $t - c(q)$ $(c', c'' > 0)$

Two approaches:

1. Revelation mechanism:
   - Firm asks agent for type θ.
   - Consumer reports $\tilde{\theta}$ (not necessarily $\tilde{\theta} = \theta$).
   - Mechanism specifies $q\left(\tilde{\theta}\right)$, $t\left(\tilde{\theta}\right)$.
   - *Objective:* Design mechanism such that the consumer finds it optimal to report $\tilde{\theta} = \theta$.

   (a) Taxation mechanism:
   - Firm offers menu of contracts $T(q)$.
   - Consumer, knowing his type θ, picks most favorable contract.
   - *Objective:* Design menu to maximize expected profit.

- **Lemma 1** (*Revelation principle & Taxation Principle*): The two approaches are equivalent.

Proof.

Revelation principle $\Rightarrow$ *Taxation principle:*

- Suppose the revelation mechanism $q(\cdot)$ and $t(\cdot)$ is incentive compatible (*i.e.*, type θ finds it optimal to report $\tilde{\theta} = \theta$).
- Let
$$T(q) = \begin{cases} t(\theta) & \text{if } q = q(\theta) \\ \infty & \text{otherwise} \end{cases}$$
- Since $q(\cdot)$ and $t(\cdot)$ is incentive compatible, type θ buys $q(\theta)$.

Taxation principle $\Rightarrow$ *Revelation principle:*

- Fix some menu $T(q)$. Now let
$$q(\theta) = \max_q \{\theta q - T(q)\} \quad \text{and} \quad t(\theta) = T(q(\theta)) \ .$$
- Type θ finds it optimal to report
$$\tilde{\theta} = \arg\max_{\theta'} \{\theta q(\theta') - t(\theta')\} = \arg\max_{\theta'} \{\theta q(\theta') - T(q(\theta'))\} = \theta$$

$\square$

- The revelation mechanism has proved to be more convenient to use!

- Utility of a type θ who reports $\tilde{\theta}$:
$$\begin{aligned} u\left(\theta; \tilde{\theta}\right) &= \theta q\left(\tilde{\theta}\right) - t\left(\tilde{\theta}\right) \\ u(\theta) &= u(\theta; \theta) \end{aligned}$$

- Principal's problem: Choose $q(\cdot)$ and $t(\cdot)$ to maximize
$$\mathbb{E}_\theta\left[t(\theta) - c(q(\theta))\right]$$
subject to
$$\begin{aligned} u(\theta; \theta) &\geq 0 && \text{(IR)} \\ u(\theta; \theta) &\geq u\left(\theta; \tilde{\theta}\right) \quad \forall \tilde{\theta} && \text{(IC)} \end{aligned}$$

First-best Outcome:

- Suppose that the principal knows the agent's type.
- Consumer gets 0 utility.
  - So $t(\theta) = \theta q(\theta)$, so the principal solves $\max_{q(\cdot)} \mathbb{E}_\theta [\theta q(\theta) - c(q(\theta))]$.
  - Pointwise maximization w.r.t q yields: $\theta = c'(q(\theta))$.
- Does this satisfy (IC) ?
  - No. Suppose $\theta \in \{\theta_L, \theta_H\}$.
  - Then θ_i receives q_i and assuming $q_L > 0$, we have

$$\begin{aligned} u(\theta_H;\theta_L) &= \theta_H q_L - t_L = (\theta_H - \theta_L) q_L + \theta_L q_L - t_L \\ &= \underbrace{(\theta_H - \theta_L) q_L}_{>0} + \underbrace{u(\theta_L;\theta_L)}_{=0} \\ &> 0 = u(\theta_H;\theta_H) \end{aligned}$$

  - *i.e.*, type θ_H wants to imitate type θ_L.

Screening with Two Types

- Suppose $\theta \in \{\theta_L, \theta_H\}$, where $\theta_H > \theta_L$
- $\Pr\{\theta = \theta_H\} = 1 - \pi$
- Principal maximizes

$$\max_{q_H, t_H, q_L, t_L} \{(1-\pi)[t_H - c(q_H)] + \pi [t_L - c(q_L)]\}$$

subject to

$$\begin{aligned} \theta_H q_H - t_H &\geq 0 && (IR_H) \\ \theta_L q_L - t_L &\geq 0 && (IR_L) \\ \theta_H q_H - t_H &\geq \theta_H q_L - t_L && (IC_H) \\ \theta_L q_L - t_L &\geq \theta_L q_H - t_H && (IC_L) \end{aligned}$$

- Assume $q_L > 0$. (Otherwise, just serve θ_H efficiently and extract all rents.)

- *First-best Outcome:*
  - IR_H and IR_L bind: $t_H = \theta_H q_H$ and $t_L = \theta_L q_L$.
  - Objective function becomes: $\max_{q_H, q_L} \{(1-\pi)[\theta_H q_H - c(q_H)] + \pi [\theta_L q_L - c(q_L)]\}$
  - First order conditions: $c'\left(q_H^{fb}\right) = \theta_H$ and $c'\left(q_L^{fb}\right) = \theta_L$; *i.e.*, marginal cost = marginal benefit.

- *Proposition:* (IR_L) and (IC_H) are binding, while the other two constraints can be replaced by the monotonicity constraint $q_H \geq q_L$.

Proof.

- (IR_H) is slack:

$$\begin{aligned} \theta_H q_H - t_H &\geq \theta_H q_L - t_L \ \text{ (by } IC_H) \\ &> \theta_L q_L - t_L \\ &\geq 0 \ \text{ (by } IR_L) \end{aligned}$$

- (IR_L) binds: If not, increase t_L until it binds to increase the principal's profit.

- Monotonicity: Adding (IC_H) and (IC_L) yields

$$\begin{aligned} \theta_H q_H - \theta_L q_H &\geq \theta_H q_L - \theta_L q_L \\ (\theta_H - \theta_L) q_H &\geq (\theta_H - \theta_L) q_L \\ q_H &\geq q_L \end{aligned}$$

- (IC_H) binds: If not, increase t_H by ϵ because (IR_H) is slack.

- (IC_L) is redundant:

$$\begin{aligned} t_H - t_L &= \theta_H (q_H - q_L) \ \text{ (by } IC_H) \\ &\geq \theta_L (q_H - q_L) \ \text{ (by monotonicity)} \end{aligned}$$

□

- Program becomes:

$$\max_{q_H, t_H, q_L, t_L} \left\{ (1-\pi)\left[t_H - c\left(q_H\right)\right] + \pi \left[t_L - c\left(q_L\right)\right]\right\}$$

subject to

$$\begin{aligned} \theta_L q_L - t_L &= 0 \qquad (IR_L) \\ \theta_H q_H - t_H &= \theta_H q_L - t_L \quad (IC_H) \\ q_H &\geq q_L \qquad \text{(monotonicity)} \end{aligned}$$

- Relax program by ignoring (monotonicity), and using (IR_L) and (IC_H):

$$\begin{aligned} & (1-\pi)\left[\theta_H q_H - (\theta_H - \theta_L) q_L - c\left(q_H\right)\right] + \pi\left[\theta_L q_L - c\left(q_L\right)\right] \\ = \;& (1-\pi)\left[\underbrace{\theta_H q_H - c\left(q_H\right)}_{\text{welfare}}\right] + \pi\left[\underbrace{\theta_L q_L - c\left(q_L\right)}_{\text{welfare}} - \underbrace{\frac{1-\pi}{\pi}\left(\theta_H - \theta_L\right) q_L}_{\text{expected info rents}}\right] \end{aligned}$$

- *First-order conditions:* $c'\left(q_H\right) = \theta_H$ and $c'\left(q_L\right) = \theta_L - \frac{1-\pi}{\pi}\left(\theta_H - \theta_L\right)$
  - Observe that $q_H = q_H^{fb}$.
  - But $c'\left(q_L\right) < \theta_L$, which implies that $q_L < q_L^{fb}$.
- Check monotonicity:

$$\begin{aligned} c'\left(q_H\right) &= \theta_H \\ &> \theta_L - \frac{1-\pi}{\pi}\left(\theta_H - \theta_L\right) = c'\left(q_L\right) \\ \Longrightarrow q_H &> q_L \end{aligned}$$

- Once we have determined q_H and q_L, we can back out the transfers from (IR_L) and (IC_H):
  - $t_L = \theta_L q_L$
  - $t_H = t_L + \theta_H\left(q_H - q_L\right)$

Properties:

- Low type is inefficiently underserved.

- Makes it less attractive for the high type to imitate the low type; *i.e.*, lowers rents of high type.
- *e.g.*, squeeze passengers in economy class.

◦ Lowest type gets no surplus (that would be a waste).

◦ Efficiency at the top.

- Low type cannot "afford" to mimic the high type.
- Serve him optimally (and tax him).

◦ High type indifferent between contracts.

- Ensured by making economy class uncomfortable.

◦ Quality / quantity increases in type.

Screening with a Continuum of Types

◦ $\theta \sim f(\cdot)$ with support on $\left[\underline{\theta}, \overline{\theta}\right]$.

- The agent's "type" θ can now take a continuum of values (instead of 2).

◦ *Recall:* Utility of a type θ who reports $\tilde{\theta}$:

$$\begin{aligned} u\left(\theta; \tilde{\theta}\right) &= \theta q\left(\tilde{\theta}\right) - t\left(\tilde{\theta}\right) \\ u(\theta) &= u(\theta; \theta) \end{aligned}$$

◦ Principal's problem:

$$\max_{q(\theta), t(\theta)} \mathbb{E}_\theta \left[t(\theta) - c(q(\theta))\right]$$

subject to

$$\begin{aligned} u(\theta;\theta) &\geq 0 && \text{(IR)} \\ u(\theta;\theta) &\geq u\left(\theta;\tilde{\theta}\right) \quad \forall \tilde{\theta} && \text{(IC)} \end{aligned}$$

Theorem. *$q(\cdot)$, $t(\cdot)$ is incentive compatible if and only if*

$$\begin{aligned} u(\theta) &= u(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(s)\,ds \qquad \textit{(Payoff Equivalence)} \\ & q(\theta) \ \textit{is increasing} \qquad \textit{(Monotonicity)} \end{aligned}$$

Proof.

- Fix $\theta' > \theta$.
- Only if:
  - (IC) implies that

$$\left.\begin{array}{r} \theta' q(\theta') - t(\theta') \geq \theta' q(\theta) - t(\theta) \\ \theta q(\theta) - t(\theta) \geq \theta q(\theta') - t(\theta') \end{array}\right\} \Longrightarrow \theta'\left[q(\theta') - q(\theta)\right] \geq t(\theta') - t(\theta) \geq \theta\left[q(\theta') - q(\theta)\right]$$

    and hence $q(\theta') - q(\theta) \geq 0$.
  - Payoff equivalence follows from the Envelope Theorem:

$$\frac{du}{d\theta}(\theta) = \frac{\partial u}{\partial \theta}(\theta;\theta) + \underbrace{\left.\frac{\partial u}{\partial \tilde{\theta}}\left(\theta;\tilde{\theta}\right)\right|_{\tilde{\theta}=\theta}}_{=0 \text{ (FOC at } \tilde{\theta}=\theta)} = q(\theta)$$

    (Real proof in Milgrom and Segal (ECTA, 2002).)

- If:

$$\begin{aligned} u(\theta') &= u(\theta) + \int_{\theta}^{\theta'} q(s)\,ds \\ &\geq u(\theta) + \int_{\theta}^{\theta'} q(\theta)\,ds \quad \text{(by monotonicity)} \\ &= u(\theta) + (\theta' - \theta)\,q(\theta) \\ &= u(\theta';\theta) \end{aligned}$$

□

- Principal's Problem:

$$\max_{q(\theta),t(\theta)} \mathbb{E}_{\theta}\left[t(\theta) - c(q(\theta))\right]$$

subject to

$$\begin{aligned} u(\theta) &= u(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(s)\,ds \qquad \text{(Payoff Equivalence)} \\ u(\underline{\theta}) &= 0 \qquad \text{(IR low)} \\ & q(\theta) \text{ is increasing} \qquad \text{(Monotonicity)} \end{aligned}$$

- $u(\theta) = \theta q(\theta) - t(\theta)$ implies that $t(\theta) = \theta q(\theta) - u(\theta)$ so that the objective function can be re-written as $\mathbb{E}[\theta q(\theta) - c(q(\theta)) - u(\theta)]$
- Notice:

$$\begin{aligned} \mathbb{E}[u(\theta)] &= \int_{\underline{\theta}}^{\bar{\theta}} u(\theta)\,dF(\theta) \\ &= -\int_{\underline{\theta}}^{\bar{\theta}} u(\theta)\,[1 - F(\theta)]'\,d\theta \\ &= -\underbrace{\left[u(\theta)\,\bar{F}(\theta)\right]_{\underline{\theta}}^{\bar{\theta}}}_{=0} + \int_{\underline{\theta}}^{\bar{\theta}} \underbrace{u'(\theta)}_{=q(\theta)} \bar{F}(\theta)\,d\theta \\ &= \mathbb{E}\left[q(\theta)\,\frac{1 - F(\theta)}{f(\theta)}\right] \end{aligned}$$

- We can write the principal's profit as

$$\mathbb{E}\left[q(\theta)\left[\theta - \frac{1 - F(\theta)}{f(\theta)}\right] - c(q(\theta))\right]$$

- Ignore the monotonicity constraint (for now), and maximize this pointwise with respect to q.
  - First-order condition: $\underbrace{\theta - \frac{1 - F(\theta)}{f(\theta)}}_{MR(\theta)} - \underbrace{c'(q(\theta))}_{MC(\theta)} = 0$
  - As long as $MR(\theta)$ is increasing, $q(\theta)$ is increasing in θ, and the monotonicity constraint is satisfied.
  - *e.g.*, true if θ follows a uniform or exponential distribution.
- Selling $q(\theta)$ generates:
  - Surplus $\theta q(\theta)$

- Costs $c(q(\theta))$
- Consumer rents $\frac{1-F(\theta)}{f(\theta)} q(\theta)$

- How to back out transfers $t(\cdot)$?
  - The above maximization problem yields $q(\theta)$.
  - We can then compute $u(\theta) = \int_{\underline{\theta}}^{\theta} q(s)\, ds$.
  - Finally, we obtain $t(\theta) = \theta q(\theta) - u(\theta)$.

Example 1: Quadratic Costs

- *Setup:* $\theta \sim U[0,1]$; $c(q) = \frac{q^2}{2}$
- Marginal Revenue: $MR(\theta) = 2\theta - 1$
- Marginal Cost: $c'(q) = q$
- Optimal Contract: $q(\theta) = [2\theta - 1]^+$
  - Then $u(\theta) = \int_{\frac{1}{2}}^{\theta} (2s-1)\, ds = \frac{1}{4} - \theta(1-\theta)$; and
  - Transfers $t(\theta) = \theta(2\theta - 1) - \frac{1}{4} + \theta(1-\theta) = \theta^2 - \frac{1}{4}$.

Example 2: Linear Costs

- *Setup:* $c(q) = cq$ with $q \in [0,1]$
- No haggling Theorem:

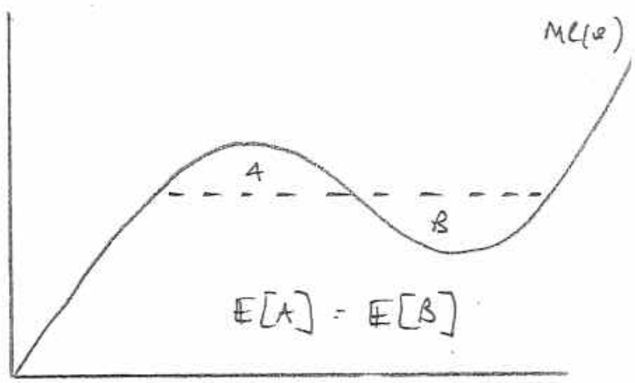
$$q(\theta) = \begin{cases} 1 & \text{if } MR(\theta) \geq c \\ 0 & \text{otherwise} \end{cases}$$

Further Remarks:

- One agent or many agents
  - It doesn't matter.
  - If types are independent, then principal cannot gain by linking games.
  - If types are correlated, then principal can fully extract surplus by linking games (Cremer and Maclean, 1985).

- Generalizes to n agents with independent information: Optimal auctions

- Quality or Quantity:
  - With one consumer, q could also be quantity.
  - With many consumers, games are linked if costs are convex, because $c\left(\sum_i q_i\right)$ (Segal, AER 2003).
- What if marginal revenue $MR(\theta)$ is not increasing?
  - Need to "iron" it.
  - Two approaches:
    * Optimal control (Guesnerie and Laffont,1984)
    * Convex hull (Myerson, 1981)
  - Convex hull approach: Replace $\int^{\theta} MR(s)\,ds$ with smallest convex envelope (draw picture).



    * x-axis: θ
    * y-axis: $MR(\theta)$
    * We choose a value to flatten the allocation function $q(\theta)$ such that $\mathbb{E}[A] = \mathbb{E}[B]$.

References

Board S., (2011), Lecture Notes.

Bolton and Dewatripont, (2005), *Contract Theory*, MIT Press.