

Web Appendix to “The Optimal Degree of Cooperation in the Repeated Prisoners’ Dilemma with Side Payments”

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July 06, 2008

In this appendix, we prove Propositions 3-5 in Section 4 of the paper.

Proof of Proposition 3. Adapting the proof of Lemma 3, we observe that in the game with transferable utilities, achieving Pareto dominant outcomes of the entire game requires achieving a Pareto dominant outcome in very period on the equilibrium path. It also implies that the probability mix over the four action profiles in each period is stationary, although different histories (particularly, different realizations of the randomization in the action stage) may lead to different side payments in the following period.

Now consider for some point in the (δ_I, δ_P) -plane, a correlated strategy profile σ that implements a payoff vector that is Pareto dominant in the SPE payoff set. The equilibrium path of this strategy is represented canonically by a sequence of possible side payments $b_1, \{b_{CC,t}, b_{DC,t}, b_{CD,t}, b_{DD,t}\}, t > 1$, and the corresponding stationary action set probability matrices

$$\begin{array}{cc}
 & \begin{array}{cc} C & D \end{array} \\
 \begin{array}{c} C \\ D \end{array} & \begin{array}{|cc|} \hline \lambda_{CC} & \lambda_{CD} \\ \hline \lambda_{DC} & \lambda_{DD} \\ \hline \end{array}
 \end{array} \tag{13}$$

for all $t \geq 1$, where $\lambda_{CC} + \lambda_{CD} + \lambda_{DC} + \lambda_{DD} = 1$.

Adapting Lemma 2, it follows that in (13), maintaining I 's incentive constraints requires that he receives expected payoff of:

$$\begin{aligned} \frac{\lambda_{CC}}{\lambda_{CC} + \lambda_{CD}} (y + \underline{b}) + \frac{\lambda_{CD}}{\lambda_{CC} + \lambda_{CD}} \frac{x}{\delta_I}; & \text{ if called upon to play action } C, \\ x; & \text{ if called upon to play action } D, \end{aligned} \quad (14)$$

and this applies to *all* periods $t > 0$.

By direct computation, the average payoffs of players I and P starting for $t \geq 2$, evaluated before the transfers are made, are

$$\begin{aligned} U_{It} &= \lambda_{CC} (y + \underline{b}) + \lambda_{CD} \left(\frac{x}{\delta_I} \right) + (\lambda_{DC} + \lambda_{DD}) x, \\ U_{Pt} &= \lambda_{CC} (2y) + (\lambda_{CD} + \lambda_{DC}) z + \lambda_{DD} (2x) - U_{It} \\ &= \lambda_{CC} (y - \underline{b}) + \delta_{CD} \left(z - \frac{x}{\delta_I} \right) + \lambda_{DC} (z - x) + \lambda_{DD} x, \end{aligned}$$

and the Pareto frontier of the set of payoffs is then characterized by the following equation:

$$\begin{aligned} W(1) &\equiv \frac{U_{I1}}{1 - \delta_I} + \frac{U_{P1}}{1 - \delta_P} \\ &= \frac{\lambda_{CC}(y + \underline{b}) + \lambda_{CD} \frac{x}{\delta_I} + (\lambda_{DC} + \lambda_{DD})x}{1 - \delta_I} + \frac{\lambda_{CC}(y - \underline{b}) + \lambda_{CD}(z - \frac{x}{\delta_I}) + \lambda_{DC}(z - x) + \lambda_{DD}x}{1 - \delta_P}. \end{aligned} \quad (15)$$

It is immediate that $W(1)$ is linear in the probabilities, λ_{CC} , λ_{CD} , λ_{DC} , and λ_{DD} .

Note that achieving Pareto dominant outcomes also requires that regardless of the action profile chosen in period t , the Total Utility in period $t+1$, W_{t+1} , remains unaffected. Let $U_{I,t+1}^{(a_{I,t}, a_{P,t})}$ and $U_{P,t+1}^{(a_{I,t}, a_{P,t})}$ denote the average utilities of players I and P in period $t+1$ after the public randomization device has picked the action profile $(a_{I,t}, a_{P,t})$ in period t .

It must hold that

$$\frac{U_{I,t+1}^{(a_{I,t}, a_{P,t})}}{1 - \delta_I} + \frac{U_{P,t+1}^{(a_{I,t}, a_{P,t})}}{1 - \delta_P} = \frac{U_{I,t+1}^{(a'_{I,t}, a'_{P,t})}}{1 - \delta_I} + \frac{U_{P,t+1}^{(a'_{I,t}, a'_{P,t})}}{1 - \delta_P} = W_{t+1} \quad (16)$$

for any $a_{I,t}, a_{P,t}, a'_{I,t}, a'_{P,t} \in \{C, D\}$.

Now, we prove that

Lemma A2 \overrightarrow{DC} is Pareto dominant in correlated strategies for $(\delta_I, \delta_P) \in \Omega_{DC}^2$ and \overrightarrow{CC} is Pareto dominant in correlated strategies for $(\delta_I, \delta_P) \in \Omega_{CC}$.

Proof. Substituting the value of \underline{b} into (15), we have:

$$W = \frac{\lambda_{CC}\left(y + \left(\frac{1-\delta_I}{\delta_I}(z-y) - (y-x)\right)\right) + \lambda_{CD}(x/\delta_I) + (\lambda_{DC} + \lambda_{DD})x}{1-\delta_I} + \frac{\lambda_{CC}\left(y - \left(\frac{1-\delta_I}{\delta_I}(z-y) - (y-x)\right)\right) + \lambda_{CD}(z-x/\delta_I) + \lambda_{DC}(z-x) + \lambda_{DD}x}{1-\delta_P} \quad (17)$$

Direct computation will give us:

$$\frac{\partial W}{\partial \lambda_{DC}} - \frac{\partial W}{\partial \lambda_{CD}} = x \frac{\delta_P - \delta_I}{(1-\delta_P)\delta_I} > 0. \quad (18)$$

This implies that W is maximized by setting $\lambda_{CD} = 0$. Also

$$\frac{\partial W}{\partial \lambda_{DC}} - \frac{\partial W}{\partial \lambda_{DD}} = \frac{z-2x}{1-\delta_P} \geq 0 \text{ if and only if } z \geq 2x. \quad (19)$$

This implies that W is maximized by setting $\lambda_{DC} = 0$ if $z < 2x$ and $\lambda_{DD} = 0$ if $z \geq 2x$.

Finally,

$$\frac{\partial W}{\partial \lambda_{CC}} - \frac{\partial W}{\partial \lambda_{DC}} = \frac{\delta_I y - \delta_P(z-y)}{\delta_I(1-\delta_P)} < 0 \text{ if and only if } \frac{\delta_P}{\delta_I} > \frac{y}{(z-y)}.$$

This implies that W is maximized by setting $\lambda_{CC} = 0$ if $\frac{\delta_P}{\delta_I} > \frac{y}{(z-y)}$ and $\lambda_{DC} = 0$ if $\frac{\delta_P}{\delta_I} \leq \frac{y}{(z-y)}$.

Consider $(\delta_I, \delta_P) \in \Omega_{DC}^1$. Since Ω_{DC}^1 is an empty set when $z < 2x$, we only have to analyze the case in which $z \geq 2x$. Since W is maximized by setting $\lambda_{CC} = \lambda_{CD} = \lambda_{DD} = 0$ and \overrightarrow{DC} is a SPE according to Proposition 2, \overrightarrow{DC} must also be Pareto dominant.

Now consider $(\delta_I, \delta_P) \in \Omega_{CC}$. Note that for all $(\delta_I, \delta_P) \in \Omega_{CC}$, $\frac{\delta_P}{\delta_I} \leq \frac{y}{(z-y)}$ so W is maximized by setting $\lambda_{DC} = \lambda_{CD} = 0$. Since

$$\frac{\partial W}{\partial \lambda_{CC}} - \frac{\partial W}{\partial \lambda_{DD}} = \frac{\delta_I(z-x) + \delta_I(y-x) - \delta_P(z-y)}{\delta_I(1-\delta_P)}$$

is independent of λ_{CC} or λ_{DD} , W is maximized at either $\lambda_{CC} = 1$ or $\lambda_{DD} = 1$, i.e., Pareto dominant strategy profile is either \overrightarrow{CC} or \overrightarrow{DD} . According to Proposition 1, \overrightarrow{CC} is indeed

an equilibrium when $(\delta_I, \delta_P) \in \Omega_{CC}$. So both players must at least receive x . Therefore, \overrightarrow{CC} must also be Pareto dominant. Q.E.D.

Next, we show that in any Pareto dominant SPE, the action profile (C, D) is never played. Our approach is to take an arbitrary period t and demonstrate that Pareto improvement can be created by shifting the probability on (C, D) to (D, C) . Recall that by moving the probabilities in period t , W_{t+1} is unaffected. Let $\tilde{U}_{I,t+1}^{(a_{I,t}, a_{P,t})}$ and $\tilde{U}_{P,t+1}^{(a_{I,t}, a_{P,t})}$ denote players I and P 's average utilities in period $t+1$ following the new randomization device's choosing the action profile $(a_{I,t}, a_{P,t})$ in period t and \widetilde{W}_{t+1} be the corresponding Total Utility in period $t+1$. Then

$$\widetilde{W}_{t+1} \equiv \frac{\tilde{U}_{I,t+1}^{(a_{I,t}, a_{P,t})}}{1 - \delta_I} + \frac{\tilde{U}_{P,t+1}^{(a_{I,t}, a_{P,t})}}{1 - \delta_P} = W_{t+1}. \quad (20)$$

Lemma A3 $\lambda_{CD} = 0$ in any Pareto dominant SPE.

Proof. We divide the proof into two cases. (i) First, consider the case of a Pareto dominant SPE where $\lambda_{CC} + \lambda_{DC} > 0$. Suppose $\lambda_{CD} > 0$. For achieving Pareto dominant outcomes, it is necessary that

$$\lambda_{CC}U_{I,t+1}^{(C,C)} + \lambda_{CD}U_{I,t+1}^{(C,D)} = \lambda_{CC}(y + \underline{b}) + \lambda_{CD}\frac{x}{\delta_I} \quad (21)$$

and

$$U_{I,t+1}^{(D,C)} = x.$$

Player P 's incentive compatibility also requires that

$$\lambda_{CC}U_{P,t+1}^{(C,C)} + \lambda_{DC}U_{P,t+1}^{(D,C)} \geq \lambda_{CC}(y - \bar{b}) + \lambda_{DC}\frac{x}{\delta_P} \quad (22)$$

and

$$U_{P,t+1}^{(C,D)} \geq x.$$

Now we move the probability on (C, D) to (D, C) in period t . In order to maintain

player I 's incentive constraint, we have to set $\tilde{U}_{I,t+1}^{(C,C)} = y + \underline{b}$. This and (21) imply

$$\begin{aligned}\tilde{U}_{I,t+1}^{(C,C)} - U_{I,t+1}^{(C,C)} &= y + \underline{b} - \frac{1}{\lambda_{CC}} \left(\lambda_{CC} (y + \underline{b}) + \lambda_{CD} \left(\frac{x}{\delta_I} - U_{I,t+1}^{(C,D)} \right) \right) \\ &= -\frac{\lambda_{CD}}{\lambda_{CC}} \left(\frac{x}{\delta_I} - U_{I,t+1}^{(C,D)} \right).\end{aligned}$$

Since according to (16) and (20) $\frac{\tilde{U}_{I,t+1}^{(C,C)}}{1-\delta_I} + \frac{\tilde{U}_{P,t+1}^{(C,C)}}{1-\delta_P} = \frac{U_{I,t+1}^{(C,C)}}{1-\delta_I} + \frac{U_{P,t+1}^{(C,C)}}{1-\delta_P}$, we have

$$\begin{aligned}\tilde{U}_{P,t+1}^{(C,C)} - U_{P,t+1}^{(C,C)} &= -\frac{1-\delta_P}{1-\delta_I} \left(\tilde{U}_{I,t+1}^{(C,C)} - U_{I,t+1}^{(C,C)} \right) \\ &= \frac{\lambda_{CD}}{\lambda_{CC}} \frac{1-\delta_P}{1-\delta_I} \left(\frac{x}{\delta_I} - U_{I,t+1}^{(C,D)} \right).\end{aligned}\tag{23}$$

Since $U_{I,t+1}^{(D,C)} = x$ and $U_{P,t+1}^{(C,D)} \geq x$, we have

$$\frac{U_{P,t+1}^{(D,C)}}{1-\delta_P} - \frac{U_{I,t+1}^{(C,D)}}{1-\delta_I} = \frac{U_{P,t+1}^{(C,D)}}{1-\delta_P} - \frac{U_{I,t+1}^{(D,C)}}{1-\delta_I} \geq \frac{x}{1-\delta_P} - \frac{x}{1-\delta_I}.\tag{24}$$

After the probability on (C, D) is moved to (D, C) , it is incentive compatible for player P to play C in period t if and only if

$$\lambda_{CC} \tilde{U}_{P,t+1}^{(C,C)} + (\lambda_{CD} + \lambda_{DC}) U_{P,t+1}^{(D,C)} \geq \lambda_{CC} (y - \bar{b}) + (\lambda_{CD} + \lambda_{DC}) \frac{x}{\delta_P}$$

According to (22), this is satisfied if

$$\begin{aligned}\lambda_{CC} \tilde{U}_{P,t+1}^{(C,C)} + (\lambda_{CD} + \lambda_{DC}) U_{P,t+1}^{(D,C)} - \left(\lambda_{CC} U_{P,t+1}^{(C,C)} + \lambda_{DC} U_{P,t+1}^{(D,C)} \right) \\ \geq \lambda_{CC} (y - \bar{b}) + (\lambda_{CD} + \lambda_{DC}) \frac{x}{\delta_P} - \left(\lambda_{CC} (y - \bar{b}) + \lambda_{DC} \frac{x}{\delta_P} \right)\end{aligned}$$

or equivalently

$$\lambda_{CC} \left(\tilde{U}_{P,t+1}^{(C,C)} - U_{P,t+1}^{(C,C)} \right) + \lambda_{CD} U_{P,t+1}^{(D,C)} \geq \lambda_{CD} \frac{x}{\delta_P}.$$

According to (23), this can be rewritten as

$$\begin{aligned}\lambda_{CD} \frac{1-\delta_P}{1-\delta_I} \left(\frac{x}{\delta_I} - U_{I,t+1}^{(C,D)} \right) + \lambda_{CD} U_{P,t+1}^{(D,C)} &\geq \lambda_{CD} \frac{x}{\delta_P} \\ \frac{U_{P,t+1}^{(D,C)}}{1-\delta_P} - \frac{U_{I,t+1}^{(C,D)}}{1-\delta_I} &\geq \frac{x}{\delta_P (1-\delta_P)} - \frac{x}{\delta_I (1-\delta_I)}.\end{aligned}$$

This sufficient condition for player P 's incentive compatibility is implied by (24) because

$$\frac{x}{1-\delta_P} - \frac{x}{1-\delta_I} - \left(\frac{x}{\delta_P(1-\delta_P)} - \frac{x}{\delta_I(1-\delta_I)} \right) = \frac{x(\delta_P - \delta_I)}{\delta_P\delta_I} > 0.$$

This proves that both players' incentive constraints remain satisfied after moving the probability on (C, D) to (D, C) . According to (18), if we move the probability on (C, D) to (D, C) in every period, Pareto improvement can be generated. This contradicts $\lambda_{CD} > 0$.

(ii) Next, consider the case that $\lambda_{CC} + \lambda_{DC} = 0$ in a Pareto dominant SPE. Suppose $\lambda_{CD} > 0$. Then Pareto dominance and incentive compatibility of player P imply that

$$\begin{aligned} U_{I,t+1}^{(C,D)} &= \frac{x}{\delta_I}, \\ U_{P,t+1}^{(C,D)} &\geq x. \end{aligned}$$

Now we move the probability on (C, D) to (D, C) . Following (16) and (20) it follows that $\frac{\tilde{U}_{I,t+1}^{(D,C)}}{1-\delta_I} + \frac{\tilde{U}_{P,t+1}^{(D,C)}}{1-\delta_P} = \frac{U_{I,t+1}^{(C,D)}}{1-\delta_I} + \frac{U_{P,t+1}^{(C,D)}}{1-\delta_P}$. By setting $\tilde{U}_{I,t+1}^{(D,C)} = x$ to satisfy player I 's IR, we have

$$\frac{\tilde{U}_{P,t+1}^{(D,C)}}{1-\delta_P} = \frac{U_{I,t+1}^{(C,D)} - x}{1-\delta_I} + \frac{U_{P,t+1}^{(C,D)}}{1-\delta_P}.$$

Since $U_{I,t+1}^{(C,D)} = x/\delta_I$ and $U_{P,t+1}^{(C,D)} \geq x$,

$$\tilde{U}_{P,t+1}^{(D,C)} \geq (1-\delta_P) \left(\frac{\frac{x}{\delta_I} - x}{1-\delta_I} + \frac{x}{1-\delta_P} \right) = \frac{1-\delta_P + \delta_I}{\delta_I} x > \frac{x}{\delta_P}. \quad (25)$$

The last inequality follows

$$\frac{1-\delta_P + \delta_I}{\delta_I} x - \frac{x}{\delta_P} = \frac{(\delta_P - \delta_I)(1-\delta_P)}{\delta_I\delta_P} x > 0.$$

(25) implies that player P 's incentive constraint will be satisfied under the new probability mix. Next, move the probability this way every period, starting from the first period. Equation (18) implies that such reshuffling of probability is Pareto improving. This again contradicts $\lambda_{CD} > 0$. Q.E.D.

Next, we also show that

Lemma A4 *There is no loss in generality in assuming that in a Pareto dominant SPE, in every period, $\lambda_{CC} + \lambda_{DC} = 1$ or $\lambda_{DD} = 1$.*

Proof. Suppose $\lambda_{CC} + \lambda_{DC} \in (0, 1)$. In this case, by lowering the probability over action (D, D) by Δ and increasing the probability over action profile (C, C) by $\frac{\lambda_{CC}\Delta}{\lambda_{CC}+\lambda_{DC}}$ and the probability over the action profile (D, C) by $\frac{\lambda_{DC}\Delta}{\lambda_{CC}+\lambda_{DC}}$, both players' IC and IR will continue to be satisfied. Since W is linear in λ_{CC} , λ_{DC} , and λ_{DD} according to (15), it is also linear in Δ . Therefore, achieving Pareto dominant SPE outcomes will require either $\Delta = \lambda_{DD}$ or $\Delta = -(1 - \lambda_{DD})$. Q.E.D.

Lemma A5 (i) *For all $(\delta_I, \delta_P) \in \Omega_{DD}$, \overrightarrow{DD} is Pareto dominant.* (ii) *For all $(\delta_I, \delta_P) \in \Omega_{DC}^2$, a Pareto dominant SPE is characterized by mixing over (C, C) with probability*

$$\hat{\lambda}_{CC} = \delta_I \frac{(z-x)\delta_P - x}{(z-y)\delta_P + (z-x-y)\delta_I - (z-x)\delta_I\delta_P} \quad (26)$$

and mixing over (D, C) with probability $1 - \hat{\lambda}_{CC}$ in every period. Also, for all $t \geq 1$, if (C, C) is played in period t , then $b_{t+1} = b_{CC}(\hat{\lambda}_{CC})$, and if (D, C) is played in period t , then $b_{t+1} = b_{DC}(\hat{\lambda}_{CC})$, where

$$b_{CC}(\hat{\lambda}_{CC}) = \underline{b} - (1 - \hat{\lambda}_{CC})(z - y), \quad (27)$$

$$b_{DC}(\hat{\lambda}_{CC}) = -[\hat{\lambda}_{CC}(y - x) + (1 - \lambda)(z - x)]. \quad (28)$$

Proof. First, we identify conditions under which $\lambda_{CC} + \lambda_{DC} = 1$ holds in a Pareto dominant SPE. By plugging $\lambda_{CD} = \lambda_{DD} = 0$ and $\lambda_{DC} = 1 - \lambda_{CC}$ into (17), it can be verified that

$$W(\lambda_{CC}) = \lambda_{CC} \left(\frac{y + \underline{b}}{1 - \delta_I} + \frac{y - \underline{b}}{1 - \delta_P} \right) + (1 - \lambda_{CC}) \left(\frac{x}{1 - \delta_I} + \frac{z - x}{1 - \delta_P} \right). \quad (29)$$

Following (16) and the stationarity of W_t , we have $\frac{U_{I,t+1}^{(D,C)}}{1 - \delta_I} + \frac{U_{P,t+1}^{(D,C)}}{1 - \delta_P} = \frac{U_{I,t+1}^{(C,C)}}{1 - \delta_I} + \frac{U_{P,t+1}^{(C,C)}}{1 - \delta_P} =$

$W_{t+1} = W(\lambda_{CC})$, which can be rewritten as

$$U_{P,t+1}^{(C,C)} = (1 - \delta_P) \left(W(\lambda_{CC}) - \frac{y + \underline{b}}{1 - \delta_I} \right), \quad (30)$$

$$U_{P,t+1}^{(D,C)} = (1 - \delta_P) \left(W(\lambda_{CC}) - \frac{x}{1 - \delta_I} \right). \quad (31)$$

Player P 's incentive constraint is satisfied if and only if

$$\lambda_{CC} U_{P,t+1}^{(C,C)} + (1 - \lambda_{CC}) U_{P,t+1}^{(D,C)} \geq \lambda_{CC} (y - \bar{b}) + (1 - \lambda_{CC}) \frac{x}{\delta_P}.$$

From (29), (30), and (31), this can be rewritten as

$$\lambda_{CC} (\bar{b} - \underline{b}) + (1 - \lambda_{CC}) \left(z - x - \frac{x}{\delta_P} \right) \geq 0. \quad (32)$$

If $(\delta_I, \delta_P) \in \Omega_{DD}$, then we know from our analysis focusing on pure strategies that $\bar{b} < \underline{b}$ and $\delta_P < \frac{x}{z-x}$. This implies that (32) necessarily fails for every $\lambda_{CC} \in [0, 1]$. In this case, \overrightarrow{DD} are the unique and thus also the Pareto dominant SPE outcomes.

Note that Ω_{DC}^2 is the set of (δ_I, δ_P) pairs with which had \overrightarrow{CC} been sustainable as SPE, it would have Pareto dominated \overrightarrow{DC} but \overrightarrow{CC} is not a SPE. If $(\delta_I, \delta_P) \in \Omega_{DC}^2 \subseteq \Omega^U \setminus \Omega_{DD}$, then we know from our analysis focusing on pure strategies that $\bar{b} \leq \underline{b}$ but $\delta_P \geq \frac{x}{z-x}$. This implies that (32) holds for some $\lambda_{CC} \in [0, 1]$. In other words, a correlated strategy SPE with $\lambda_{CC} + \lambda_{DC} = 1$ exists and \overrightarrow{DD} is necessarily Pareto dominated.

Next, we derive the Pareto dominant $(\lambda_{CC}, \lambda_{DC})$ for $(\delta_I, \delta_P) \in \Omega_{DC}^2$. In other words, we are looking for the $\lambda_{CC} \in [0, 1]$ which maximizes W as specified in (29) subject to (32). By definition, for all $(\delta_I, \delta_P) \in \Omega_{DC}^2$, $\delta_I > \frac{z-y}{y} \delta_P$, which according to the equivalence between (8) and (9), implies

$$\frac{y + \underline{b}}{1 - \delta_I} + \frac{y - \underline{b}}{1 - \delta_P} > \frac{x}{1 - \delta_I} + \frac{z - x}{1 - \delta_P}.$$

Therefore, $W(\lambda_{CC})$ increases in λ_{CC} . On the other hand, $\bar{b} \leq \underline{b}$ and $\delta_P \geq \frac{x}{z-x}$ imply that (32) is easier to satisfy with smaller λ_{CC} . These together imply that the solution to the

constrained maximization problem is to choose the λ_{CC} such that (32) holds in equality.

Let such λ_{CC} be $\hat{\lambda}_{CC}$; then

$$\hat{\lambda}_{CC} (\bar{b} - \underline{b}) + \left(1 - \hat{\lambda}_{CC}\right) \left(z - x - \frac{x}{\delta_P}\right) = 0,$$

which gives rise to (26).

Given $\lambda_{CC} = \hat{\lambda}_{CC}$ and $\lambda_{DC} = (1 - \hat{\lambda}_{CC})$, the expected utility of player I from period 2 onward is $\hat{\lambda}_{CC} (y + \underline{b}) + (1 - \hat{\lambda}_{CC})x$. Alternatively, player I 's expected utility can also be calculated using the following. Let b_{CC} (b_{DC}) be the net side payment player I receives in a period if the action profile (C, C) [(D, C)] was chosen in the previous period, which happened with probability $\hat{\lambda}_{CC}$ [$(1 - \hat{\lambda}_{CC})$]. At the current period action stage, player I receives an instantaneous payoff y with probability $\hat{\lambda}_{CC}$ and receives z with probability $(1 - \hat{\lambda}_{CC})$; in the following period, he will again receive $\hat{\lambda}_{CC} (y + \underline{b}) + (1 - \hat{\lambda}_{CC})x$. Therefore,

$$\begin{aligned} & \hat{\lambda}_{CC} (y + \underline{b}) + (1 - \hat{\lambda}_{CC})x \\ &= \hat{\lambda}_{CC} ((1 - \delta_I) (b_{CC} + y) + \delta_I (y + \underline{b})) + (1 - \hat{\lambda}_{CC}) ((1 - \delta_I) (b_{DC} + z) + \delta_I \cdot x). \end{aligned}$$

This can be simplified as

$$\hat{\lambda}_{CC} (\underline{b} - b_{CC}) = (1 - \hat{\lambda}_{CC})(b_{DC} + z - x). \quad (33)$$

On the other hand, $U_I^{(C,C)} - U_I^{(D,C)} = b_{CC} - b_{DC}$, or

$$b_{CC} = (y + \underline{b}) - x + b_{DC}. \quad (34)$$

Solving (33) and (34) for b_{CC} and b_{DC} gives rise to (27) and (28). Q.E.D.

The proposition follows Lemmas A2 and A5. ■

Proof of Proposition 4. We first note that after a unilateral deviation in some period t from any fixed, arbitrary play path implemented by \overrightarrow{CC} , renegotiation-proofness of the play path entails reverting to \overrightarrow{CC} by period $t + 1$ (with an appropriately scaled period

($t + 1$) side payment that ensures deterrence against the deviation). For if not, then by the Pareto dominance of play paths in \overrightarrow{CC} (Theorem 1), the proposed punishment is not renegotiation-proof, a contradiction. With this fact in our pocket, we now proceed with the proof:

Step 1 (Deviations at the action stage in period $t \geq 1$). First assume that if player $i \in \{I, P\}$ chooses an action other than C , players can simply restart \overrightarrow{CC} in period ($t + 1$) but with $U_{it+1} = x$, the same individual payoff Nash reversion would have brought about. In Steps 2-3, we characterize conditions under which players have the incentives to make such side payments.

Step 2 (Deviations by player I at the side payment stage in period $t \geq 1$ when $b_{It} > 0 = b_{Pt}$). Suppose player I is required to make a side payment such that $U_{It} = x$. Consider four possible punishments each of which begins with a different action profile but in all future periods players return to \overrightarrow{CC} . The following are the action profiles and the lowest U_{It+1} which support the action profiles:

$$a_t = (D, D) \text{ and } U_{It+1} = x; \tag{I1}$$

$$a_t = (C, D) \text{ and } U_{It+1} = \frac{x}{\delta_I}; \tag{I2}$$

$$a_t = (C, C) \text{ and } U_{It+1} = y + \underline{b}; \tag{I3}$$

$$a_t = (D, C) \text{ and } U_{It+1} = x. \tag{I4}$$

By definition, only (I1) or (I2) lead to $U_{It+1} = x$ and may constitute a renegotiation-proof punishment ensuring I 's incentive to make the side payment. More specifically, there exists a deviation-detering, renegotiation-proof punishment if and only if either (I1) or (I2) (or both) are Pareto undominated by (I3) and (I4).

Denote by \mathcal{P} a ‘‘Pareto correspondence’’: for any subset, Λ , of SPE strategy profiles of the supergame, $\mathcal{P}(\Lambda)$ denotes those SPE strategy profiles in Λ that are Pareto undominated by any other SPE strategy in Λ . Our objective in this step is focus on $(\delta_I, \delta_P) \in$

Ω_{CC} and characterize conditions under which either (i) $(I1) \in \mathcal{P}((I1), (I3), (I4))$ conditioned on $(I1) \in \mathcal{P}((I1), (I2))$; or (ii) $(I2) \in \mathcal{P}((I2), (I3), (I4))$ conditioned on $(I1) \notin \mathcal{P}((I1), (I2))$.

Claim 4.1: $(I1) \in \mathcal{P}((I1), (I2))$ if and only if $\delta_P/\delta_I \geq (z-x)/x$.

Proof: Conditioned on $U_{It}((I2)) = U_{It}((I1)) = x$, we have

$$\begin{aligned}
\frac{U_{Pt}((I2)) - U_{Pt}((I1))}{1 - \delta_P} &= u_{Pt}((I2)) - u_{Pt}((I1)) + \delta_P \frac{U_{Pt+1}((I2)) - U_{Pt+1}((I1))}{1 - \delta_P} \\
&= u_{Pt}((I2)) - u_{Pt}((I1)) + \delta_P \frac{U_{It+1}((I1)) - U_{It+1}((I2))}{1 - \delta_I} \\
&= (z-x) - \delta_P \frac{\frac{x}{\delta_I} - x}{1 - \delta_I} \\
&= (z-x) - \frac{\delta_P}{\delta_I} x \leq 0 \text{ if and only if } \frac{\delta_P}{\delta_I} \geq \frac{z-x}{x}.
\end{aligned}$$

This proves the claim.

Claim 4.2: $(I1) \in \mathcal{P}((I1), (I3), (I4))$ if and only if $\delta_P/\delta_I \geq (y-x)/(z-y)$.

Proof: Note that $u_{Pt}((I4)) - u_{Pt}((I1)) = -x$ and $U_{Pt+1}((I4)) = U_{Pt+1}((I1))$. This implies that $U_{Pt}((I4)) - U_{Pt}((I1)) = -x < 0$, which proves that $I1 \in \mathcal{P}((I1), (I4))$.

Next,

$$\begin{aligned}
\frac{U_{Pt}((I3)) - U_{Pt}((I1))}{1 - \delta_P} &= u_{Pt}((I3)) - u_{Pt}((I1)) + \delta_P \frac{U_{It+1}((I1)) - U_{It+1}((I3))}{1 - \delta_I} \\
&= (y-x) + \delta_P \frac{x - (y+b)}{1 - \delta_I} = (y-x) - \frac{\delta_P}{\delta_I} (z-y).
\end{aligned}$$

Therefore, $(I1) \in \mathcal{P}((I1), (I3))$ if $\delta_P/\delta_I \geq (y-x)/(z-y)$. This proves the claim.

Claim 4.3: Suppose $(I1) \notin \mathcal{P}((I1), (I2))$. Then $(I2) \in \mathcal{P}((I2), (I3), (I4))$ if and only if $\delta_P/\delta_I \leq (z-y)/(x+y-z)$.

Proof: By direct computation, we have

$$\frac{U_{Pt}((I4)) - U_{Pt}((I2))}{1 - \delta_P} = -z - \delta_P \frac{x - x/\delta_I}{1 - \delta_I} = \frac{\delta_P}{\delta_I} x - z.$$

Therefore, $(I2) \in \mathcal{P}((I2), (I4))$ if and only if $\delta_P/\delta_I \leq z/x$, which is satisfied whenever $(I1) \notin \mathcal{P}((I1), (I2))$, i.e., $\delta_P/\delta_I < (z-x)/x < z/x$. Next,

$$\frac{U_{Pt}((I3)) - U_{Pt}((I2))}{1 - \delta_P} = (y - z) - \delta_P \frac{y + \frac{b}{x} - x/\delta_I}{1 - \delta_I} = \frac{\delta_P}{\delta_I} (x + y - z) - (z - y).$$

Therefore, $(I2) \in \mathcal{P}((I2), (I3))$ if and only if $\delta_P/\delta_I \leq (z-y)/(x+y-z)$. This proves the claim.

With direct computation, we can also establish the following:

Claim 4.4: If $y \leq z - x + \frac{x^2}{z}$, then $\frac{y-x}{z-y} \leq \frac{z-x}{x} \leq \frac{z-y}{x+y-z}$; if $z - x + \frac{x^2}{z}$, then $\frac{z-y}{x+y-z} < \frac{z-x}{x} < \frac{y-x}{z-y}$.

Suppose $y \leq z - x + \frac{x^2}{z}$. Then according to Claims 4.1, 4.2, and 4.4, whenever $(I1) \in \mathcal{P}((I1), (I2))$, i.e., $\delta_P/\delta_I \geq (z-x)/x$, it follows that $\delta_P/\delta_I \geq (y-x)/(z-y)$, implying $(I1) \in \mathcal{P}((I1), (I3), (I4))$. Similarly, according to Claims 4.1, 4.3, and 4.4, whenever $(I1) \notin \mathcal{P}((I1), (I2))$, it follows that $(I2) \in \mathcal{P}((I2), (I3), (I4))$.

Suppose $y > z - x + \frac{x^2}{z}$. Then for $(\delta_I, \delta_P) \in \Omega_{CC}$, $\{(I1), (I2)\} \in \mathcal{P}((I1), (I2), (I3), (I4))$ if and only if $\frac{\delta_P}{\delta_I} \notin \left(\frac{z-y}{x+y-z}, \frac{y-x}{z-y} \right)$. Note that if $\frac{y-x}{z-y} \leq 1$, i.e., $y \leq \frac{x+z}{2}$, then for all $(\delta_I, \delta_P) \in \Omega_{CC}$, $\frac{\delta_P}{\delta_I} \notin \left(\frac{z-y}{x+y-z}, \frac{y-x}{z-y} \right)$.

Summing up, there exist renegotiation proof punishments that deter I from refusing to make a side payment leading to $U_{It} = x$ for all $(\delta_I, \delta_P) \in \Omega_{CC}$ if:

$$y \leq z - x + \frac{x^2}{z} \text{ or } y \leq \frac{z+x}{2}. \quad (35)$$

Step 3 (Deviations by player P at the bribing stage in period $t \geq 1$ when $b_{Pt} > 0 = b_{It}$).

Suppose P is required to make a side payment at time t such that $U_{Pt} = x$. Consider four possible punishments each begins with a different action profile but in all future periods players return to \overrightarrow{CC} . The following are the action profiles and the lowest U_{It+1} which

support the action profiles:

$$a_t = (D, D) \text{ and } U_{P_{t+1}} = x; \quad (P1)$$

$$a_t = (D, C) \text{ and } U_{P_{t+1}} = \frac{x}{\delta_P}; \quad (P2)$$

$$a_t = (C, C) \text{ and } U_{P_{t+1}} = y - \bar{b}; \quad (P3)$$

$$a_t = (C, D) \text{ and } U_{P_{t+1}} = x. \quad (P4)$$

It is obvious that only (P1) and (P2) can be used to push player P 's continuation payoff to x upon her deviation in the side payment stage.

Claim 4.5: $(P1) \in \mathcal{P}((P1), (P4))$ and $(P2) \in \mathcal{P}((P2), (P4))$.

Proof: It can be directly computed that

$$\begin{aligned} \frac{U_{It}(P4) - U_{It}(P1)}{1 - \delta_I} &= 0 - x - \delta_I \frac{U_{P_{t+1}}(P4) - U_{P_{t+1}}(P1)}{1 - \delta_P} = -x < 0, \\ \frac{U_{It}(P4) - U_{It}(P2)}{1 - \delta_I} &= 0 - z - \delta_I \frac{x - x/\delta_P}{1 - \delta_P} = -z + x \frac{\delta_I}{\delta_P} < 0. \end{aligned}$$

Claim 4.6: $(P1) \in \mathcal{P}((P1), (P2))$ if and only if $\frac{\delta_P}{\delta_I} \leq \frac{x}{z-x}$.

Proof:

$$\begin{aligned} \frac{U_{It}(P2) - U_{It}(P1)}{1 - \delta_I} &= (z - x) - \delta_I \frac{U_{P_{t+1}}(P2) - U_{P_{t+1}}(P1)}{1 - \delta_P} \\ &= (z - x) - \frac{\delta_I}{\delta_P} x \leq 0 \text{ if and only if } \frac{\delta_P}{\delta_I} \leq \frac{x}{z-x}. \end{aligned}$$

Claim 4.7: $(P1) \in \mathcal{P}((P1), (P3))$ if and only if $\frac{\delta_P}{\delta_I} \leq \frac{z-y}{y-x}$.

Proof:

$$\begin{aligned} \frac{U_{It}(P3) - U_{It}(P1)}{1 - \delta_I} &= (y - x) - \delta_I \frac{(y - \bar{b}) - x}{1 - \delta_P} \\ &= (y - x) - (z - y) \frac{\delta_I}{\delta_P} \leq 0 \text{ if and only if } \frac{\delta_P}{\delta_I} \leq \frac{z-y}{y-x}. \end{aligned}$$

Claim 4.8: Suppose $(P1) \notin \mathcal{P}((P1), (P2))$. Then $(P2) \in \mathcal{P}((P2), (P3))$ if and only if

$$\frac{\delta_P}{\delta_I} \geq \frac{x+y-z}{z-y}.$$

Proof:

$$\begin{aligned}\frac{U_{It}(P3) - U_{It}(P2)}{1 - \delta_I} &= (y - z) - \delta_I \frac{(y - \bar{b}) - x/\delta_P}{1 - \delta_P} \\ &= (x + y - z) \frac{\delta_I}{\delta_P} - (z - y) \leq 0 \text{ if and only if } \frac{x + y - z}{z - y} \leq \frac{\delta_P}{\delta_I}.\end{aligned}$$

The following is an immediate corollary of Claim 5.4:

Claim 4.9: If $y \leq z - x + \frac{x^2}{z}$, then $\frac{x+y-z}{z-y} \leq \frac{x}{z-x} \leq \frac{z-y}{y-x}$; if $y > z - x + \frac{x^2}{z}$, then $\frac{z-y}{y-x} < \frac{x}{z-x} < \frac{x+y-z}{z-y}$.

Suppose $y \leq z - x + \frac{x^2}{z}$. Then according to Claims 4.5-4.7, 4.9, whenever $(P1) \in \mathcal{P}((P1), (P2))$, i.e., $\frac{\delta_P}{\delta_I} \leq \frac{x}{z-x}$, it follows that $\frac{\delta_P}{\delta_I} \leq \frac{z-y}{y-x}$, implying $(P1) \in \mathcal{P}((P1), (P3), (P4))$. Similarly, according to Claims 4.5-4.6, 4.8-4.9, whenever $(P1) \notin \mathcal{P}((P1), (P2))$, i.e., $\frac{\delta_P}{\delta_I} > \frac{x}{z-x}$, it follows that $\frac{\delta_P}{\delta_I} > \frac{x+y-z}{z-y}$, implying $(P2) \in \mathcal{P}((P2), (P3), (P4))$.

Suppose $y > z - x + \frac{x^2}{z}$. Then for $(\delta_I, \delta_P) \in \Omega_{CC}$, $\{(P1), (P2)\} \in \mathcal{P}((P1), (P2), (P3), (P4))$ if and only if $\delta_P/\delta_I \notin (\frac{z-y}{y-x}, \frac{x+y-z}{z-y})$. Finally, for all $(\delta_I, \delta_P) \in \Omega_{CC}$, $\delta_P/\delta_I \notin (\frac{z-y}{y-x}, \frac{x+y-z}{z-y})$ if $\frac{x+y-z}{z-y} \leq 1$, i.e., $y \leq z - x/2$.

Summing up, there exist renegotiation proof punishments that deter I from refusing to make a side payment leading to $U_{It} = x$ for all $(\delta_I, \delta_P) \in \Omega_{CC}$ if:

$$y \leq z - x + \frac{x^2}{z} \text{ or } y \leq z - \frac{x}{2}. \quad (36)$$

Combining (35) in Step 2 and (36) in Step 3 completes the proof of Part (i) of the proposition. If $y > z - x + \frac{x^2}{z}$ and $y > \min\{z - \frac{x}{2}, \frac{x+z}{2}\}$, then the entire set of Pareto dominant SPE outcomes implemented by \overrightarrow{CC} survives renegotiation proofness if and only if $\delta_P/\delta_I \notin (\frac{z-y}{x+y-z}, \frac{y-x}{z-y})$ and $\delta_P/\delta_I \notin (\frac{z-y}{y-x}, \frac{x+y-z}{z-y})$. ■

Proof of Proposition 5. We start by noting that given a unilateral deviation in period t from any fixed, arbitrary play path implemented by \overrightarrow{DC} , renegotiation-proofness of the punishments corresponding to this deviation must entail reverting to \overrightarrow{DC} by period $t + 1$

(with an appropriately scaled period $(t + 1)$ net side payment in order to ensure deterrence against the deviation).

Step 1 (Deviations at the action stage in period $t \geq 1$). It is obvious that player I has no incentive to deviate from playing D . If player P deviates from playing C , players can simply restart an equilibrium play path implemented by \overrightarrow{DC} from period $t + 1$ at the beginning of which player P is required to make a side payment low enough such that $U_{P_{t+1}} = x$, the same payoff Nash reversion would have brought about. In Step 2, we will verify player P 's incentive to make such side payment.

Step 2 (Deviations by player P at the bribing stage when $b_{P_t} > 0 = b_{I_1}$). Consider the side payment stage of a period t at which player P is required to make a side payment which lowers her continuation payoff to $U_{P_t} = x$. To ensure player P 's incentive to pay such amount, we propose the punishment that if player P pays less than the specified amount, players will play (D, C) in the action stage of period t and play \overrightarrow{DC} starting from period $t + 1$ at the beginning of which player P will be required to pay an amount such that $\tilde{U}_{P_{t+1}} = x/\delta_P$. In every subsequent period $t + s$, $s \geq 1$, in which P refuses to pay, players continue to play (D, C) and in the following period P is required to pay the amount such that $\tilde{U}_{P_{t+s+1}} = x/\delta_P$. Given the punishment, player P has the incentive to pay in period t . Now check if playing (D, C) on this punishment path is renegotiation proof. Given that \overrightarrow{DC} will be played starting from period $t + 1$, the only room for Pareto improving renegotiation is to conduct intertemporal trade by raising $U_{P_{t+1}}$ and lowering U_{P_t} . However, the side payment stage has passed and player P 's instantaneous payoff is 0, such renegotiation is infeasible. With the same logic, the same punishment proposed will provide an enough incentive for player P to make any lower amount of side payment.

Step 3 (Deviations by player I at the bribing stage in period $t \geq 1$ when $b_{I_t} > 0 = b_{P_t}$).

Consider the side payment stage of a period t at which player I is required to make a side payment which lowers his continuation payoff to $U_{I_t} = x$. We focus on four pun-

ishments, which we label as $(\widetilde{I1}) - (\widetilde{I4})$, defined similar to $(I1) - (I4)$ in the proof of Proposition 3 except that here one period after this punishment players revert to \overrightarrow{DC} instead of \overrightarrow{CC} . The following are the action profiles and the lowest U_{It+1} which support the action profiles:

$$a_t = (D, D) \text{ and } U_{It+1} = x; \quad (\widetilde{I1})$$

$$a_t = (C, D) \text{ and } U_{It+1} = \frac{x}{\delta_I}; \quad (\widetilde{I2})$$

$$a_t = (C, C) \text{ and } U_{It+1} = y + \underline{b}; \quad (\widetilde{I3})$$

$$a_t = (D, C) \text{ and } U_{It+1} = x. \quad (\widetilde{I4})$$

Obviously, only $(\widetilde{I1})$ or $(\widetilde{I2})$ lead to $U_{It+1} = x$ and may constitute a renegotiation-proof punishment ensuring I 's incentive to make the side payment. More specifically, there exists a deviation-detering, renegotiation-proof punishment if and only if either $(\widetilde{I1})$ or $(\widetilde{I2})$ (or both) are Pareto undominated by $(\widetilde{I3})$ and $(\widetilde{I4})$. Moreover, note that $(\widetilde{I3})$ is not feasible for $(\delta_I, \delta_P) \in \Omega_{DC}^2$ for the reason that \overrightarrow{DC} is dominated by \overrightarrow{CC} and (C, C) could not be supported even \overrightarrow{CC} would be played in the following period.

Claim 5.1: $(\widetilde{I1}) \in \mathcal{P}((\widetilde{I1}), (\widetilde{I2}))$ if and only if $\delta_P/\delta_I \geq (z - x)/x$.

Proof: Same argument that proves Claim 4.1.

Claim 5.2: $(\widetilde{I1}) \in \mathcal{P}((\widetilde{I1}), (\widetilde{I3}), (\widetilde{I4}))$.

Proof: Adapting the proof of Claim 4.2, it follows that $\widetilde{I1} \in \mathcal{P}((\widetilde{I1}), (\widetilde{I4}))$, and $(\widetilde{I1}) \in \mathcal{P}((\widetilde{I1}), (\widetilde{I3}))$ if $\delta_I/\delta_I \geq (y - x)/(z - y)$. If $\delta_I/\delta_I < (y - x)/(z - y)$ ($< \frac{y}{z-y}$), then $(\delta_I, \delta_P) \in \Omega_{DC}^2$ and $(\widetilde{I3})$ is not sustainable. So $(\widetilde{I1}) \in \mathcal{P}((\widetilde{I1}), (\widetilde{I3}))$ as well. This proves the claim.

Claim 5.3: If $(\widetilde{I1}) \notin \mathcal{P}((\widetilde{I1}), (\widetilde{I2}))$, then $(\widetilde{I2}) \in \mathcal{P}((\widetilde{I2}), (\widetilde{I3}), (\widetilde{I4}))$.

Proof. If $(\widetilde{I1}) \notin \mathcal{P}((\widetilde{I1}), (\widetilde{I2}))$, then $U_{Pt}(\widetilde{I2}) > U_{Pt}(\widetilde{I1})$. This with Claim 5.2 imply $(\widetilde{I2}) \in \mathcal{P}((\widetilde{I2}), (\widetilde{I3}), (\widetilde{I4}))$.

Steps 1-2 and Claims 5.2-5.3 together complete the proof of the proposition. ■