B For Online Publication: Proofs for Sections 5

First, we prove some preliminary properties of optimal incentives schemes. If $\underline{u} > -\infty$, Lemma 1 shows that any profit-maximizing incentive scheme $v(\cdot)$ must be unique, and Online Appendix D shows the same for $\underline{u} = -\infty$. We prove that $v(\cdot)$ must be monotonically increasing and satisfy (IC-FOC) with equality.

Suppose $v(\cdot)$ is concave and not everywhere increasing. Then, we can find $\tilde{y} \in \mathcal{Y}$ such that if we replace $v(y)$ by a constant $v(\tilde{y})$ to the right of $\tilde{y}$, the resultant contract is concave, gives the same utility to the agent, is cheaper, and, using MLRP and Beesack's inequality makes (IC-FOC) slack. So any optimal $v(\cdot)$ must be increasing.

Suppose $v(\cdot)$ does not satisfy (IC-FOC) with equality. Then, a convex combination of v and the contract which gives utility constant and equal to $\max\{\underline{u}, u_0 + c(a)\} \geq 0$ implements a, is strictly cheaper than v, and satisfies (IC-FOC) with equality. So any optimal $v(\cdot)$ must satisfy (IC-FOC) with equality.

Consider an interval $[y_L, y_H]$. The initial impact of *raising* the agent's utility on this interval is given by

$$r_{y_L,y_H}(y) = \begin{cases} 1 & y \in [y_L, y_H] \\ 0 & else \end{cases} .$$

Similarly, *tilting* this interval has an initial impact on the agent's utility given by

$$t_{y_L,y_H}(y) = \begin{cases} 0 & y \leq y_L \\ y - y_L & y \in (y_L, y_H) \\ y_H - y_L & y \geq y_H \end{cases} .$$

We will carefully define the perturbations *raise* and *tilt* and show that they respect concavity in Section B.3.2.

Our first result proves two useful properties of any contract that is GHM.

Lemma 3. *Let v be GHM, and let $[y_L, y_H]$ be a linear segment of v. Then, for* *each $\hat{y} \in (y_L, y_H)$, there is $\tilde{y} \in (\hat{y}, y_H)$ such that*

$$n(\tilde{y}) \leq 0.$$

If $v(y_L) > \underline{u}$, then such a $\tilde{y}$ exists in $(y_L, \hat{y})$ as well. But, somewhere on (y_L, y_H), $n(y) \geq 0$.

Proof. Note that for $y > y_H$, $t_{\hat{y},y_H}(y) = y_H - \hat{y} = (y_H - \hat{y}) r_{y_H,\bar{y}}(y)$. Since v satisfies IC, since $a > 0$, and since v is concave and weakly increasing, v must be strictly increasing near $\underline{y}$. Hence, since $y_H > \underline{y}$, $v(y_H) > \underline{u}$. We thus have $\int n(y) r_{y_H,\bar{y}}(y) f(y|a) dy = 0$ by Definition 2.1. Hence, by Definition 2.3, we have

$$\begin{aligned} 0 &\geq \int n(y) t_{\hat{y},y_H}(y) f(y|a) dy \\ &= \int n(y) t_{\hat{y},y_H}(y) f(y|a) dy - (y_H - \hat{y}) \int n(y) r_{y_H,\bar{y}}(y) f(y|a) dy \\ &= \int_{\hat{y}}^{y_H} n(y) t_{\hat{y},y_H}(y) f(y|a) dy, \end{aligned}$$

and so at some point $\tilde{y} \in (\hat{y}, y_H)$, the integrand is weakly negative. Since $t_{\hat{y},y_H}(\tilde{y}) > 0$, it follows that $n(\tilde{y}) \leq 0$.

Similarly, note that if $v(y_L) > \underline{u}$, then $\int n(y) r_{y_L,\bar{y}}(y) f(y|a) dy = 0$ by Definition 2.1, and so by Definition 2.2,

$$\begin{aligned} 0 &\leq \int n(y) t_{y_L,\hat{y}}(y) f(y|a) dy \\ &= \int n(y) t_{y_L,\hat{y}}(y) f(y|a) dy - (\hat{y} - y_L) \int n(y) r_{y_L,\bar{y}}(y) f(y|a) dy \\ &= \int_{y_L}^{\hat{y}} n(y) [t_{y_L,\hat{y}}(y) - (\hat{y} - y_L)] f(y|a) dy, \end{aligned}$$

where, since the bracketed term is strictly negative on $(y_L, \hat{y})$, it follows that $n(y)$ is somewhere weakly negative on $(y_L, \hat{y})$.

Finally, since $\int n(y) r_{y_L,y_H}(y) f(y|a) dy \geq 0$, and since we have established that $n(y)$ is weakly negative somewhere on (y_L, y_H), we must also have $n(y)$

weakly positive somewhere on the same interval.

□

B.1 Proof of Proposition 3

This proof uses both Proposition 5 and Corollary 2, which are proven below. Suppose that there exists some $y_I \in [\underline{y}, \bar{y}]$ such that $\rho(\lambda + \mu l(\cdot|a))$ is convex on $[\underline{y}, y_I]$ and concave on $[y_I, \bar{y}]$, let $v^*(\cdot)$ implement $a \geq 0$ at maximum profit, and suppose $v^*(\underline{y}) > \underline{u}$. Since $v^*(\cdot)$ is increasing, (LL) must be slack.

First, we show that $v^*(\cdot)$ has no more than one linear segment. Since $v^*(\cdot)$ implements a at maximum profit, it is GHM by Proposition 5. Consequently, if $v^*(\cdot)$ has more than one linear segment, then Lemma 3 implies that $n(\cdot)$ must be positive, then negative, then positive over each segment. Hence, $v^*(\cdot) - \rho(\lambda + \mu l(\cdot|a))$ must be negative, then positive, then negative over each linear segment. But then $\rho(\lambda + \mu l(\cdot|a))$ must have two disjoint non-concave regions, which is ruled out by assumption.

If $y_I = \underline{y}$, then $v^*(\cdot)$ cannot have any linear segments, since on any such segment $v^*(\cdot) - \rho(\lambda + \mu l(\cdot|a))$ would be positive, then negative, then positive. But then any interior free point must be a point of normal concavity, and so Corollary 2 implies that $v^*(\cdot) = \rho(\lambda + \mu l(\cdot|a))$ over $(\underline{y}, \bar{y})$.

If $y_I > \underline{y}$, then $v^*(\cdot)$ must have a linear segment because it cannot coincide with $\rho(\lambda + \mu l(\cdot|a))$ everywhere. We claim that this linear segment must be $[\underline{y}, \hat{y}]$ for some $\hat{y} \geq y_I$. If the linear segment starts at some $\tilde{y} > \underline{y}$, then every $y \in (\underline{y}, \tilde{y})$ must be a point of normal concavity. But then $v^*(\cdot) = \rho(\lambda + \mu l(\cdot|a))$ on $(\underline{y}, \tilde{y})$, which violates (NG) because $\rho(\lambda + \mu l(\cdot|a)$ is convex on that region by assumption. Similarly, if $\hat{y} < y_I$, then every $y \in (\hat{y}, y_I)$ must be a point of normal concavity, which again violates (NG). So $v^*(\cdot)$ has a single linear segment $[\underline{y}, \hat{y}]$, where $\hat{y} \geq y_I$. Since $v^*(\cdot)$ is GHM and $v^*(\underline{y}) > \underline{u}$, (3) holds with equality on this linear segment and so $\int_{\underline{y}}^{\hat{y}} n(y) f(y) dy = 0$.

Finally, any $y \in (\hat{y}, \bar{y})$ is again a point of normal concavity, and so $v^*(\cdot) = \rho(\lambda + \mu l(\cdot|a))$ at all such points. This proves the result. ■

B.2 Proof of Proposition 4

Let $v(\cdot)$ be an optimal incentive scheme, and suppose (IR) does not bind. Towards a contradiction, suppose that $v(\cdot)$ is strictly concave at some $y < y_0$. Consider the alternative contract

$$\tilde{v}(y) = \begin{cases} \alpha v(y) + (1-\alpha)\left[v(\underline{y}) + (y-\underline{y})\frac{v(y_0)-v(\underline{y})}{y_0-\underline{y}}\right] & y \le y_0 \\ v(y) & y > y_0 \end{cases}.$$

Note that $\tilde{v}(\cdot)$ is concave, $\tilde{v}(y) \le v(y)$ for all $y \in \mathcal{Y}$, $\tilde{v}(\underline{y}) \ge \underline{u}$, and there exists an interval in $[\underline{y}, y_0]$ such that $\tilde{v}(y) < v(y)$ on that interval. Therefore, $\tilde{v}(\cdot)$ is strictly less expensive than $v(\cdot)$ to the principal. Since (IR) does not bind, there exists some $\alpha \in [0,1)$ such that $\tilde{v}(\cdot)$ satisfies (IR). Furthermore,

$$\begin{gathered}\int \tilde{v}(y) f_a(y|a)dy = \int_{\underline{y}}^{y_0} \tilde{v}(y) f_a(y|a)dy + \int_{y_0}^{\bar{y}} v(y) f_a(y|a)dy > \\ \int_{\underline{y}}^{y_0} v(y) f_a(y|a)dy + \int_{y_0}^{\bar{y}} v(y) f_a(y|a)dy = \int v(y) f_a(y|a)dy,\end{gathered}$$

where the strict inequality follows because $f_a(y|a)$ is negative on $y \in [\underline{y}, y_0]$. Hence, $\tilde{v}(\cdot)$ satisfies (IC-FOC). So $\tilde{v}(\cdot)$ implements a, contradicting that $v(\cdot)$ is optimal. ■

B.3 Proof of Proposition 5

The discussion prior to the statement of Proposition 5 proves necessity, given well-defined perturbations that satisfy concavity, and well-defined shadow values. This section begins by formally defining the relevant perturbations, showing that they preserve concavity, and then showing how they can be used to establish shadow values for (IR) and (IC-FOC). We then turn to sufficiency.

B.3.1 Preliminaries

Definition 2 and Proposition 5 are phrased in terms of free points. But, not every free point is a convenient place to define a perturbation. Instead, for any given v, let C_v be the set of points y at which there exists a supporting plane L such that $L(y') > v(y')$ for all $y' \neq y$.

Clearly any kink point (see the discussion immediately before Corollary 2) is an element of C_v. The next claim shows that for every other free point, there is an arbitrarily close-by element of C_v.

Claim 1. *Let $\hat{y}$ be any point of normal concavity. Then, for each δ, there is a point in $\{(\hat{y}-\delta, \hat{y}+\delta) \setminus \hat{y}\} \cap C_v$. From this, it follows that for each $\varepsilon > 0$, there exists $y_L < y_H$ such that $y_L, y_H \in C_v$, and such that $y_L, y_H \in [\hat{y}-\varepsilon, \hat{y}+\varepsilon]$.*

Proof of Claim. We will show first that for each δ, there is a point in $\{(\hat{y}-\delta, \hat{y}+\delta) \setminus \hat{y}\} \cap C_v$. To see that this suffices to show the second part, apply the result first to find a point y_1 in $\{(\hat{y}-\varepsilon, \hat{y}+\varepsilon) \setminus \hat{y}\} \cap C_v$. Apply the result again to find y_2 in $\{(\hat{y}-\delta, \hat{y}+\delta) \setminus \hat{y}\} \cap C_v$ where $\delta = (1/2)\,|y_1 - \hat{y}|$, and finally take y_L and y_H as the smaller and larger of y_1 and y_2.

So, fix $\delta > 0$. Since $\hat{y}$ is not on the interior of a linear segment and not a kink point, there is at least one side of $\hat{y}$, without loss of generality the right side, such that $v(\cdot)$ is not linear on $(\hat{y}, \hat{y}+\delta)$. Let $S(\cdot)$ be the correspondence which for each y assigns the set of slopes of supporting planes at y, and let $s(\cdot)$ be any selection from $S(\cdot)$. Note that since v is concave, for any $y'' > y'$, $\max\{S(y'')\} \leq \min\{S(y')\}$, and hence s is decreasing. Assume first that there is a point $\tilde{y} \in (\hat{y}, \hat{y}+\delta)$ where $s(\cdot)$ jumps downward, say from s'' to $s' < s''$. Then, the supporting plane at $\tilde{y}$ with slope $(s'+s'')/2$ qualifies. Assume instead that $s(\cdot)$ is continuous on $(\hat{y}, \hat{y}+\delta)$. It cannot be everywhere constant, since $v(\cdot)$ is not linear on $(\hat{y}, \hat{y}+\delta)$. Hence, since $s(\cdot)$ is continuous, there is a point $\tilde{y}$ at which it is strictly decreasing, so that in specific, $s(\tilde{y}) < s(y)$ for all $y < \tilde{y}$, and $s(\tilde{y}) > s(y)$ for all $y > \tilde{y}$. The supporting plane at $\tilde{y}$ with slope $s(\tilde{y})$ then qualifies.

□

To see that why Claim 1 is helpful, assume that some part of Definition 2 is violated. For example, assume some optimal contract has a pair of free points y_L and y_H such that $\int n(y)\, r_{y_L, y_H} f(y)\, dy < 0$. If either y_L or y_H is a kink point, then it is also an element of C_v. If not, then we can apply Claim 1 to replace each relevant point by a sufficiently close-by element of C_v that the strict inequality is maintained. Hence, it is enough to prove Proposition 5 when each restriction to a free point is tightened to a restriction to C_v.

B.3.2 Formal Definition and Properties of the Perturbations

This section defines *raise* and *tilt*, being careful in particular to maintain concavity at the endpoints of the perturbed interval. We will need to consider as many as three perturbations at once, where, given the previous discussion, we will require the relevant points to be in C_v. First, we will have some small amount ε_p of a perturbation p where p could be r_{y_L,y_H} or t_{y_L,y_H} in each case with ε_p positive or negative. Second, for some $\hat{y} \in C_v$, we will need to consider some amount ε_t of $t_{\hat{y},\bar{y}}$ and ε_r of $r_{\hat{y},\bar{y}}$. Intuitively, we will use $t_{\hat{y},\bar{y}}$ and $r_{\hat{y},\bar{y}}$ to establish shadow values for (IC-FOC) and (IR), and then, for any particular perturbation p, consider the three deviations together where one uses $t_{\hat{y},\bar{y}}$ and $r_{\hat{y},\bar{y}}$ to undo the effect of p on (IC-FOC) and (IR).

Fix y_L, y_H, and $\hat{y}$. *A priori*, $\hat{y}$ may have arbitrary position relative to y_L and y_H, and moreover, in the case where p is t_{y_L,y_H}, one of y_L or y_H may not be in C_v, depending on whether ε_p is negative or positive. Define $y_0 < y_1 < \cdots < y_K$, $K \leq 4$, as elements of the set $\left\{\underline{y}, y_L, y_H, \hat{y}, \bar{y}\right\} \cap C_v$. For any given $\boldsymbol{\varepsilon} = (\varepsilon_p, \varepsilon_t, \varepsilon_r)$, let $d(\cdot;\boldsymbol{\varepsilon}) : [\underline{y}, \bar{y}] \to \mathbb{R}$ be given by

$$d(\cdot;\boldsymbol{\varepsilon}) = \varepsilon_p p(\cdot) + \varepsilon_t t_{\hat{y},\bar{y}}(\cdot) + \varepsilon_r r_{\hat{y},\bar{y}}(\cdot).$$

If y_L and y_H are both elements of $\{y_0, \cdots, y_K\}$, as must be true if p is r_{y_L,y_H}, then it follows that d is linear on each interval of the form (y_{k-1}, y_k). Assume that $y_H \notin \{y_0, \cdots, y_K\}$. Then, it must be that p is t_{y_L,y_H} with $\varepsilon_p \geq 0$. In this case, if $y_H \notin (y_{k-1}, y_k)$, then $d(\cdot;\boldsymbol{\varepsilon})$ is linear on (y_{k-1}, y_k), while if $y_H \in (y_{k-1}, y_k)$, then, since $\varepsilon_p \geq 0$, $d(\cdot;\boldsymbol{\varepsilon})$ is concave with two linear segments on (y_{k-1}, y_k). Finally, assume $y_L \notin \{y_0, \cdots, y_K\}$. Then, p is t_{y_L,y_H} with $\varepsilon_p \leq 0$, and once again, if $y_L \notin (y_{k-1}, y_k)$, then $d(\cdot;\boldsymbol{\varepsilon})$ is linear on (y_{k-1}, y_k), while if $y_L \in (y_{k-1}, y_k)$, then since $\varepsilon_p \leq 0$, $d(\cdot;\boldsymbol{\varepsilon})$ is once again concave with two linear segments on (y_{k-1}, y_k).

For each k, let $L_k^-(\cdot;\boldsymbol{\varepsilon})$ be the line that coincides with the linear segment of $d(\cdot;\boldsymbol{\varepsilon})$ immediately to the right of y_{k-1} and let $L_k^+(\cdot;\boldsymbol{\varepsilon})$ be the line that coincides with the linear segment immediately to the left of y_k (these are the same line if d is linear on (y_{k-1},y_k)), and let

$$d_k(y;\boldsymbol{\varepsilon}) = \begin{cases} L_k^-(y;\boldsymbol{\varepsilon}) & y \le y_{k-1} \\ d(y;\boldsymbol{\varepsilon}) & y \in (y_{k-1},y_k) \\ L_k^+(y;\boldsymbol{\varepsilon}) & y \ge y_k \end{cases}.$$

Note that d_k is concave, and that as $|\boldsymbol{\varepsilon}| \equiv |\varepsilon_p| + |\varepsilon_t| + |\varepsilon_r| \to 0$, d_k converges uniformly to the function that is constant at 0.

For each k, let L_k be a supporting line to v at y_k, where since $y_k \in C_v$, we can choose L_k such that $L_k(y) > v(y)$ for all $y \neq y_k$, and let

$$v_k(y) = \begin{cases} L_{k-1}(y) & y \le y_{k-1} \\ v(y) & y \in (y_{k-1},y_k) \\ L_k(y) & y \ge y_k \end{cases},$$

so that $v_k(\cdot)$ is concave. Define $\hat{v}(\cdot;\boldsymbol{\varepsilon})$ by

$$\hat{v}(y;\boldsymbol{\varepsilon}) = \min_{k\in\{1,\cdots,K\}} (v_k(y) + d_k(y;\boldsymbol{\varepsilon})).$$

As the minimum over concave functions, $\hat{v}(\cdot;\boldsymbol{\varepsilon})$ is concave.

Fix k and consider any $y \in (y_{k-1},y_k)$. Since $d_k(y,\mathbf{0}) = 0$, and by the fact that for each k', $L_{k'}(y) > v(y)$ for all $y \neq y_{k'}$, k is the unique minimizer of $v_k(y) + d_k(y;\mathbf{0})$. From this, it follows first that $\hat{v}(y;\mathbf{0}) = v_k(y) = v(y)$, and second, that for all $\boldsymbol{\varepsilon}$ in some neighborhood of $\mathbf{0}$ (where ε_p is restricted in sign if $p = t_{y_L,y_H}$ and if one of y_L or y_H is not in C_v),

$$\begin{aligned} \hat{v}_{\varepsilon_p}(y;\boldsymbol{\varepsilon}) &= d_{\varepsilon_p}(y;\boldsymbol{\varepsilon}) = p(y), \\ \hat{v}_{\varepsilon_t}(y;\boldsymbol{\varepsilon}) &= d_{\varepsilon_t}(y;\boldsymbol{\varepsilon}) = t_{\hat{y},\bar{y}}(y), \text{ and} \\ \hat{v}_{\varepsilon_r}(y;\boldsymbol{\varepsilon}) &= d_{\varepsilon_r}(y;\boldsymbol{\varepsilon}) = r_{\hat{y},\bar{y}}(y). \end{aligned}$$

But then, except on the zero-measure set of points $\{y_0, \cdots, y_K\}$,

$$\begin{aligned}\hat{v}_{\varepsilon_p}(\cdot;\mathbf{0}) &= p(\cdot), \\ \hat{v}_{\varepsilon_t}(\cdot;\mathbf{0}) &= t_{\hat{y},\bar{y}}(\cdot), \text{ and} \\ \hat{v}_{\varepsilon_r}(\cdot;\mathbf{0}) &= r_{\hat{y},\bar{y}}(\cdot).\end{aligned} \tag{9}$$

B.3.3 Shadow Values

We need to establish that starting from $\varepsilon = \mathbf{0}$ the effects of perturbation p can be undone via $t_{\hat{y},\bar{y}}$ and $r_{\hat{y},\bar{y}}$. To do so, let

$$Q(\boldsymbol{\varepsilon}) = \begin{bmatrix} \int \hat{v}_{\varepsilon_t}(y,\boldsymbol{\varepsilon}) f_a(y|a)\,dy & \int \hat{v}_{\varepsilon_r}(y,\boldsymbol{\varepsilon}) f_a(y|a)\,dy \\ \int \hat{v}_{\varepsilon_t}(y,\boldsymbol{\varepsilon}) f(y|a)\,dy & \int \hat{v}_{\varepsilon_r}(y,\boldsymbol{\varepsilon}) f(y|a)\,dy \end{bmatrix}.$$

The top row of Q tracks the rate at which ε_t and ε_r respectively affect (IC-FOC), while the bottom row tracks the rate at which ε_t and ε_r respectively affect (IR). Then, from (9),

$$\begin{aligned} Q(\mathbf{0}) &= \begin{bmatrix} \int t_{\hat{y},\bar{y}} f_a(y|a)\,dy & \int r_{\hat{y},\bar{y}} f_a(y|a)\,dy \\ \int t_{\hat{y},\bar{y}} f(y|a)\,dy & \int r_{\hat{y},\bar{y}} f(y|a)\,dy \end{bmatrix} \\ &= \begin{bmatrix} \int_{\hat{y}}^{\bar{y}} (y-\hat{y}) f_a(y|a)\,dy & \int_{\hat{y}}^{\bar{y}} f_a(y|a)\,dy \\ \int_{\hat{y}}^{\bar{y}} (y-\hat{y}) f(y|a)\,dy & \int_{\hat{y}}^{\bar{y}} f(y|a)\,dy \end{bmatrix}, \end{aligned}$$

and so

$$\begin{aligned} |Q(\mathbf{0})| &= \int_{\hat{y}}^{\bar{y}} (y-\hat{y}) f_a(y|a)\,dy \int_{\hat{y}}^{\bar{y}} f(y|a)\,dy - \int_{\hat{y}}^{\bar{y}} (y-\hat{y}) f(y|a)\,dy \int_{\hat{y}}^{\bar{y}} f_a(y|a)\,dy \\ &\underset{s}{=} \frac{\int_{\hat{y}}^{\bar{y}} (y-\hat{y}) f_a(y|a)\,dy}{\int_{\hat{y}}^{\bar{y}} (y-\hat{y}) f(y|a)\,dy} - \frac{\int_{\hat{y}}^{\bar{y}} f_a(y|a)\,dy}{\int_{\hat{y}}^{\bar{y}} f(y|a)\,dy} \\ &= \int_{\hat{y}}^{\bar{y}} l(y|a) \frac{(y-\hat{y}) f(y|a)}{\int_{\hat{y}}^{\bar{y}} (y-\hat{y}) f(y|a)\,dy} dy - \int_{\hat{y}}^{\bar{y}} l(y|a) \frac{f(y|a)}{\int_{\hat{y}}^{\bar{y}} f(y|a)\,dy} dy, \end{aligned}$$

where the symbol $\underset{s}{=}$ means "has (strictly) the same sign as."

Thus, $|Q(\mathbf{0})|$ has the same sign as the difference between two expectations of $l(\cdot|a)$. Using that $(y - \hat{y})$ is strictly increasing, the density in the first integral strictly likelihood-ratio dominates the density in the second integral. Since $l(\cdot|a)$ is strictly increasing, it follows that $|Q(\mathbf{0})|$ is strictly positive (and so remains so for all $\boldsymbol{\varepsilon}$ in some ball around $\mathbf{0}$.) But then by the implicit function theorem, for each $p \in \{t_{y_L, y_H}, r_{y_L, y_H}\}$, we can on the appropriate neighborhood implicitly define $\varepsilon_r(\cdot)$ and $\varepsilon_t(\cdot)$ by

$$\begin{aligned} \int \hat{v}(y; \varepsilon_p, \varepsilon_t(\varepsilon_p), \varepsilon_r(\varepsilon_p)) f(y|a)\, dy &= c(a) + u_0, \text{ and} \\ \int \hat{v}(y; \varepsilon_p, \varepsilon_t(\varepsilon_p), \varepsilon_r(\varepsilon_p)) f_a(y|a)\, dy &= c'(a), \end{aligned}$$

so that starting from $\boldsymbol{\varepsilon} = \mathbf{0}$, if we make the small perturbation ε_p to v, we can restore (IC-FOC) and (IR) by a suitable combination of small applications ε_t and ε_r of $t_{\hat{y},\bar{y}}$ and $r_{\hat{y},\bar{y}}$.

Let λ be the rate of change of costs as one relaxes (IR) using $t_{\hat{y},\bar{y}}$ and $r_{\hat{y},\bar{y}}$. That is, if we let

$$\begin{pmatrix} q_t^{IR} \\ q_r^{IR} \end{pmatrix} = [Q(\mathbf{0})]^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then

$$\lambda = \int \rho^{-1}(v(y)) \left(q_t^{IR} t_{\hat{y},\bar{y}}(y) + q_r^{IR} r_{\hat{y},\bar{y}}(y) \right) f(y|a)\, dy.$$

Similarly, if

$$\begin{pmatrix} q_t^{IC} \\ q_r^{IC} \end{pmatrix} = [Q(\mathbf{0})]^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then the rate of change of costs as one relaxes (IC-FOC) using $t_{\hat{y},\bar{y}}$ and $r_{\hat{y},\bar{y}}$ is

$$\mu = \int \rho^{-1}(v(y)) \left(q_t^{IC} t_{\hat{y},\bar{y}}(y) + q_r^{IC} r_{\hat{y},\bar{y}}(y) \right) f(y|a)\, dy.$$

Given the shadow values λ and μ, the argument in Section 5 (prior to Definition 2) completes the proof of necessity in Proposition 5. ■

B.3.4 Proof of Sufficiency

We begin by proving the following useful result.

Lemma 4. *Let $v(\cdot)$ be GHM and suppose $y \in (\underline{y}, \bar{y})$ is free. Then $n(y) \leq 0$, and $n(y) = 0$ if y is a point of normal concavity (as defined immediately before Corollary 2).*

Proof. If y is a kink point, then Lemma 3 applied to the left of y implies that $n(y) \leq 0$. If y is a point of normal concavity, then by Lemma 1 there exist sequences of points $\{y_k^L\}, \{y_k^H\} \in C_v$ such that $y_k^L < y < y_k^H$ for all $k \in \mathbb{N}$ and $\lim_k y_k^L = \lim_k y_k^H = y$. These points are free, so 3 holds with equality on each interval $[y_k^L, y_k^H]$. Hence, in the limit, $n(y) = 0$.

□

Now, let v, with associated λ and μ, be GHM. Let us show that v is optimal. We will argue by contradiction. Assume v is not optimal, and let v^* be a lower cost contract satisfying (IC-FOC) and (IR)-(NG). As in the argument at the beginning of Appendix B, v^* can be taken to be increasing, satisfy (IC-FOC) exactly, and as in the proof of Lemma 5 in Appendix D, $v^*(\bar{y})$ and $v^*(\underline{y})$ can be taken to be finite.

Enumerate the closed linear segments $S_1, S_2, \cdots$, of v, and let $S = \cup S_i$. Let $\delta(y) = v^*(y) - v(y)$, and let $\hat{v}(y;\varepsilon) = v(y) + \varepsilon\delta(y)$, so that $\hat{v}(\cdot, 0) = v(\cdot)$ and $\hat{v}(\cdot, 1) = v^*(\cdot)$. Then, for each ε, $\hat{v}(\cdot;\varepsilon)$ is a convex combination of the concave contracts v and v^*. Hence, $\hat{v}(\cdot;\varepsilon)$ satisfies (IC-FOC) and (IR)-(NG). Since $u^{-1}(\cdot)$ is convex, and since for each y, $\hat{v}(y;\varepsilon)$ is linear in ε, it follows that $\int u^{-1}(\hat{v}(y;\varepsilon)) f(y|a)\, dy$ is convex in ε. Thus, since

$$\begin{aligned}
\int u^{-1}(\hat{v}(y;0)) f(y|a)\, dx &= \int u^{-1}(v(y)) f(y|a)\, dy \\
&> \int u^{-1}(v^*(y)) f(y|a)\, dy \\
&= \int u^{-1}(\hat{v}(y;1)) f(y|a)\, dy,
\end{aligned}$$

it follows that

$$\begin{aligned}
0 &> \frac{d}{d\varepsilon}\int u^{-1}\left(\hat{v}\left(y;0\right)\right) f\left(y|a\right) dy \\
&= \int \frac{1}{u'(u^{-1}\left(\hat{v}\left(y;0\right)\right))}\delta\left(y\right) f\left(y|a\right) dy \\
&= \int \rho^{-1}\left(v\left(y\right)\right)\delta\left(y\right) f\left(y|a\right) dy \\
&= \int_S \rho^{-1}\left(v\left(y\right)\right)\delta\left(y\right) f\left(y|a\right) dy + \int_{\mathcal{Y}\backslash S} \rho^{-1}\left(v\left(y\right)\right)\delta\left(y\right) f\left(y|a\right) dy,
\end{aligned}$$

and so, since every point in $\mathcal{Y}\backslash S$ is a point of normal concavity (noting that we took the sets S_i to be closed, and so any kink point is in S), we have

$$\begin{aligned}
\int_S \rho^{-1}\left(v\left(y\right)\right)\delta\left(y\right) f\left(y|a\right) dy &< -\int_{\mathcal{Y}\backslash S} \rho^{-1}\left(v\left(y\right)\right)\delta\left(y\right) f\left(y|a\right) dy \\
&= -\int_{\mathcal{Y}\backslash S} \left(\lambda + \mu l\left(y|a\right)\right)\delta\left(y\right) f\left(y|a\right) dy \\
&= -\lambda\int_{\mathcal{Y}\backslash S} \delta\left(y\right) f\left(y|a\right) dy - \mu\int_{\mathcal{Y}\backslash S} \delta\left(y\right) f_a\left(y|a\right) dy.
\end{aligned}$$

where the first equality follows by Lemma 4.

Both v and v^* satisfy (IC-FOC) with equality, and hence $\int \delta\left(y\right) f_a\left(y|a\right) dy = 0$, from which

$$-\mu\int_{\mathcal{Y}\backslash S} \delta\left(y\right) f_a\left(y|a\right) dy = \mu\int_S \delta\left(y\right) f_a\left(y|a\right) dy.$$

Similarly, either (IR) is binding at v, in which case $\int \delta\left(y\right) f\left(y|a\right) dy \geq 0$, or (IR) does not bind at v, in which case $\lambda = 0$, and hence in either case

$$-\lambda\int_{\mathcal{Y}\backslash S} \delta\left(y\right) f\left(y|a\right) dy \leq \lambda\int_S \delta\left(y\right) f\left(y|a\right) dy.$$

Making these two substitutions thus yields

$$\int_S \rho^{-1}\left(v\left(y\right)\right)\delta\left(y\right) f\left(y|a\right) dy < \lambda\int_S \delta\left(y\right) f\left(y|a\right) dy + \mu\int_S \delta\left(y\right) f_a\left(y|a\right) dy.$$

Hence, since $S = \cup S_i$, where the S_i's are disjoint except possibly at their zero-measure boundaries, there must be some i such that

$$\int_{S_i} \rho^{-1}(v(y))\,\delta(y)\,f(y|a)\,dy < \lambda \int_{S_i} \delta(y)\,f(y|a)\,dy + \mu \int_{S_i} \delta(y)\,f_a(y|a)\,dy,$$

or equivalently,

$$\int_{S_i} n(y)\,\delta(y)\,f(y|a)\,dy < 0.$$

Fix such an i, and consider δ_1, the restriction of δ to $S_i = [y_L, y_H]$. Since v is linear on S_i, and v^* is concave, δ_1 is concave. For any given K, let $\Delta = (y_H - y_L)/2^K$, and consider the function δ_K on $[y_L, y_H]$ that agrees with δ_1 on the set of points $\{y_L, y_L + \Delta, \cdots, y_H\}$, and is linear between these points. Note that δ_K is concave and continuous on $[y_L, y_H]$, and that for each y, $\delta_K(y)$ is monotonically increasing in K with limit $\delta(y)$. Hence, we can choose $\hat{K}$ large enough that

$$\int_{S_i} n(y)\,\delta_{\hat{K}}(y)\,f(y|a)\,dy < 0.$$

Finally, define $\tilde{\delta}$ on $\left[\underline{y}, \bar{y}\right]$ by

$$\tilde{\delta}(y) = \begin{cases} 0 & y \leq y_L \\ \delta_{\hat{K}}(y) & y \in [y_L, y_H] \\ \delta_{\hat{K}}(y_H) & y > y_H \end{cases}.$$

Note that y_H and $\bar{y}$ are free. Note also that as in the proof of Lemma 3, $v(y_H) > \underline{u}$. It follows from Definition 2.1 that since $\tilde{\delta}$ is constant on $[y_H, \bar{y}]$,

$$\int_{y_H}^{\bar{y}} n(y)\,\tilde{\delta}(y)\,f(y|a)\,dy = 0,$$

and hence,

$$\int n(y)\,\tilde{\delta}(y)\,f(y|a)\,dy < 0.$$

Let us next argue that $\tilde{\delta}$ can be expressed as a sum of raises and tilts. For $k \in \left\{0, \ldots, 2^{\hat{K}}\right\}$, let $y_k = y_L + k\Delta$, and let s_k be the slope of $\tilde{\delta}$ on (y_{k-1}, y_k).

Then, we claim that for all y in $[y_L, y_H]$,

$$\tilde{\delta}(y) = \delta(y_0)\, r_{y_0,\bar{y}}(y) + \sum_{k=1}^{2^{\hat{K}}-1} (s_k - s_{k+1})\, t_{y_0,y_k}(y) + s_{2^{\hat{K}}} t_{y_0,y_{2^{\hat{K}}}}(y). \qquad (10)$$

To see (10) note first that for $y < y_0 = y_L$, both sides of the equation are 0. At y_0, each side is $\delta(y_0)$, since $r_{y_0,\bar{y}}(y_0) = 1$, and since $t_{y_0,y_k}(y_0) = 0$ for all k. Thus, since both sides are continuous and piecewise linear on $[y_0, \bar{y}]$, it is enough that the two sides have that same derivative where defined. So, fix $\hat{k} \in \left\{1, \ldots, 2^{\hat{K}}\right\}$, and let $y \in \left(y_{\hat{k}-1}, y_{\hat{k}}\right)$. Note that for $k < \hat{k}$, $t'_{y_0,y_k}(y) = 0$, and for $k \geq \hat{k}$, $t'_{y_0,y_k}(y) = 1$. Hence, the derivative of the right-hand side is

$$\sum_{k=\hat{k}}^{2^{\hat{K}}-1} (s_k - s_{k+1}) + s_{2^{\hat{K}}} = s_{\hat{k}},$$

as desired, and so, noting that $\tilde{\delta}'(y) = 0$ for $y > y_K = y_H$, we have established (10).

Since $\int n(y)\, \tilde{\delta}(y)\, f(y|a)\, dy < 0$, we must thus have at least one of

1. $\delta(y_0) \int n(y)\, r_{y_0,\bar{y}}(y)\, f(y|a)\, dy < 0$,
2. for some $k < 2^{\hat{K}}$, $(s_k - s_{k+1}) \int n(y)\, t_{y_0,y_k}(y)\, f(y|a)\, dy < 0$, or
3. $s_{2^{\hat{K}}} \int n(y)\, t_{y_0,y_{2^{\hat{K}}}}(y)\, f(y|a)\, dy < 0$.

By Definition 2.1, and since y_0 is free, $\int n(y)\, r_{y_0,\bar{y}}(y)\, f(y|a)\, dy = \int_{y_0}^{\bar{y}} n(y)\, f(y|a)\, dy \geq 0$, and so 1 cannot hold. Since $\tilde{\delta}$ is concave on $[y_L, y_H]$, it follows that $s_k - s_{k+1} \geq 0$, and so, since y_0 is free, it follows by Definition 2.2 that 2 cannot hold either. Finally, since y_0 and $y_{2^{\hat{K}}}$ are both free, the integral in 3 is in fact 0 by Definition 2.2 and Definition 2.3. We thus have the required contradiction, and v is in fact optimal. ■

B.4 Proof of Corollary 2

This result follows immediately from Proposition 5 and Lemma 4. ■

C For Online Publication: Proofs for Section 6

C.1 Proof of Proposition 6

Since $s(\cdot) \geq -M$, $V_s(x) = \int s(y) f(y|x) dy \geq -M$ and so $V_s^c(\cdot) \geq -M$. Consider relaxing (7) so that the principal can choose any $V_s(\cdot)$ that is concave and satisfies $V_s(\cdot) \geq -M$. In this relaxed problem, the principal solves

$$
\begin{aligned}
\max_{a, V_s(\cdot)} \quad & a - V_s(a) \\
\text{s.t.} \quad & a \in \arg\max_{\tilde{a}} \{V_s(\tilde{a}) - c(\tilde{a})\} \\
& V_s(a) - c(a) \geq u_0 \\
& V_s(y) \geq -M \text{ for all } y \in \mathcal{Y} \\
& V_s(\cdot) \text{ is weakly concave.}
\end{aligned}
$$

This problem is identical to (Obj)-(NG) with a degenerate distribution over intermediate output.

Suppose $(a^*, V_s(\cdot))$ is optimal in this relaxed program. Note that $s_{a^*}^L(\cdot)$ is feasible in this relaxed problem, so $V_s(a^*) \leq s_{a^*}^L(a^*)$. Suppose $s_{a^*}^L(\cdot)$ is not optimal, so $V_s(a^*) < s_{a^*}^L(a^*)$. Then $s_{a^*}^L(a^*) - c(a^*) > u_0$, and so $s_{a^*}^L(\underline{y}) = -M$. Define $s^L(\cdot)$ as the linear function that intersects $V_s(\cdot)$ at $\underline{y}$ and a^*, so

$$
s^L(y) = V_s(\underline{y}) + \frac{V_s(a^*) - V_s(\underline{y})}{a^* - \underline{y}} \left(y - \underline{y}\right).
$$

Since $V_s(a^*)$ is concave, $s^L(y) \leq V_s(y)$ for all $y \in [\underline{y}, a^*]$.

For the agent to be willing to choose a^* under $V_s(\cdot)$, it must be that $\partial^- V_s(a^*) \geq c'(a^*)$, where $\partial^- V_s(y)$ is the left derivative of $V_s(\cdot)$ at y. Since V_s is concave,

$$
\frac{V_s(a^*) - V_s(\underline{y})}{a^* - \underline{y}} \geq \partial^- V_s(y) \geq c'(a^*).
$$

Since $V_s(\underline{y}) \geq M$, we conclude that $s^L(y) \geq s_{a^*}^L(y)$ for all $y \in \mathcal{Y}$. But then $V_s(a^*) = s^L(a^*) \geq s_{a^*}^L(a^*)$, which gives a contradiction. So $(a^*, s_{a^*}^L(\cdot))$ is also optimal. Note that for any $a^* > a^{FB}$, $(a^*, s^L_{a^*}(\cdot))$ is strictly dominated by $(a^{FB}, s^L_{a^{FB}}(\cdot))$, which generates higher total surplus and gives a (weakly) lower payment to the agent. So $a^* \leq a^{FB}$ and $s^L_{a^*}(\cdot)$ is optimal in this relaxed problem.

Finally, note that for any $a \geq 0$ and $x \in \mathcal{Y}$, $V_{s^L_a}(x) = \mathbb{E}_{F(\cdot|x)}\left[s^L_a(y)\right] = s^L_a(x)$ because $\mathbb{E}_{F(\cdot|x)}[y] = x$. But then $V^c_{s^L_a}(a) = V_{s^L_a}(a) = s^L_a(a)$, and so the optimal linear $V_s(\cdot)$ in the relaxed problem can be implemented in the full problem by $s^L_{a^*}(\cdot)$. ■

C.2 Proof of Proposition 7

Given the definition of $\tilde{v}(\cdot)$, $\tilde{c}$, and $\tilde{\pi}$, the optimal a and $\tilde{v}(\cdot)$ solve

$$\max_{a,G\in\mathcal{G},\tilde{v}(\cdot)} \mathbb{E}_{F(\cdot|a)}\left[\mathbb{E}_{G_x}\left[\tilde{\pi}(y) - \tilde{v}(y)\right]\right] \tag{11}$$
$$\text{s.t. } a, G \in \arg\max_{\tilde{a},\tilde{G}\in\mathcal{G}} \left\{\mathbb{E}_{F(\cdot|\tilde{a})}\left[\mathbb{E}_{\tilde{G}_x}\left[\tilde{v}(y)\right]\right] - \tilde{c}(\tilde{a})\right\}$$
$$\mathbb{E}_{F(\cdot|a)}\left[\mathbb{E}_{G_x}\left[\tilde{v}\left(y\right)\right]\right] - \tilde{c}(a) \geq u_0$$
$$\tilde{v}(y) \geq -M - d(y)\ \forall y \in \mathcal{Y}.$$

As in Proposition 1, following any intermediate output x, the agent optimally chooses G_x so that $\mathbb{E}_{G_x}[\tilde{v}(x)] = \tilde{v}^c(x)$, where $\tilde{v}^c(\cdot)$ is the concave closure of $\tilde{v}(\cdot)$. Therefore, the principal's payoff following x equals $\mathbb{E}_{G_x}[\tilde{\pi}(y) - \tilde{v}(y)] \leq \tilde{\pi}(x) - \tilde{v}^c(x)$. Since $\tilde{\pi}(\cdot)$ is strictly concave, this inequality holds with equality only if G_x is degenerate. Consequently, we can restrict attention to contracts in which $\tilde{v}(\cdot)$ is concave, and hence for every x, the agent will optimally choose $G_x(y) = \mathbb{I}_{\{y\geq x\}}$.

Relax the limited liability constraint so that it must be satisfied only at

$y = \underline{y}$. Then (11) can be written as

$$\begin{aligned} &\max_{a,\tilde{v}(\cdot)} \mathbb{E}_{F(\cdot|a)}\left[\tilde{\pi}(y) - \tilde{v}(y)\right] \\ &\text{s.t. } a \in \arg\max_{\tilde{a}} \left\{\mathbb{E}_{F(\cdot|\tilde{a})}\left[\tilde{v}(y)\right] - \tilde{c}(\tilde{a})\right\} \\ &\qquad \mathbb{E}_{F(\cdot|a)}\left[\tilde{v}\left(y\right)\right] - \tilde{c}(a) \geq u_0 \\ &\qquad \tilde{v}(\underline{y}) \geq -M \\ &\qquad \tilde{v}(\cdot) \text{ is concave.} \end{aligned}$$

Fix any effort $a \geq 0$ and any concave incentive scheme $\tilde{v}(\cdot)$ that implements a. As in the proof of Proposition 2, let $\tilde{v}^L(\cdot)$ be the unique linear incentive scheme that satisfies $\tilde{v}^L(\underline{y}) = \tilde{v}(\underline{y})$ and $\mathbb{E}_{F(\cdot|a)}\left[\tilde{v}^L(y)\right] = \mathbb{E}_{F(\cdot|a)}\left[\tilde{v}(y)\right]$. Then $\tilde{v}^L(\cdot) - \tilde{v}(\cdot)$ single-crosses 0 from below and hence Beesack's inequality implies

$$\int (\tilde{v}^L(y) - \tilde{v}(y)) \frac{f_a(y|a)}{f(y|a)} f(y|a) dx \geq 0$$

with strict inequality if $\tilde{v}^L(y) \neq \tilde{v}(y)$ for some y. Consequently, $\tilde{v}^L(\cdot)$ implements some $\tilde{a} \geq a$, with $\tilde{a} > a$ if $\tilde{v}^L(y) \neq \tilde{v}(y)$ for some y.

Define $\tilde{v}^*(y) = \tilde{c}'(a)(y - \underline{y}) - \tilde{w}$, where $\tilde{w} = \min\left\{M, \tilde{c}'(a)(a - \underline{y}) - \tilde{c}(a) - u_0\right\}$, and suppose that $\tilde{v}^*(\underline{y}) = -M$. Then $\tilde{v}^*(y) \leq \tilde{v}^L(y)$ for all $y \geq \underline{y}$ and strictly so if $\tilde{a} > a$. Therefore, $\tilde{v}^*(\cdot)$ uniquely implements $a \geq 0$ at maximum profit in the relaxed problem. But $\tilde{v}^*(y) \geq -M \geq -M - d(y)$ for all $y \in \mathcal{Y}$, so $\tilde{v}^*(\cdot)$ satisfies the limited liability constraint, and hence implements a in the original problem.

Next, suppose that $\tilde{v}^*(\underline{y}) > -M$. Then by construction, $\mathbb{E}_{F(\cdot|a)}\left[\tilde{v}^*(y)\right] = u_0 + \tilde{c}(a) \leq \mathbb{E}_{F(\cdot|a)}\left[\tilde{v}(y)\right]$, which implies that $\mathbb{E}_{F(\cdot|a)}\left[\tilde{\pi}(y) - \tilde{v}^*(y)\right] \geq \mathbb{E}_{F(\cdot|a)}\left[\tilde{\pi}(y) - \tilde{v}(y)\right]$; *i.e.*, $\tilde{v}^*(\cdot)$ implements a at maximum profit.

Finally, note that the preceding holds for any $a \geq 0$, proving that $\tilde{v}^*(\cdot)$, or equivalently, $s^*(y) = \tilde{c}'(a)(y - \underline{y}) - d(y) - \tilde{w}$ is optimal. ■

C.3 Proof of Proposition 8

It suffices to prove that for any total output x,

$$\max_{y_x:[0,1]\to[\underline{y},\bar{y}]}\left\{\int_0^1 s(y_x(t))\,dt \quad \text{s.t.} \quad \int_0^1 y_x(t)dt = x\right\} = s^c(x).$$

Consider the following y_x: if $s(x) = s^c(x)$, then $y_x(t) = x$ for all t. If $s(x) < s^c(x)$, then there exist w, z, and $\alpha \in [0,1]$ such that $\alpha w + (1-\alpha)z = x$ and $\alpha s(w) + (1-\alpha)s(z) = s^c(x)$. For $t \leq \alpha$, $y_x(t) = w$, with $y_x(t) = z$ for $t > \alpha$. This function y_x guarantees that the agent earns $s^c(x)$.
Now, $s(y_x(t)) \leq s^c(y_x(t))$ for all $y_x(t)$. Since s^c is weakly concave and $\int_0^1 y_x(t)dt = x$, we conclude that $\int_0^1 s(y_x(t))dt \leq \int_0^1 s^c(y_x(t))dt \leq \int_0^1 s^c(x)dt = s^c(x)$. So the agent earns (and the principal pays) $s^c(x)$ following intermediate output x, which proves the claim. ■

D For Online Publication: Existence, Uniqueness, and Continuity

The first part of this section proves existence and some properties of the optimal contract for the case of a finite limited liability constraint. The second part gives conditions under which an optimal contract exists and is unique if there is no limited liability constraint. While the statement of the latter case requires only some mild extra structure on the convexity of the utility function, the proof is embarrassingly complex.

D.1 Proof of existence, uniqueness, and continuity for $\underline{u}$ finite

Proposition 9. *Let U and Π be the set of increasing concave utility functions for the agent and principal satisfying our assumptions and let V be the set of concave (but not necessarily increasing) functions from $[\underline{y}, \bar{y}]$ to $\mathbb{R}$, where each of U, Π, and V has the topology of almost everywhere pointwise convergence.*

Fix a. Then, (i) for each $z = (M, u_0, \pi, u)$, there exists an optimal contract v that implements a given z and (ii) at any point z where at least one of π or u is strictly concave, the optimal contract implementing a is unique and continuous in z.

Proof. The proof relies on Berge's Theorem. Fix a. For any given $z = (M, u_0, u, \pi)$, let $v^L(\cdot|z)$ be given by $v^L(y|z) = c'(a)(y - \underline{y}) + \beta$, where $\beta = \min\left(u(-M), c(a) + u_0 - c'(a)(a - \underline{y})\right)$ be the maximum-profit linear (in utils) contract that implements a. In particular $v^L(\cdot|z)$ satisfies IC, since, under our assumptions, the agent's utility from income given $v^L(\cdot|z)$ is linear in effort while $-c(\cdot)$ is concave and so the first order condition implies (IC)

Let $B : \mathbb{R} \times \mathbb{R} \times \Pi \times U \rightarrow\rightarrow V$ be the correspondence which for each $M \in \mathbb{R}$, $u_0 \in \mathbb{R}$, $\pi \in \Pi$, and $u \in U$ gives the set of contracts v such that

$$\begin{aligned} E_{F(\cdot|a)}\left[\pi\left(y - u^{-1}(v(y))\right)\right] &\geq E_{F(\cdot|a)}\left[\pi\left(y - u^{-1}\left(v^L(y|z)\right)\right)\right] - 1, \quad (12) \\ a &\in \arg\max_{\tilde{a}}\left\{E_{F(\cdot|\tilde{a})}\left[v(y) - c(\tilde{a})\right]\right\}, \\ E_{F(\cdot|a)}\left[v(y) - c(a)\right] &\geq u_0, \\ v(\underline{y}) &\geq u(-M), \text{ and} \\ v &\in V, \end{aligned}$$

where the second through fifth constraints are simply the translations of (IC)-(NG) when z is a parameter, and the first constraint restricts attention to contracts that come within 1 util for the principal of $v^L(\cdot|z)$. Since $v^L(\cdot|z) \in B(z)$, this constraint is innocuous, and it also follows that B is non-empty valued.

For any given $v \in V$, define $v_{\max} = \max_{y \in [\underline{y}, \bar{y}]} v(y)$. We begin by proving

(*) For each compact subset $Z \subseteq \mathbb{R} \times \mathbb{R} \times \Pi \times U$, there is $\bar{u}$ such that $v_{\max} \leq \bar{u}$ for all $z \in Z$ and $v \in B(z)$.

To see $(*)$, begin by noting that $v^L(\cdot|\cdot)$ is continuous on the compact set $[\underline{y}, \bar{y}] \times Z$, and so $-\infty < m \equiv \min_{y \in [\underline{y}, \bar{y}] \times Z}\left\{\pi\left(y - u^{-1}\left(v^L(y|z)\right)\right)\right\}$. Using that Z is compact, let $u^* < \infty$ satisfy that for all $z \in Z$, $\pi\left(\bar{y} - u^{-1}(u^*)\right) \leq m - 2$, so that anytime the principal gives the agent utility u^* or above, the principal is at least two utils worse off than under $v^L(\cdot|z)$.

Fix $z \in Z$, and $v \in B(z)$. Choose $y_{\max}$ so that $v(y_{\max}) = v_{\max}$. Let $u_{\min} = \min_{z\in Z} u(-M)$, and define $\hat{v}$ as the function that equals $u_{\min}$ at $\underline{y}$ and $\bar{y}$, equals $v_{\max}$ at $y_{\max}$, and is linear to the left and right of $y_{\max}$. That is, $\hat{v}(y_{\max}) = y_{\max,}$ and

$$\hat{v}(y) = \left\{ \begin{array}{ll} u_{\min} + \frac{v_{\max}-u_{\min}}{y_{\max}-\underline{y}}\left(y-\underline{y}\right) & y \in [\underline{y}, y_{\max}) \\ u_{\min} + \frac{v_{\max}-u_{\min}}{\bar{y}-y_{\max}}\left(\bar{y}-y\right) & y \in (y_{\max}, \bar{y}] \end{array} \right\}.$$

Note that

$$E_{F(\cdot|a)}\left[\pi\left(y-u^{-1}\left(\hat{v}(y)\right)\right)\right] \geq E_{F(\cdot|a)}\left[\pi\left(y-u^{-1}\left(v^L(y|z)\right)\right)\right]-1, \qquad (13)$$

using that the concave function v is everywhere at or above $\hat{v}$, and the first constraint in (12).

We will show that (13) implies a uniform bound on $v_{\max}$. Intuitively, when $v_{\max}$ is large, the piece-wise linear function $\hat{v}(y)$ is above u^* for nearly all of $[\underline{y}, \bar{y}]$, implying losses compared to $v^L(\cdot|z)$ that contradict (13).

A uniform bound on $v_{\max}$ is of course trivial for v such that $v_{\max} \leq u^*$. So, assume $v_{\max} > u^*$. Let $y_L \in [\underline{y}, y_{\max})$ solve $\hat{v}(y_L) = u^*$, where if $y_{\max} = \underline{y}$, we let $y_L = \underline{y}$, and similarly, define $y_H \in (y_{\max}, \bar{y}]$ by $\hat{v}(y_H) = u^*$, where if $y_{\max} = \bar{y}$, $y_H = \bar{y}$.

Since $\hat{v}(\cdot)$ is concave, $\hat{v}(y) \geq u^*$ for all $y \in [y_L, y_H]$, and hence,

$$\pi\left(y-u^{-1}\left(\hat{v}(y)\right)\right)-\pi\left(y-u^{-1}\left(v^L(y|z)\right)\right) \leq -2.$$

while for any y,

$$\pi\left(y-u^{-1}\left(\hat{v}(y)\right)\right)-\pi\left(y-u^{-1}\left(v^L(y|z)\right)\right) \leq b,$$

where $b \equiv \pi(\bar{y}+\max_{z\in Z} M)-m$. So from (13) we must have

$$\left(F(y_H|a)-F(y_L|a)\right)(-2)+\left(1-\left(F(y_H|a)-F(y_L|a)\right)\right)b \geq -1,$$

or equivalently,

$$F(y_H|a) - F(y_L|a) \leq \frac{1+b}{2+b}, \tag{14}$$

where the RHS is strictly less than one because $\infty > b > 0$. But, if $y_L \neq \underline{y}$, then

$$\begin{aligned} y_L &= \underline{y} + \frac{u^* - u_{\min}}{v_{\max} - u_{\min}} \left(y_{\max} - \underline{y} \right) \\ &\leq \underline{y} + \frac{u^* - u_{\min}}{v_{\max} - u_{\min}} \left(\bar{y} - \underline{y} \right), \end{aligned}$$

and so as $v_{\max} \to \infty$, $y_L \to \underline{y}$. Similarly, if $y_H \neq \bar{y}$, then

$$\begin{aligned} y_H &= \bar{y} - \frac{u^* - u_{\min}}{v_{\max} - u_{\min}} \left(\bar{y} - y_{\max} \right) \\ &\geq \bar{y} - \frac{u^* - u_{\min}}{v_{\max} - u_{\min}} \left(\bar{y} - \underline{y} \right), \end{aligned}$$

and so as $v_{\max} \to \infty$, $y_H \to \bar{y}$. But then by (14), $v_{\max}$ is bounded, establishing $(*)$.

From $(*)$ and the dominated convergence theorem, each expectation in (12) is continuous in z, and hence, noting that each of (IC) and (NG) can be expressed as a collection of weak inequalities, $B(\cdot)$ is upper hemicontinuous.

Let us next show that $B(\cdot)$ is lower hemicontinuous. To see this, fix z, let $v \in B(z)$, and let $z_k \to z$. For $\varepsilon \in (0,1)$ and $\delta > 0$, define $\tilde{v}(\cdot|\varepsilon,\delta)$ by $\tilde{v}(y|\varepsilon,\delta) = (1-\varepsilon) v(y) + \varepsilon v^L(y|M-\delta, u_0+\delta, u, \pi)$.

By Jensen's inequality, for each y,

$$y - u^{-1}\left((1-\varepsilon) v(y) + \varepsilon v^L(y|z)\right) \geq (1-\varepsilon)\left(y - u^{-1}(v(y))\right) + \varepsilon\left(y - u^{-1}\left(v^L(y|z)\right)\right),$$

since $-u^{-1}$ is concave. Hence, since π is increasing and concave,

$$\begin{aligned} \pi\left(y - u^{-1}\left((1-\varepsilon) v(y) + \varepsilon v^L(y|z)\right)\right) &\geq \pi\left((1-\varepsilon)\left(y - u^{-1}(v(y))\right) + \varepsilon\left(y - u^{-1} v^L(y|z)\right)\right) \\ &\geq (1-\varepsilon)\pi\left(y - u^{-1}(v(y))\right) + \varepsilon\pi\left(y - u^{-1}\left(v^L(y|z)\right)\right). \end{aligned}$$

and so the same is true in expectation. Since the first constraint in (12) holds weakly for $v(\cdot)$ and strictly for $v^L(\cdot|z)$, we have that for each $\varepsilon \in (0,1)$,

$$E_{F(\cdot|a)}\left[\pi\left(y-u^{-1}\left((1-\varepsilon)v(y)+\varepsilon v^L(y|z)\right)\right)\right] > E_{F(\cdot|a)}\left[\pi\left(y-u^{-1}\left(v^L(y|z)\right)\right)\right]-1.$$

It follows from continuity that for each $n \in \{1,2,\cdots\}$ there exists $\frac{1}{n} > \delta_n > 0$ sufficiently small that the first constraint in (12) is slack for $v_n \equiv \tilde{v}\left(\cdot|\frac{1}{n},\delta_n\right)$.

It is immediate that the third and fourth constraints in (12) hold strictly at v_n, while the second and fifth constraints (which do not depend on z) continue to hold, since v_n is a convex combination of concave contracts satisfying IC. Hence, for each n, there is K_n such that for all $k \geq K_n$, $v_n \in B(z_k)$. Let $k_n = \max\{n, \max_{n' \leq n} K_n\}$. Then, for each n, $v_{k_n} \in B(z_{k_n})$, and, since $k_n \to \infty$ and $\delta_n \to 0$, $v_{k_n} \to v$. Hence, $B(\cdot)$ is lower hemicontinuous, and thus continuous.

Fix z, and let $\{v_k\}$ be a sequence in $B(z)$. Since each v_k is concave, and thus has variation at most $2(\bar{u}-u(-M))$, it follows from Helly's Selection Theorem that $\{v_k\}$ has a convergent subsequence. Thus, B is compact-valued, and from Berge's theorem the set of maximizers of $E_{F(\cdot|a)}\left[\pi\left(y-u^{-1}(v(y))\right)\right]$ on $B(\cdot)$ is non-empty valued and upper hemicontinous.

Finally, consider any z where at least one of π and u is strictly concave. Then, if v_1,$v_2 \in B(z)$, it is direct that $(v_1+v_2)/2 \in B(z)$ is strictly more profitable that either v_1 or v_2. Thus the maximum is unique, and hence continuous in z.

□

D.2 Proof of existence for $\underline{u} = -\infty$

Suppose that $\underline{u} = -\infty$, which corresponds to the agent having no liability constraint. This section gives conditions under which a unique solution to (P) exists and satisfies certain properties. Say that $u(\cdot)$ is *regular* if $\underline{w} = -\infty$ and $\frac{u'(w)u'''(w)}{u''(w)} < 3$ for all $w \in \mathbb{R}$. These conditions are quite mild; in particular, the second condition means that $u'(\cdot)$ is not excessively convex, in the sense that it has local concavity everywhere greater than -2. See Prekopa (1973) and Borell (1975) for details.

Proposition 10. *Suppose $\pi(y) \equiv y$, $u(\cdot)$ is strictly concave and regular, and $\underline{u} = -\infty$. Then for any $a \geq 0$, there exists a unique contract $v(\cdot)$ that implements a at maximum profit. Furthermore, there exists $\bar{u} < \infty$ independent of $\underline{u}$ such that $v(\bar{y}) < \bar{u}$ and $v(\underline{y}) > -\bar{u}$.*

We begin this argument with a lemma that shows that if $v_{\underline{u}}(\cdot)$ is an optimal contract for some limited liability constraint $\underline{u}$ and $v_{\underline{u}}(\underline{y}) > \underline{u}$, then $v_{\underline{u}}(\cdot)$ remains optimal in the problem with any less binding limited-liability constraint $\underline{u}'$, including $\underline{u}' = -\infty$.

Lemma 5. *Assume that for some $\underline{u} > -\infty$, $v_{\underline{u}}(\underline{y}) > \underline{u}$. Let $\underline{u}' < \underline{u}$. Then, $v_{\underline{u}'} = v_{\underline{u}}$.*

Proof. Assume $v_{\underline{u}}$ has $v_{\underline{u}}(\underline{y}) > \underline{u}$, but that when the limited liability constraint is some $\underline{u}' < \underline{u}$ there exists a superior concave contract $\hat{v}$ that implements a. We will show that this leads to a contradiction.

Assume first that $\hat{v}(\underline{y}) > -\infty$ (as is automatic if $\underline{u}'$ is finite). Then, for small enough ε, the contract $(1-\varepsilon)\, v_{\underline{u}}(\cdot) + \varepsilon \hat{v}(\cdot)$ is both strictly cheaper than $v_{\underline{u}}$ (since u is strictly concave), and implements a subject to liability constraint $\underline{u}$, yielding the desired contradiction.

Assume instead that $\hat{v}(\underline{y}) = -\infty$. Begin by picking any point $x' > \underline{y}$ where $x' \in C_{\hat{v}}$ (since $\hat{v}(\underline{y}) = -\infty$, such points exist), and construct $\tilde{v}$ by applying a sufficiently small positive amount of $t_{x',\bar{y}}$ such that $\tilde{v}$ remains strictly cheaper than $v_{\underline{u}}$. Since this adds a positive increasing function to $\hat{v}$, both (IC-FOC) and (IR) are strictly slack at $\tilde{v}$.

For each $y \in [\underline{y}, \bar{y}]$, let $h_y(\cdot)$ be a supporting plane to $\tilde{v}$ at y. Let the concave contract $v_y(\cdot)$ be given by $v_y(x) = \tilde{v}(x)$ for $x > y$, and $v_y(x) = h_y(x)$ for $x \leq y$. For each x, $v_y(x)$ is weakly decreasing in y, with $\lim_{y \to \underline{y}} v_y(x) = \tilde{v}(x)$. Thus, by the monotone convergence theorem, as $y \to \underline{y}$, $\int v_y(x) f(x|a)\, dx \to \int \tilde{v}(x) f(x|a)\, dx$, $\int v_y(x) f_a(x|a)\, dx \to \int \tilde{v}(x) f_a(x|a)\, dx$, and $\int u^{-1}(v_y(x)) f(x|a)\, dx \to \int u^{-1}(\tilde{v}(x)) f(x|a)\, dx$. Hence, for y close enough to $\underline{y}$, v_y implements a and is cheaper than $v_{\underline{u}}$. For any such y, $v_y(\underline{y})$ is finite, and we are back to the previous case.

□

Proof. Given Lemma 5, it is enough to show that for some $\underline{u}$, $v_{\underline{u}}(\underline{y}) > \underline{u}$. Assume not, so that in particular, for all $\underline{u}$, $v_{\underline{u}}(\underline{y}) = \underline{u}$. We will show that this leads to a contradiction. We will henceforth restrict attention to $\underline{u} \leq 0$. For $\underline{u}$ sufficiently negative, it cannot be the case that $v_{\underline{u}}$ is linear. In particular, if $v_{\underline{u}}$ is linear, then since $v_{\underline{u}}(\bar{y}) > u_0 + c(a)$, we have that

$$\int v_{\underline{u}}(x) f_a(x|a)\, dx = \int v'_{\underline{u}}(x)(-F_a(x|a))\, dx \geq \frac{u_0 + c(a) - \underline{u}}{\bar{y} - \underline{y}}$$

which diverges in $\underline{u}$, contradicting that $v_{\underline{u}}$ must satisfy (IC-FOC) with equality. Hence, for each $\underline{u}$, we can take a point $x_{\underline{u}} \in C_{v_{\underline{u}}}$, and derive $\lambda_{\underline{u}}$ and $\mu_{\underline{u}}$ as in the proof of Proposition 5.

Let $z_{\underline{u}}(\cdot) = \rho(\lambda_{\underline{u}} + \mu_{\underline{u}} l(\cdot|a))$, where we follow the convention that $\rho(s) = -\infty$ for $s \leq 0$. The contract $v_{\underline{u}}$ will in general differ from $z_{\underline{u}}$ since $z_{\underline{u}}$ need be neither concave nor satisfy the limited liability constraint. Note that $n_{\underline{u}}(\cdot) = \rho^{-1}(v_{\underline{u}}(\cdot)) - (\lambda_{\underline{u}} + \mu_{\underline{u}} l(\cdot|a)) =_s v_{\underline{u}}(\cdot) - z_{\underline{u}}(\cdot)$.

Step 1 There is $\bar{\mu} < \infty$ such that $\mu_{\underline{u}} \leq \bar{\mu}$ for all $\underline{u}$.

Proof of Step 1 Applying a small positive amount of $t_{x_{\underline{u}},\bar{y}}$ adds cost at rate at most $\rho^{-1}(\bar{u}) \int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f(x|a)\, dx$, adds incentives at rate $\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f_a(x|a)\, dx$, and relaxes (IR). It follows that

$$\mu_{\underline{u}} \leq \rho^{-1}(\bar{u}) \frac{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f(x|a)\, dx}{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f_a(x|a)\, dx}.$$

But, as in the proof that $|Q(\mathbf{0})| > 0$,

$$\begin{aligned}
& \frac{\partial}{\partial x_{\underline{u}}} \frac{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f(x|a)\, dx}{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f_a(x|a)\, dx} \\
&\underset{s}{=} -\frac{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f_a(x|a)\, dx}{\int_{x_{\underline{u}}}^{\bar{y}} (x - x_{\underline{u}}) f(x|a)\, dx} + \frac{\int_{x_{\underline{u}}}^{\bar{y}} f_a(x|a)\, dx}{\int_{x_{\underline{u}}}^{\bar{y}} f(x|a)\, dx} \\
&\leq 0,
\end{aligned}$$

and so we can take

$$\bar{\mu} = \rho^{-1}(\bar{u}) \frac{\int (x - \underline{y}) f(x|a)\, dx}{\int (x - \underline{y}) f_a(x|a)\, dx} < \infty.$$

Step 2 There is $\underline{\mu} > 0$ and $\underline{u}^* > -\infty$ such that $\mu_{\underline{u}} \geq \underline{\mu}$ for all $\underline{u} < \underline{u}^*$.

Proof of Step 2 Choose $-\infty < \underline{u}^* \leq 0$ such that

$$\rho^{-1}(\underline{u}^*) < \frac{1}{2}\rho^{-1}(u_0 + c(a)), \text{ and} \tag{15}$$

$$c'(a) < \frac{u_0 + c(a) - \underline{u}^*}{\bar{y} - \underline{y}}, \tag{16}$$

where such a $\underline{u}^*$ exists since by assumption $\lim_{w \to -\infty} \frac{1}{u'(w)} = 0$. Let

$$r \equiv \sup_{\tau \in \left[\frac{1}{2}\rho^{-1}(u_0 + c(a)), \infty\right)} \rho'(\tau).$$

Since $\rho(1/u'(w)) = u(w)$, we have that

$$\rho'\left(\frac{1}{u'(w)}\right) = \frac{(u')^3}{-u''}(w),$$

from which

$$\frac{\rho''\left(\frac{1}{u'(w)}\right)}{\rho'\left(\frac{1}{u'(w)}\right)} = u'(w)\left(\frac{u'''(w)\,u'(w)}{(u''(w))^2} - 3\right). \tag{17}$$

Since u is regular, it follows that $\rho'' < 0$, and so $r < \infty$. Let $\bar{l}_x = \max_x l_x(x|a)$, and choose $\underline{\mu} > 0$ such that

$$\underline{\mu} < \frac{1}{2}\frac{\rho^{-1}(u_0 + c(a))}{l(\bar{y}|a) - l(\underline{y}|a)}, \text{ and} \tag{18}$$

$$\underline{\mu} < \frac{1}{r\bar{l}_x}\frac{u_0 + c(a)}{\bar{y} - \underline{y}}. \tag{19}$$

Assume that for some $\underline{u} < \underline{u}^*$, $\mu_{\underline{u}} < \underline{\mu}$. We will show that this leads to a contradiction, establishing the result.

Using Corollary 2 (which depends only on the necessity part of the proof of Proposition 5, which is proven in Appendix B), and the fact that $\bar{y}$ is free, $n(\bar{y}) \leq 0$, and so $\lambda_{\underline{u}} + \mu_{\underline{u}} l(\bar{y}|a) \geq \rho^{-1}(v_{\underline{u}}(\bar{y})) \geq \rho^{-1}(u_0 + c(a))$. Thus,

$$\begin{aligned} \lambda_{\underline{u}} + \mu_{\underline{u}} l(\underline{y}|a) &= \lambda_{\underline{u}} + \mu_{\underline{u}} l(\bar{y}|a) - \mu_{\underline{u}}\left(l(\bar{y}|a) - l(\underline{y}|a)\right) \\ &\geq \rho^{-1}(u_0 + c(a)) - \frac{1}{2}\frac{\rho^{-1}(u_0 + c(a))}{l(\bar{y}|a) - l(\underline{y}|a)}\left(l(\bar{y}|a) - l(\underline{y}|a)\right) \\ &= \frac{1}{2}\rho^{-1}(u_0 + c(a)), \end{aligned} \tag{20}$$

where the inequality follows from $\mu_{\underline{u}} < \underline{\mu}$ and (18).

Since $\underline{u} < \underline{u}^*$, and by (15), $\rho^{-1}(v_{\underline{u}}(\underline{y})) = \rho^{-1}(\underline{u}) < \frac{1}{2}\rho^{-1}(u_0 + c(a))$. Thus, using (20), $n(\underline{y})$ is strictly positive, and it follows by Corollary 2 that $v_{\underline{u}}$ begins with a linear segment, the slope of which (by concavity) is at least

$$\frac{u_0 + c(a) - \underline{u}}{\bar{y} - \underline{y}} \geq \frac{u_0 + c(a)}{\bar{y} - \underline{y}}. \tag{21}$$

But, using (20) and the definition of r, we have that for all x,

$$\begin{aligned} z'_{\underline{u}}(x) &= \rho'(\lambda_{\underline{u}} + \mu_{\underline{u}} l(x|a))\, \mu_{\underline{u}} l_x(x|a) \\ &\leq r\mu_{\underline{u}} \bar{l}_x \\ &< \frac{u_0 + c(a)}{\bar{y} - \underline{y}}, \end{aligned}$$

where the strict inequality follows from (19). Hence, the initial linear segment of $v_{\underline{u}}$ crosses $z_{\underline{u}}$ at most once (from below). This implies that the entire contract is in fact linear with slope at least $(u_0 + c(a) - \underline{u}) / (\bar{y} - \underline{y})$. In particular, let x_H be the right end of the linear segment. If x_H is at or before the crossing point, then $v_{\underline{u}}$ violates (3) and so cannot be optimal by part 1 of Proposition 5. If $x_H < \bar{y}$ is after the crossing, then we violate Corollary 2. It follows that $v_{\underline{u}}$ generates incentives at least

$$\frac{u_0 + c(a) - \underline{u}^*}{\bar{y} - \underline{y}} > c'(a)$$

using (16). But we have shown that (IC-FOC) binds at $v_{\underline{u}}$, leading to the desired contradiction.

Step 3 There is $u_0 + c(a) > u_c > -\infty$ such that if $\underline{u} < \underline{u}^*$ and $\rho(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a))) < u_c$, then $z_{\underline{u}}(\cdot)$ is concave at x.

Proof of Step 3 Note first that ρ is trivially concave anywhere that it is equal to $-\infty$, and that by assumption, $\lim_{s \to 0} \rho(s) = -\infty$. Hence, it is enough to prove concavity where $\rho(\lambda_{\underline{u}} + \mu_{\underline{u}}(l(x|a)))$ is finite. But, it follows from (17) and the fact that u is regular that $\lim_{t \downarrow 0} \rho''(t) / \rho'(t) = -\infty$, and so $\rho''(t) / \rho'(t)$

is negative for t below some t'. Assume $\lambda_{\underline{u}} + \mu_{\underline{u}} (l (x|a)) < t'$. Then,

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} \rho (\lambda_{\underline{u}} + \mu_{\underline{u}} (l (x|a))) \quad &= \frac{\partial}{\partial x} (\rho' (\lambda_{\underline{u}} + \mu_{\underline{u}} (l (x|a))) \mu_{\underline{u}} l_x (x|a)) \\
&= \rho'' (\lambda_{\underline{u}} + \mu_{\underline{u}} (l (x|a))) (\mu_{\underline{u}} l_x (x|a))^2 + \rho' (\lambda_{\underline{u}} + \mu_{\underline{u}} (l (x|a))) \mu_{\underline{u}} l_{xx} (x|a) \\
&\underset{s}{=} \frac{\rho''}{\rho'} (\lambda_{\underline{u}} + \mu_{\underline{u}} (l (x|a))) \mu_{\underline{u}} + \frac{l_{xx}}{l_x^2} (x|a) \\
&\leq \frac{\rho''}{\rho'} (\lambda_{\underline{u}} + \mu_{\underline{u}} (l (x|a))) \underline{\mu} + \frac{l_{xx}}{l_x^2} (x|a) .
\end{aligned}$$

The second term is bounded by assumption. The first term diverges to $-\infty$ as $\lambda_{\underline{u}} + \mu_{\underline{u}} (l (x|a)) \to 0$. Hence, since ρ is monotone, and since $\lim_{w \to -\infty} u' (w) = \infty$, the result follows.

Step 4 As in the derivation of r in Step 2, let $\hat{r}$ be such that for all $t \geq \rho^{-1} (u_c)$, $\rho' (t) \leq \hat{r}$. Let $-\infty < \hat{\underline{u}} \leq \underline{u}^*$ satisfy

$$\hat{s} \equiv \frac{u_0 + c (a) - \hat{\underline{u}}}{\bar{y} - \underline{y}} \geq \max \left\{ c' (a) , \bar{\mu} \bar{l}_x \hat{r} \right\} ,$$

and assume that $\underline{u} < \hat{\underline{u}}$. Then, $z_{\underline{u}}(\underline{y}) \leq \underline{u}$.

Proof of Step 4 Assume that $z_{\underline{u}}(\underline{y}) > \underline{u}$. Then, since $v_{\underline{u}}(\underline{y}) = \underline{u}$, $v_{\underline{u}}$ begins with a linear segment of positive length of slope at least $\hat{s}$, and so by Proposition 5 and Part 1 of Definition 2, crosses $z_{\underline{u}}$ from below, and is strictly above $z_{\underline{u}}$ for an interval of positive length as well. Let $x_{\underline{u},c}$ be defined by $z_{\underline{u}} (x_{\underline{u},c}) = u_c$. If $v_{\underline{u}}$ has its initial crossing of $z_{\underline{u}}$ at or before $x_{\underline{u},c}$, then since $z_{\underline{u}}$ is concave until $x_{\underline{u},c}$, $v_{\underline{u}}$ remains above $z_{\underline{u}}$ until $x_{\underline{u},c}$. But then, since for $x > x_{\underline{u},c}$, $\hat{s} \geq z'_{\underline{u}}$, $v_{\underline{u}}$ in fact never recrosses $z_{\underline{u}}$. On the other hand, if the initial crossing of $z_{\underline{u}}$ by $v_{\underline{u}}$ is after $x_{\underline{u},c}$, then again, since $v_{\underline{u}}$ has slope greater than $z'_{\underline{u}}$ for $x > x_{\underline{u},c}$, $v_{\underline{u}}$ never recrosses $z_{\underline{u}}$. In either case, by Corollary 2, $v_{\underline{u}}$ is thus linear on all of $[\underline{y}, \bar{y}]$, a contradiction.

Step 5 Let $u_{y_0} = u_0 + c(a) - c'(a)(\bar{y} - y_0) > -\infty$. Then, $v_{\underline{u}}(y_0) \geq u_{y_0}$.

Proof of Step 5 Since $v_{\underline{u}}(\bar{y}) \geq u_0 + c(a)$, it follows that everywhere on $[\underline{y}, y_0)$, $v_{\underline{u}}(\cdot)$ is below the line $L(\cdot)$ that goes through $(y_0, v_{\underline{u}}(y_0))$ and $(\bar{y}, u_0 + c(a))$, and everywhere on $(y_0, \bar{y}]$, $v_{\underline{u}}(\cdot)$ is above $L(\cdot)$. Hence, since $f_a < 0$ on $[\underline{y}, y_0)$ and $f_a > 0$ on $(y_0, \bar{y}]$,

$$\begin{aligned} c'(a) &= \int v_{\underline{u}}(x) f_a(x|a)\, dx \\ &\geq \int L(x) f_a(x|a)\, dx \\ &= \frac{u_0 + c(a) - v_{\underline{u}}(y_0)}{\bar{y} - y_0}. \end{aligned}$$

Rearranging yields the desired result.

Step 6 Choose $\infty < u_s < \min\left\{u_{y_0}, u_c, \rho(-\bar{\mu} l(\underline{y}|a)), \hat{\underline{u}}\right\}$ small enough that for all $t \leq u_s$,

$$\rho'\left(\frac{1}{u'(u^{-1}(t))}\right)\bar{\mu}\underline{l_x} \geq \hat{s}, \tag{22}$$

where $\underline{l_x} = \min_x l_x(x|a) > 0$. Since $\rho'(\tau)$ diverges to ∞ as $\tau \downarrow 0$, and since $1/u'(u^{-1}(t))$ goes to 0 as $t \downarrow -\infty$, such a u_s is guaranteed to exist.

Step 7 Choose $\underline{u} < u_s$. Let $\bar{z}_{\underline{u}}(\cdot) = \rho\left(\bar{\lambda}_{\underline{u}} + \bar{\mu} l(\cdot|a)\right)$, where $\bar{\lambda}_{\underline{u}}$ solves $\rho(\bar{\lambda}_{\underline{u}} + \bar{\mu} l(\underline{y}|a)) = \underline{u}$. By Step 4, $z_{\underline{u}}(\underline{y}) \leq \underline{u}$, and so, since $\mu_{\underline{u}} \leq \bar{\mu}$, $z_{\underline{u}}(\cdot) \leq \bar{z}_{\underline{u}}(\cdot)$. Let $x_{\underline{u},s}$ be defined by $\bar{z}_{\underline{u}}(x_{\underline{u},s}) = u_s$. Since $\bar{\lambda}_{\underline{u}} + \bar{\mu} l(\underline{y}|a) = 1/u'(u^{-1}(\underline{u})) > 0$, it follows that $\bar{\lambda}_{\underline{u}} + \bar{\mu} l(y_0|a) \geq -\bar{\mu} l(\underline{y}|a)$, and hence

$$\rho\left(\bar{\lambda}_{\underline{u}} + \bar{\mu} l(y_0|a)\right) > \rho(-\bar{\mu} l(\underline{y}|a)) > u_s,$$

where the last inequality is by definition of u_s in Step 6. Thus, $x_{\underline{u},s} < y_0$.

Step 8 For all $x < x_{\underline{u},s}$, $v_{\underline{u}}(x) \leq \bar{z}_{\underline{u}}(x)$.

Proof of Step 8 Let $x_{\underline{u},c}$ be defined by $z_{\underline{u}}(x_{\underline{u},c}) = u_c$. By construction, $\bar{z}_{\underline{u}}(\cdot)$ is concave where $x \leq x_{\underline{u},c}$. Using (22), $\bar{z}'_{\underline{u}}(\cdot) > \hat{s}$ for $x < x_{\underline{u},s}$, and $\bar{z}'_{\underline{u}}(\cdot) < \hat{s}$ for $x \geq x_{\underline{u},c}$. Assume that for some $\tilde{x} < x_{\underline{u},s}$, $v_{\underline{u}}(\tilde{x}) > \bar{z}_{\underline{u}}(\tilde{x}) \geq z_{\underline{u}}(\tilde{x})$. By Corollary 2, $v_{\underline{u}}$ is linear at $\tilde{x}$. If $v'_{\underline{u}}(\tilde{x}) \leq \bar{z}'_{\underline{u}}(\tilde{x})$, then, since $\bar{z}_{\underline{u}}$ is concave on $[\underline{y}, x_{\underline{u},s}]$, and again using Corollary 2, $v_{\underline{u}}$ is also above $\bar{z}_{\underline{u}}$, and hence linear, for all x in $[\underline{y}, \tilde{x}]$. But then,

$$v_{\underline{u}}(\underline{y}) - \bar{z}_{\underline{u}}(\underline{y}) \geq v_{\underline{u}}(\tilde{x}) - \bar{z}_{\underline{u}}(\tilde{x}) > 0,$$

contradicting that $v_{\underline{u}}(\underline{y}) = \underline{u}$. Thus, $v'_{\underline{u}}(\tilde{x}) > \bar{z}'_{\underline{u}}(\tilde{x}) > \hat{s}$. But then, $v_{\underline{u}}$ remains linear, and hence strictly above the concave function $z_{\underline{u}}$ at least until $x_{\underline{u},c}$. For $x \geq x_{\underline{u},c}$, $\bar{z}'_{\underline{u}}(\tilde{x}) \leq \hat{s}$, and so as before v can never re-cross $\bar{z}_{\underline{u}}$, and so *a fortiori* can never re-cross $z_{\underline{u}}$. Hence $v_{\underline{u}}$ is linear on $[\tilde{x}, \bar{y}]$, with slope at least $\hat{s}$. Let L be the line that agrees with $v_{\underline{u}}$ on $[\tilde{x}, \bar{y}]$. To the left of $\tilde{x}$, $v_{\underline{u}}$, being concave, lies below L. But, $\tilde{x} < x_{\underline{u},s} < y_0$, and so, since $f_a(\cdot|a)$ is negative on $[\underline{y}, \tilde{x}]$,

$$\int v_{\underline{u}}(x) f_a(x|a)\,dx \geq \int L(x) f_a(x|a)\,dx \geq \hat{s} > c'(a),$$

again a contradiction.

Step 9 We show that $\lim_{\underline{u}\to-\infty} \int v_{\underline{u}}(x) f_a(x|a)\,dx = \infty$. For $\underline{u}$ sufficiently negative, this provides the necessary contradiction to the original supposition that $v_{\underline{u}}(\underline{y}) = \underline{u}$ for all $\underline{u}$, proving the result.

Proof of Step 9 By Step 8, for $\underline{u}$ sufficiently negative, $v_{\underline{u}}(x) \leq \bar{z}_{\underline{u}}(x)$ for all $x \leq x_{\underline{u},s}$. Let $v^T_{\underline{u}}$ truncate $v_{\underline{u}}$ to never pay more than u_s. Since $\max(0, v_{\underline{u}}(x) - u_s)$ is an increasing function, $\int \max(0, v_{\underline{u}}(x) - u_s) f_a(x|a)\,dx \geq 0$, and hence, $\int v_{\underline{u}}(x) f_a(x|a)\,dx \geq \int v^T_{\underline{u}}(x) f_a(x|a)\,dx$. Note also that since $v_{\underline{u}}(y_0) > u_s$, $v^T_{\underline{u}}(x) = u_s$ for all $x \geq y_0$. Let $\bar{z}^T_{\underline{u}}$ similarly truncate $\bar{z}_{\underline{u}}$ to pay u_s to the right of $x_{\underline{u},s}$. Then, $\bar{z}^T_{\underline{u}}$ is everywhere at least as large as $v^T_{\underline{u}}$, but equal to $v^T_{\underline{u}}$ everywhere to right of y_0. Hence, since f_a is negative to the left of y_0, we have

$$\int v_{\underline{u}}(x) f_a(x|a)\, dx \geq \int v^T_{\underline{u}}(x) f_a(x|a)\, dx \geq \int \bar{z}^T_{\underline{u}}(x) f_a(x|a)\, dx.$$

To arrive at a contradiction, it would thus be enough to show that $\int \bar{z}^T_{\underline{u}}(x) f_a(x|a)\, dx$ diverges as $\underline{u} \to -\infty$. But, by Moroni and Swinkels (2014, Lemma 4), under our regularity conditions, $\int \bar{z}_{\underline{u}}(x) f_a(x|a)\, dx$ does diverge as $\underline{u} \to -\infty$.

Let

$$u^{**} = \rho\left(1 + \bar{\mu}\left(l(\bar{y}|a) - l(\underline{y}|a)\right)\right) < \infty.$$

Then, for all $\underline{u}$ sufficiently negative that $\frac{1}{u'(u^{-1}(\underline{u}))} \leq 1$, $\bar{z}_{\underline{u}}(\bar{y}) \leq u^{**}$. Hence,

$$\begin{aligned}
\int \bar{z}_{\underline{u}}(x) f_a(x|a)\, dx - \int \bar{z}^T_{\underline{u}}(x) f_a(x|a)\, dx &= \int \left(\bar{z}_{\underline{u}}(x) - \bar{z}^T_{\underline{u}}(x)\right) f_a(x|a)\, dx. \\
&\leq \int_{y_0}^{\bar{y}} \left(\bar{z}_{\underline{u}}(x) - \bar{z}^T_{\underline{u}}(x)\right) f_a(x|a)\, dx \\
&\leq (u^{**} - u_s) \int_{y_0}^{\bar{y}} f_a(x|a)\, dx \\
&< \infty,
\end{aligned}$$

where the first inequality follows because $\bar{z}_{\underline{u}}(x) - \bar{z}^T_{\underline{u}}$ is weakly positive, and the second because it is bounded above by $u^{**} - u_s$.

□

E For Online Publication: Additional Results

E.1 Agent reports x

In this section, we allow the agent to send a contractible message $\tilde{x}$ after he observes x but before y is realized. Payments can therefore depend on both $\tilde{x}$ and y, which allows the principal to discipline the agent from engaging in risk-taking. Restricting attention to the case where both parties are risk-neutral, we show that a linear contract is optimal in this setting.

Since the principal does not benefit from risk-taking, it is without loss to restrict attention to mechanisms that punish the agent as much as possible whenever his report does not match the final output: $s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}$ for some upper-semicontinuous function $s(\cdot)$. Then the principal's problem is

$$\begin{aligned}
&\max_{a,\, s(\cdot)} \; \mathbb{E}_{F(\cdot|a),G}\left[y - s(y)\mathbb{I}_{\{y=\tilde{x}\}} + M\mathbb{I}_{\{y\neq\tilde{x}\}}\right] \\
&\quad \text{s.t. } a, G, \tilde{x} \in \arg\max_{\tilde{a},\tilde{G}\in\mathcal{G},\tilde{\tilde{x}}} \left\{\mathbb{E}_{F(\cdot|\tilde{a}),\tilde{G}}\left[s(y)\mathbb{I}_{\{y=\tilde{\tilde{x}}\}} - M\mathbb{I}_{\{y\neq\tilde{\tilde{x}}\}}\right] - c(\tilde{a})\right\} \\
&\qquad \mathbb{E}_{F(\cdot|a),G}\left[s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}\right] - c(a) \geq u_0 \\
&\qquad s(\cdot) \geq -M,
\end{aligned} \tag{23}$$

where $\tilde{x}$ maps x to a report made to the principal.

Fix $s(\cdot)$, and consider the agent's choice of G_x and $\tilde{x}$ following any intermediate output $x > \underline{y}$. Define

$$\lambda_s(x) = \max\left\{\lambda \,:\, \lambda(y - \underline{y}) - M = s(y) \text{ for some } y \geq x\right\}. \tag{24}$$

Intuitively, $\lambda_s(x)$ is the smallest slope such that $\lambda_s(x)(y-\underline{y})-M \geq s(y)$ for all $y \geq x$. We show that following intermediate output $x > \underline{y}$, the agent optimally chooses G_x and $\tilde{x}$ so that his expected payoff is $\lambda_s(x)(x - \underline{y}) - M$.[21]

Lemma 6. *For any $s(\cdot)$ and $x \in \mathcal{Y}$, the principal's expected payment to the agent equals:*

$$\sigma_s(x) \equiv \max_{G_x,\,\tilde{x}}\left\{\mathbb{E}_{G_x}\left[s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}\right]\right\} = \begin{cases} s(\underline{y}) & \text{if } x = \underline{y} \\ \lambda_s(x)(x - \underline{y}) - M & \text{if } x > \underline{y} \end{cases}. \tag{25}$$

Proof. Fix $s(\cdot)$ and $x > \underline{y}$. First, we show that there exists some G_x and $\tilde{x}$ such that $\mathbb{E}_{G_x}\left[s(y)\mathbb{I}_{\{y=\tilde{x}\}} - M\mathbb{I}_{\{y\neq\tilde{x}\}}\right] = \lambda_s(x)(x - \underline{y}) - M$. By definition of $\lambda_s(\cdot)$, there exists a $\hat{y} \geq x$ such that $\lambda_s(x)(\hat{y} - \underline{y}) - M = s(\hat{y})$. Let $\tilde{x} = \hat{y}$

[21] If $x = \underline{y}$, then the agent is compelled to choose $G_{\underline{y}}(y) = 1$, so his expected payoff is equal to $s(\underline{y})$.

and $G_x(y) = (1-p_{\hat{y}}) + p_{\hat{y}}\mathbb{I}_{\{y \geq \hat{y}\}}$, where $p_{\hat{y}} = \frac{x-\underline{y}}{\hat{y}-\underline{y}}$; *i.e.*, $y = \underline{y}$ with probability $1-p_{\hat{y}}$, and $y = \hat{y}$ with probability $p_{\hat{y}}$. Then the agent's expected payoff is

$$\begin{aligned} p_{\hat{y}} s\left(\hat{y}\right) - (1-p_{\hat{y}})M &= \frac{x-\underline{y}}{\hat{y}-\underline{y}} s\left(\hat{y}\right) - \frac{\hat{y}-x}{\hat{y}-\underline{y}} M \\ &= \frac{x-\underline{y}}{\hat{y}-\underline{y}} \left[\lambda_s(x)(\hat{y}-\underline{y}) - M\right] - \frac{\hat{y}-x}{\hat{y}-\underline{y}} M \\ &= \lambda_s(x)(x-\underline{y}) - M. \end{aligned}$$

Next, we show that the agent cannot earn more than $\lambda_s(x)(x-\underline{y}) - M$ following intermediate output x. For any report $\tilde{x}$, the agent earns more than $-M$ only if $y = \tilde{x}$, so his optimal distribution G_x maximizes the probability that $y = \tilde{x}$ subject to the constraint that $\mathbb{E}_{G_x}[y] = x$. This is accomplished by choosing $G_x(\cdot)$ such that $y = \tilde{x}$ with some probability $p_{\tilde{x}}$ and $y = \underline{y}$ with probability $1-p_{\tilde{x}}$, where $p_{\tilde{x}}\tilde{x} + (1-p_{\tilde{x}})\underline{y} = x$. It suffices to show that the agent's expected payoff under this distribution is maximized if $\tilde{x} = \hat{y}$.

Suppose that there exists some $\tilde{x} \neq \hat{y}$ such that $p_{\tilde{x}} s(\tilde{x}) - (1-p_{\tilde{x}})M > p_{\hat{y}} s\left(\hat{y}\right) - (1-p_{\hat{y}})M = \lambda_s(x)(x-\underline{y}) - M$. Then there must exist some $\tilde{\lambda} > \lambda_s(x)$ such that $\tilde{\lambda}(\tilde{x}-\underline{y}) - M = s(\tilde{x})$, which contradicts the definition of $\lambda_s(x)$. Therefore, for all x, the agent's expected payoff equals $\lambda_s(x)(x-\underline{y}) - M$. $\square$

To see this result, recall that the agent earns $-M$ whenever his report does not equal the realized output. Therefore, if he misreports $\tilde{x} \neq x$, then he chooses G_x to maximize the probability that $y = \tilde{x}$. In particular, it is optimal for G_x to put weight on only two points, $\tilde{x}$ and $\underline{y}$. Given this $\tilde{x}$, the agent's payoff can be written as $p_{\tilde{x}} s(\tilde{x}) - (1-p_{\tilde{x}})M$, where $p_{\tilde{x}}\tilde{x} + (1-p_{\tilde{x}})\underline{y} = x$. It can be shown that the agent's payoff can be rewritten as $\lambda(x-\underline{y}) - M$, where $\lambda \leq \lambda_s(x)$. There exists some report $\tilde{x}$ that sets $\lambda = \lambda_s(x)$, proving the result.

Using Lemma 6, we can rewrite the principal's problem as

$$\begin{aligned}
&\max_{a,\, s(\cdot)} \mathbb{E}_{F(\cdot|a)}\left[x - \sigma_s(x)\right] \\
&\quad \text{s.t. } a \in \arg\max_{\tilde{a}} \left\{\mathbb{E}_{F(\cdot|\tilde{a})}\left[\sigma_s(x)\right] - c(\tilde{a})\right\} \\
&\qquad \mathbb{E}_{F(\cdot|a)}\left[\sigma_s(x)\right] - c(a) \geq u_0 \\
&\qquad s(\cdot) \geq -M
\end{aligned}$$

where for any contract $s(\cdot)$, $\sigma_s(\cdot)$ is given by (25).

Recall the definition of $s_a^L(\cdot)$ from Section 4. We show that if $a \geq 0$ is such that (LL) holds with equality after $\underline{y}$ under $s_a^L(\cdot)$, then $s_a^L(\cdot)$ implements a at maximum profit in this setting. Consequently, if (LL) binds for the optimal $a \geq 0$, then a linear contract is optimal as in Proposition 2.

Proposition 11. *Fix any effort $a \geq 0$. If $s_a^L(\underline{y}) = -M$, then $s_a^L(\cdot)$ implements a at maximum profit.*

Proof. Note that $\lambda_s(\cdot)$ is decreasing for any $s(\cdot)$, and moreover is constant for all $x \in \mathcal{Y}$ if $s(\cdot)$ is affine.Let $\hat{s}(\cdot)$ implement a at maximum profit, and suppose there exists $x_L < x_H$ such that $\lambda_{\hat{s}}(x_L) > \lambda_{\hat{s}}(x_H)$.

Define $s_L(y) = \beta(y - \underline{y}) - M$, where β is chosen such that $\mathbb{E}_{F(\cdot|a)}\left[s_L(y) - \lambda_{\hat{s}}(y)(y - \underline{y}) + M\right] = 0$. Such a β exists by the intermediate value theorem because $\lambda_{\hat{s}}(y) \geq 0$ is finite. Since $\lambda_{\hat{s}}(\cdot)$ is strictly decreasing over some interval, there exists some $y^* \in (\underline{y}, \bar{y})$ such that $\lambda_{\hat{s}}(y) \geq \beta$ if and only if $y \leq y^*$. Then $\beta - \lambda_{\hat{s}}(y)$ is first negative and then positive, $\int \left[\beta - \lambda_{\hat{s}}(y)\right](y - \underline{y}) f(y|a) dy = 0$ by construction, and $\frac{f_a(\cdot|a)}{f(\cdot|a)}$ is strictly increasing, so Beesack's inequality implies that

$$\int \left[\beta - \lambda_{\hat{s}}(y)\right](y - \underline{y}) f_a(y|a) dy > 0 \,.$$

Therefore, $s_L(\cdot)$ implements some effort level $a' > a$, which implies that $\beta > c'(a)$.

Observe that $s_a^L(y) < s_L(y)$ for all $y > \underline{y}$, because $s_a^L(\underline{y}) = -M$ by assumption and $c'(a) < \beta$. Moreover, $s_a^L(\cdot)$ implements a and satisfies both the individual rationality and limited liability constraints. Therefore, $s_a^L(\cdot)$ implements effort a at strictly higher profit than $\hat{s}(\cdot)$. So $\lambda_{\hat{s}}(\cdot)$ must be constant and $\sigma_{\hat{s}}(\underline{y}) = -M$, in which case $s_a^L(\cdot)$ is also optimal. $\square$

E.2 Mild Sufficient Conditions for Proposition 3

This Appendix gives sufficient conditions under which $\rho(\lambda + \mu l(\cdot|a))$ is first convex and then concave. We show that this case obtains if $con\,(\rho') + con\,(l_y) > -1$, where for an interval $X \subseteq \mathbb{R}$ and analytic function $h : X \to \mathbb{R}_+$, $con(h) = \inf_X \left\{1 - (hh'')/(h')^2\right\}$. For any analytic function q with domain a subset of the reals, let $q^{(k)}$ be the k^{th} derivative of q.

Lemma 7. *Assume $q > 0$ is not everywhere a constant, is analytic, and has $con\,(q) = \omega > -\infty$. Assume also that for some $\hat{y}$ on the interior of its domain, $q'\,(\hat{y}) = 0$. Let $\hat{k} = \min\left\{k | q^{(k)}\,(\hat{y}) \neq 0\right\}$. Then, $q^{(\hat{k})}\,(\hat{y}) < 0$.*

Proof. Note that $\hat{k} \geq 2$. Recall that q has concavity ω if q^{ω}/ω is concave, or, equivalently (cancelling the strictly positive term $q^{\omega-2}$), if for all y in the domain of q,

$$\xi\,(y) \equiv (\omega - 1)\,(q'\,(y))^2 + q\,(y)\,q''\,(y) \leq 0.$$

So, in particular, if $\hat{k} = 2$, then we must have $q''\,(\hat{y}) < 0$, since $\xi\,(\hat{y}) \leq 0$. Note that for $k \in \{0, 1, 2, \cdots\}$

$$\xi^{(k)}\,(\hat{y}) = d\,(\hat{y}) + q\,(\hat{y})\,q^{(k+2)}\,(\hat{y})\,,$$

where d is an expression involving derivatives of q of order less than $k + 2$. So, the first non-zero term of the Taylor expansion of ξ is $\frac{\xi^{(\hat{k}-2)}(\hat{y})}{(\hat{k}-2)!}\,(y - \hat{y})^{\hat{k}-2}$, where $\xi^{(\hat{k}-2)}\,(\hat{y}) = q\,(\hat{y})\,q^{(\hat{k})}\,(\hat{y})$. Hence, since $(y - \hat{y})^{\hat{k}-2} > 0$ for $y > \hat{y}$, while $\xi(y) \leq 0$, $q^{(\hat{k})}\,(\hat{y})$, which is non-zero by assumption, must be strictly negative. $\square$

Using this lemma, we can prove the following claim, from which our sufficient condition is immediate.

Claim 2. *Let g and h be strictly positive analytic functions with $con\,(g') + con\,(h') > -1$, and g' and h' everywhere strictly positive. Then, $(g\,(h\,(\cdot)))$ is never first strictly concave and then weakly convex.*

Proof. Let

$$\theta\,(\cdot) = (g\,(h\,(\cdot)))'' = g''\,(h')^2 + g'h''. \tag{26}$$

If both g and h are linear, then $\theta \equiv 0$, and we are done. Assume g and h are not both linear, and consider any point $\hat{y}$ at which $\theta = 0$. We will show that immediately to the right of $\hat{y}$, $\theta < 0$. This rules out that θ is ever first strictly negative and then weakly positive over any interval of non-zero length.

To see this, note that

$$\theta' = g'''\,(h')^3 + 3g''h'h'' + g'h'''. \tag{27}$$

Consider any point $\hat{y}$ at which $\theta = 0$. Consider first the case that $g''\,(\hat{y})\,h''\,(\hat{y}) \neq 0$. Then, since $g' > 0$, it follows by (26) that $g''(\hat{y})$ and $h''(\hat{y})$ have opposite sign. Hence, $g''\,(\hat{y})\,h''\,(\hat{y})\,h'\,(\hat{y}) < 0$, and so, evaluated at $\hat{y}$,

$$\begin{aligned}
\theta' \quad &\underset{s}{=} -\frac{g'''\,(h')^2}{g''h''} - 3 - \frac{g'h'''}{g''h''h'} \\
&= \frac{g'g'''}{(g'')^2} - 3 + \frac{h'h'''}{(h'')^2} \\
&\leq -con\,(g') - con\,(h') - 1 \\
&< 0
\end{aligned}$$

where in the second line we substitute for $(h')^2$ in the first term using (26) and that $\theta\,(\hat{y}) = 0$, and similarly for g' in the third term. Hence, θ is negative on an interval to the right of $\hat{y}$.

Assume instead that $g''\,(\hat{y})\,h''\,(\hat{y}) = 0$, where, since $\theta\,(\hat{y}) = 0$, it follows that $g''\,(\hat{y}) = h''\,(\hat{y}) = 0$. Thus, since $con\,(g') > -\infty$, it follows from Lemma 7 applied to $q = g'$ that the first non-zero derivative of g' is strictly negative, and similarly for h'. But then, the first non-zero derivative of θ will be of the form $g^{(k)}\,(h')^k + g'h^{(k)}$ with $k \geq 3$, and at least one term strictly negative, and so, taking a Taylor expansion, θ is strictly negative on an interval to the right of $\hat{y}$, and we are done.

$\square$

E.3 Comparative Static of Optimal Contract with respect to $\underline{y}$

This Appendix considers how a^* changes with the lower bound $\underline{y}$ on output. A decrease in $\underline{y}$ implies that the agent can take on more severe left-tail risk by gambling over worse outcomes. We prove that a lower $\underline{y}$ makes it costlier for the principal to induce any non-zero effort level. As $\underline{y}$ approaches $-\infty$, inducing any positive effort becomes arbitrarily expensive and so the agent exerts no effort in the optimal contract.

Corollary 3. *Consider a decreasing sequence $\{\underline{y}_k\}_{k=0}^{\infty}$ with $\lim_{k\to\infty} \underline{y}_k = -\infty$. For each $k \geq 0$, consider $\mathcal{Y} = [\underline{y}_k, \bar{y}]$ and some output distribution $F_k(\cdot|a)$ that satisfies our assumptions (i.e., has full support on $[\underline{y}_k, \bar{y}]$, satisfies $\mathbb{E}_{F_k(\cdot|a)}[x] = a$, etc), and let a_k^* be the corresponding optimal effort. Then $\lim_{k\to\infty} a_k^* = 0$, and if $\pi(y) \equiv y$, then a_k^* is decreasing in k.*

Proposition 2 implies that the principal's expected payment from inducing $a^* \geq 0$ equals $E_{F(\cdot|a^*)}[\pi(y - c'(a^*)(y - \underline{y}) + w)]$. For small enough $\underline{y}$, $s_{a^*}^L(\underline{y}) = -M$. But then implementing $a^* > 0$ becomes arbitrarily costly as $\underline{y} \to -\infty$, in which case the principal is better off not motivating the agent at all. If the principal is risk-neutral, then we can show that the principal's profit under $s_{a^*}^L(\cdot)$ is supermodular in a^* and $\underline{y}$, so that a^* is increasing in $\underline{y}$.

Proof. Fix $\hat{a} > 0$. Define

$$y_1 \equiv \min_{a\in[\hat{a},a^{FB}]} \left\{ a - \frac{c(a) + u_0 + M}{c'(a)} \right\},$$

and

$$y_2 \equiv \min_{a\in[\hat{a},a^{FB}]} \left\{ \frac{u^{-1}(u_0) - (1 - c'(a)a) - M}{c'(\hat{a})} \right\},$$

and note that since $c'(a) \geq c'(\hat{a}) > 0$ for all $a \geq \hat{a}$, $y_{min} \equiv \min\{0, y_1, y_2\} > -\infty$.

Let $\underline{y} < y_{min}$, and suppose towards a contradiction that there exists a distribution $F(\cdot|a)$ on $[\underline{y}, \bar{y}]$ such that effort $a^* \geq \hat{a}$ is optimal under $F(\cdot|a)$. Note first that Proposition 2 implies that the principal's expected payoff equals

$$\mathbb{E}_{F(\cdot|a^*)}\left[\pi(y - s^L_{a^*}(y))\right] = \mathbb{E}_{F(\cdot|a^*)}\left[\pi\left(y - c'(a^*)(y - \underline{y}) + \min\left\{M, c'(a^*)(a^* - \underline{y}) - c(a^*) - u_0\right\}\right)\right].$$

Since $\underline{y} < y_1$, $c'(a^*)(a^* - \underline{y}) - c(a^*) - u_0 > M$. Furthermore, the principal's payoff is bounded above by

$$\pi\left((1 - c'(a^*))a^* + c'(a^*)\underline{y} + M\right)$$

by Jensen's inequality. Since $\underline{y} < \min\{0, y_2\}$, $(1-c'(a))a + c'(a)\underline{y} + M < u^{-1}(u_0)$ for any $a \in [\hat{a}, a^{FB}]$. But then $a^* \geq \hat{a}$ cannot be optimal because it is strictly dominated by $a^* = 0$ and $s(\cdot) \equiv u^{-1}(u_0)$, a contradiction. Hence, for $\underline{y} < y_{min}$, any distribution $F(\cdot|a)$, and any optimal a^*, it must be that $a^* < \hat{a}$. Since $\hat{a} > 0$ is arbitrary, $\lim_{\underline{y} \to -\infty} a^* = 0$.

Suppose $\pi(y) \equiv y$. To prove that a^* is increasing in $\underline{y}$, it suffices to show that the principal's payoff from implementing a in an optimal contract, $\Pi(a, \underline{y}) = a - c'(a)(a - \underline{y}) + w$, is supermodular in a and $\underline{y}$.

Recall that $w = \min\left\{M,\ c'(a)(a - \underline{y}) - c(a) - u_0\right\}$ is a function of $(a, \underline{y})$. Therefore,

$$\frac{\partial \Pi}{\partial a} = 1 - c''(a)(a - \underline{y}) - c'(a) + \frac{\partial w}{\partial a}$$

and so

$$\frac{\partial^2 \Pi}{\partial \underline{y} \partial a} = c''(a) + \frac{\partial^2 w}{\partial \underline{y} \partial a}.$$

But $\frac{\partial^2 w}{\partial \underline{y} \partial a} = 0$ if $M < c'(a)(a - \underline{y}) - c(a) - u_0$ and $\frac{\partial^2 w}{\partial \underline{y} \partial a} = -c''(a)$ otherwise. In either case, $\frac{\partial^2 \Pi}{\partial \underline{y} \partial a} \geq 0$ and so optimal effort a^* is increasing in $\underline{y}$, as desired. $\square$