A TWO-SIDED REPUTATION RESULT WITH LONG RUN PLAYERS

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Abstract. Cripps et al. (2005) conjectured that in an infinitely repeated game with two equally patient players, if there is positive probability that the players could be Stackelberg types, then equilibrium behavior would resemble a war of attrition, i.e., a two-sided reputation result would hold. In this note we show that this conjecture is indeed true for a wide set of stage games for which the one-sided reputation result of Atakan and Ekmekci (2008) holds.

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1. Introduction and Related Literature

One-sided reputation results have been established for infinitely repeated games played by equally patient agents in Cripps et al. (2005) and Atakan and Ekmekci (2008). Under certain restrictions on the stage game, these two papers show that if there is positive probability that player 1 is a type committed to playing a (dynamic) Stackelberg strategy, then, in any equilibrium of the repeated game, player 1 gets his highest payoff compatible with the individual rationality of player 2, if the agents are sufficiently patient.[1] In both these papers there is incomplete information only about the type of player 1, and player 2's type is known with certainty. Cripps et al. (2005) conjectured that if there were incomplete information about the type of both players, and both players could be Stackelberg types with positive probability, then equilibrium behavior would resemble a war of attrition. In this note we show that this conjecture is indeed true: we maintain either the set of assumptions on the stage

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[1] The dynamic Stackelberg payoff of a player is the highest payoff that he can guarantee in the repeated game through public pre-commitment to a repeated game strategy (a dynamic Stackelberg strategy); and a dynamic Stackelberg type is a commitment type that plays such a strategy.

game that gives the one-sided reputation result of Cripps et al. (2005), or, the set of assumptions that imply the result of Atakan and Ekmekci (2008). We show that if there is positive probability that both players are Stackelberg types, then equilibrium play converges to the unique equilibrium of a continuous time war of attrition, as the stage game is repeated increasingly frequently.

The one-sided reputation result in Cripps et al. (2005) focuses on simultaneous move stage games and restricts attention to stage games where these exists an action which is the best for player 1 and is the worst for his opponent (*a strictly conflicting interest game*). In contrast, Atakan and Ekmekci (2008) focuses on extensive form stage games of perfect information. In addition to strictly conflicting interest games, the reputation result of Atakan and Ekmekci (2008) covers any stage game where player 2 receives a payoff that strictly exceeds her worst payoff in any profile where player 1 receives his best payoff (*a locally non-conflicting interest game*). Our two-sided reputation result holds under either of the set of assumptions detailed above that imply a one-sided reputation result.

Our two-sided reputation result is closely related to previous work by Abreu and Gul (2000).[2] Abreu and Gul (2000) show that in a two player bargaining game, as the frequency of offers increases, the equilibria of the (two-sided) incomplete information game converges to the unique equilibrium of a continuous time war of attrition. Their two-sided reputation result builds on a one-sided reputation result for bargaining games due to Myerson (1991). Likewise, our two-sided reputation result builds on our one-sided reputation result presented in Atakan and Ekmekci (2008) that ensures that there is a unique equilibrium payoff in any continuation game with one-sided incomplete information.

The only other two-sided reputation result for repeated games with long run players that we are aware of is by Abreu and Pearce (2007). In this paper, the authors allow for multiple types and show that the equilibrium payoff profile coincides with the Nash bargaining solution with endogenous threats. However, their paper studies a different economic environment than ours and is not directly comparable. Specifically, in Abreu and Pearce (2007) agents write binding contracts and commitment types announce their inflexible demands truthfully at the start of the repeated game. These enforceable contracts uniquely determine payoffs in the continuation game with

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[2] Also see Kreps and Wilson (1982) for an earlier reference.

one-sided incomplete information. In our paper, in contrast, continuation payoffs are unique as a consequence of a one-sided reputation result and no extra communication is assumed. Uniqueness in the one-sided incomplete information game is a key component for the two-sided reputation result. Without uniqueness, many equilibria can be generated in the game with two-sided incomplete information by leveraging the multiplicity of equilibria in the continuation game with one-sided incomplete information.

2. The Model

We consider a repeated game $\Gamma^{\infty}(\Delta)$ in which a stage game Γ is played by players 1 and 2 in periods $t \in \{0, 1, 2, ...\}$. The players discount payoffs using a common discount factor $e^{-r\Delta}$, where $\Delta > 0$ is the period length, and r is the instantaneous rate of time preference. *The stage game* Γ *is a two-player finite game. Also, we frequently assume that* Γ *is a game of perfect information, that is, all information sets of* Γ *are singletons* **(perfect information)**.

For the stage game, D is the set of nodes (decision nodes and terminal nodes), d is a typical element of D, $Y \subset D$ is the set of terminal nodes and y is a typical element of Y. The payoff function of player i is $g_i : Y \to \mathbb{R}$. The finite set of pure stage game actions for player i is A_i and the set of mixed stage game actions is $\mathcal{A}_i$.[3] For any action profile $a = (a_1, a_2) \in A_1 \times A_2$ there is a unique terminal history $y(a) \in Y$ under the path of play induced by a. With a slight abuse of notation we let $g_i(a) = g_i(y(a))$ for any $a \in A_1 \times A_2$.

In the repeated game Γ^{∞} players have perfect recall and can observe past outcomes. $Y^t \times D$ is the set of period $t \geq 0$ public histories and $\{y^0, y^1, ..., y^{t-1}, d\}$ is a typical element. $H^t \equiv Y^t$ is the set of period $t \geq 0$ public histories of terminal nodes and $\{y^0, y^1, ..., y^{t-1}\}$ is a typical element.

The stage game. The minimax payoff for player i, $\hat{g}_i = \min_{\alpha_j} \max_{\alpha_i} g_i(\alpha_i, \alpha_j)$. For games of perfect information there exists $a_1^p \in A_1$ such that $g_2(a_1^p, a_2) \leq \hat{g}_2$ for all $a_2 \in A_2$. The set of feasible payoffs $F = co\{g_1(a_1, a_2), g_2(a_1, a_2) : (a_1, a_2) \in A_1 \times A_2\}$; and the set of feasible and individually rational payoffs $G = F \cap \{(g_1, g_2) : g_1 \geq \hat{g}_1, g_2 \geq \hat{g}_2\}$. Let $\bar{g}_1 = \max\{g_1 : (g_1, g_2) \in G\}$, and $M = \max\{\max\{|g_1|, |g_2|\} : (g_1, g_2) \in F\}$.

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[3] An action $a_i \in A_i$ is a contingent plan that specifies a move from the set of feasible moves for player i at any decision node d where player i is called upon to move.

Assumption 1. *For $i = 1, 2$, the stage game Γ satisfies either of the following*

- (i) *(Locally non-conflicting interest game for player i) For any $g \in G$ and $g' \in G$, if $g_i = g'_i = \bar{g}_i$, then $g_j = g'_j > \hat{g}_j$, or*
- (ii) *(Strictly conflicting interest game for player i) There exists $a_i \in A_i$ such that any best response to a_i yields payoffs $(\bar{g}_i, \hat{g}_j)$. Also, $g_j = \hat{g}_j$ for all $(\bar{g}_i, g_j) \in G$.*[4]

Assumption 1 implies that there exist action profiles $(a_1^s, a_2^b) \in A_1 \times A_2$ and $(a_1^b, a_2^s) \in A_1 \times A_2$ such that $g_i(a_i^s, a_j^b) = \bar{g}_i$. If Γ is a strictly conflicting interest game for player i, then a_j^b denotes a best response to a_i^s.[5] If Γ satisfies Assumption 1, then there exists $\rho \geq 0$ such that

$$\left| \frac{g_j - g_j(a_i^s, a_j^b)}{\bar{g}_i - g_i} \right| \leq \rho, \text{ for any } (g_i, g_j) \in G. \tag{1}$$

We normalize payoffs, without loss of generality, such that

$$\bar{g}_1 = \bar{g}_2 = \bar{g}, \text{ also, if } g_i(a_i^b, a_j^s) \neq \bar{g}, \text{ then normalize } g_1(a_1^b, a_2^s) = g_2(a_1^s, a_2^b) = 0. \tag{2}$$

Types and Strategies. Before time 0 nature selects player i as a normal type N with probability $1 - z_i$ or a Stackelberg type S_i with probability z_i. Player i's belief over player j's types, $z_j : \bigcup_{t=0}^{\infty} Y^t \times D \to [0, 1]$, is the probability that j is the Stackelberg type given any history h^t.

A behavior strategy for player i is a function $\sigma_i : \bigcup_{t=0}^{\infty} H^t \to \mathcal{A}_i$. A behavior strategy chooses a mixed stage game action given any period t public history of terminal nodes. A strategy profile $\sigma = (\sigma_1, \sigma_2)$ lists the behavior strategies for the normal types of the players.

The Stackelberg type S_i plays repeated game strategy $\sigma_i(S_i)$. The strategy $\sigma_i(S_i)$ has a profit phase and a punishment phase. In the profit phase the strategy plays a_i^s and in the punishment phase the strategy plays a_i^p. The strategy begins the game in the profit phase. The strategy remains in the profit phase in period t if it was in the profit phase in period $t-1$ and $g_i(y_{t-1}) = \bar{g}$. The strategy moves to the punishment phase in period t if it was in the profit phase in period $t-1$ and $g_i(y_{t-1}) \neq \bar{g}$. If the strategy moves to the punishment phase in period t, then it remains in the punishment phase for $n_i^p - 1$ periods and then moves to the profit phase. Intuitively,

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[4] Also see Cripps et al. (2005), or Mailath and Samuelson (2006), page 541.

[5] If Γ is a locally non-conflicting interest game for player i, then a_j^b is not necessarily a best response to a_i^s.

$\sigma_i(S_i)$ punishes player j, by minimaxing her for the next $n_i^p - 1$ periods, if she does not allow player i to obtain a payoff of one. The number of punishment periods $n_1^p - 1$ is the smallest integer such that

$$g_2(a_1^s, a_2) + (n_1^p - 1)\hat{g}_2 < n_1^p g_2(a_1^s, a_2^b) = 0 \tag{3}$$

for any $a_2 \in A_2$ such that $g_1(a_1^s, a_2) < g_1(a_1^s, a_2^b) = \bar{g}$. We define n_2^p similarly. Assumption 1 implies that $n_i^p \geq 1$ exists. The number of punishment periods is chosen to ensure that it is a best response for a sufficiently patient player j to play a_j^b in every period against $\sigma_i(S_i)$ in the repeated game of complete information $\Gamma^\infty(\Delta)$. That is, if $\sigma_j \in BR(\sigma_i(S_i), \Delta)$, then $U_i(\sigma_i(S_i), \sigma_j, \Delta) = \bar{g}$, for sufficiently small Δ, in the repeated game $\Gamma^\infty(\Delta)$.

Payoffs. A player's repeated game payoff is the normalized discounted sum of the stage game payoffs. For any infinite public history $h = \{y^0, y^1, ..., \}$, $u_i(h, \Delta) = \left(1 - e^{-r\Delta}\right) \sum_{k=0}^{\infty} e^{-rk\Delta} g_i(y^k)$, and $u_i(h^{-t}, \Delta) = \left(1 - e^{-r\Delta}\right) \sum_{k=t}^{\infty} e^{-r(k-t)\Delta} g_i(y^k)$ where $h^{-t} = \{y^t, y^{t+1}, ...\}$. Player i's expected continuation payoff, following a period t public history, under strategy profile σ

$$U_i(\sigma, \Delta | h^t) = z_j(h^t) U_i(\sigma_i(N), \sigma_j(S_j), \Delta | h^t) + (1 - z_j(h^t)) U_i(\sigma_i(N), \sigma_j(N), \Delta | h^t),$$

where $U_i(\sigma_1(N), \sigma_2(N), \Delta | h^t) = \mathbb{E}_{(\sigma_1(N), \sigma_2(N))}[u_i(h^{-t}, \Delta) | h^t]$.

Equilibrium and beliefs. The repeated game with incomplete information where $z = (z_1, z_2)$ is denoted $\Gamma^\infty(z, \Delta)$. The analysis in the paper focuses on the perfect Bayesian equilibria (PBE) of the game of incomplete information $\Gamma^\infty(z, \Delta)$. In equilibrium, beliefs are obtained, where possible, using Bayes' rule given $z_i(\cdot | h^0) = z_i(\cdot)$ and conditioning on players' equilibrium strategies.

3. Two-Sided Reputation and a War of Attrition

The two-sided reputation result that we prove in this paper builds on a one-sided reputation result that we state as Theorem 1. Theorem 1 assumes perfect information and maintains that the stage game satisfies Assumption 1, or alternatively, drops the perfect information assumption but maintains that the stage game is a strictly conflicting interest game for both players (Assumption 1 (*ii*)). The theorem shows that if player j is known to be the normal type with certainty, and player i is the Stackelberg type with positive probability, then player i can guarantee a payoff arbitrarily close to $\bar{g}$, in any PBE of the repeated game, for sufficiently small Δ.

Theorem 1. *Suppose that Γ is a game of perfect information and satisfies Assumption 1; or alternatively suppose that Γ satisfies Assumption 1 (ii) for both player. There is a non-increasing function $K : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that for any $\Delta > 0$, any $z_i > 0$ and $z_j = 0$, and any PBE profile σ of $\Gamma^{\infty}(z, \Delta)$*

$$U_i(\sigma, \Delta) > \bar{g} - K(z_i)(1 - e^{-r\Delta}).$$

Proof. See Atakan and Ekmekci (2008) Theorem 1. □

In our main two-sided reputation result (Theorem 2) we also assume perfect information and maintain that the stage game satisfies Assumption 1, or alternatively, we drop the perfect information assumption but assume that the stage game is a strictly conflicting interest game for both players (Assumption 1 (ii)). We show that as $\Delta_n \to 0$, any sequence of PBE payoff profiles $\{(U_1(\sigma^n, \Delta^n), U_2(\sigma^n, \Delta^n)\}$ of the repeated game $\Gamma^{\infty}(z, \Delta^n)$ converges to a unique limit. This limit is the unique equilibrium payoff profile of a continuous time war of attrition that we define in detail below. Furthermore, the equilibrium path of play of the repeated game converges, in distribution, to the equilibrium play of the war of attrition.

3.1. A continuous time war of attrition.

Let n^p denote the smallest common multiple of n_1^p and n_2^p; and let

$$l_i = -\frac{1}{n^p} \sum_{t=0}^{n^p - 1} g_i(y^t(\sigma_1(S_1), \sigma_2(S_2))),$$

where $y^t(\sigma_1(S_1), \sigma_2(S_2))$ is period t terminal node if the players use the strategy profile $(\sigma_1(S_1), \sigma_2(S_2))$. So l_i is the time average loss of playing the Stackelberg strategy against the Stackelberg strategy.[6] Note that for $n^p > 1$,

$$|\sum_{t=0}^{n^p - 1} e^{-r\Delta t} l_i - \sum_{t=0}^{n^p - 1} e^{-r\Delta t} g_1(y^t(\sigma_1(S_1), \sigma_2(S_2)))| \leq M n^p (1 - e^{-r\Delta(n^p - 1)}). \tag{4}$$

Consequently, as $\Delta \to 0$, i.e., as we converge to continuous time, the cost of using the Stackelberg strategy against the Stackelberg strategy for player i converges to l_i.

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[6] To be exact $l_1 = -\frac{1}{n^p} g_1(a_1^s, a_2^s) + (\frac{n^p}{n_1^p} - 1) g_1(a_1^s, a_2^p) + (\frac{n^p}{n_2^p} - 1) g_1(a_1^p, a_2^s) + (n_p + 1 - \frac{n^p}{n_1^p} - \frac{n^p}{n_2^p}) g_1(a_1^p, a_2^p)$.

At time zero of the continuous time war of attrition, both players simultaneously choose either to concede or insist. If both players choose to insist, then a continuous time war ensues. The game continues until one of the two players concedes. Each player can concede at any time $t \in [0, \infty]$. If player i concedes at time $t \in [0, \infty]$ and player j continues to play insist through time t, then player i's payoff is $-l_i\left(1 - e^{-rt}\right)$ and player j's payoff is $e^{-rt}\bar{g} - l_j\left(1 - e^{-rt}\right)$. If both players concede concurrently at time t, then they receive payoff $e^{-rt}g_i(t) - l_i\left(1 - e^{-rt}\right)$ and $e^{-rt}g_j(t) - l_j\left(1 - e^{-rt}\right)$ where $(g_i(t), g_j(t)) \in G$, and consequently, $-\rho(\bar{g} - g_j(t)) \leq g_i(t) \leq \rho(\bar{g} - g_j(t))$. Before the game begins at time 0, nature chooses a type for each player independently. A player is chosen as either a Stackelberg type that never concedes, with probability $z_i > 0$, or a normal type, with probability $1 - z_i$.

This war of attrition is closely related to the repeated game $\Gamma^{\infty}(z, \Delta)$ for $\Delta \approx 0$: insisting corresponds to playing the Stackelberg strategy, and conceding corresponds deviating from the Stackelberg strategy. Player i incurs cost l_i if he insists on playing the Stackelberg strategy against the Stackelberg strategy. If one of the players deviates from the Stackelberg strategy and the other does not, then the player that deviated is known to be normal with certainty. After such a history, Theorem 1 implies that the player known as normal receives a payoff equal to zero and the rival receives a payoff equal to $\bar{g}$. This corresponds exactly to the payoffs when one of the players concedes at time t in the war of attrition. Players incur the costs (l_i, l_j) for t units of time, i.e., $\left(l_i\left(1 - e^{-rt}\right), l_j\left(1 - e^{-rt}\right)\right)$; the conceding player receives a continuation payoff equal to zero; and the player that wins receives continuation payoff equal to $\bar{g}$. If both players reveal rationality (concede) concurrently in period t, then Theorem 1 puts no restrictions on continuation payoffs. So, agents receive arbitrary payoffs from the set G.

In contrast, to Abreu and Gul (2000), in the war of attrition presented here, the payoffs that the players receive, if they concede concurrently, depend on t and are potentially non-stationary. Nevertheless, the argument for condition (i) below shows that Abreu and Gul's (2000) analysis applies without alteration. In particular, the unique equilibrium of the war of attrition satisfies three conditions: (i) at most one agent concedes with positive probability at time zero, (ii) after time zero each player concedes with constant hazard rate λ_i, (iii) the normal types finish conceding at some finite time T. Consequently, at time T the posterior probability that an agent faces a Stackelberg type equals one.

Suppose that condition (i) is violated and both players concede with positive probability at time t. If they concede concurrently, then player 1's payoff is $g_1(t)$ and player 2's payoff is $g_2(t)$. By equation (2), $g_i(t) \leq \rho(\bar{g} - g_j(t))$. Consequently, for one of the two players $g_i(t) < \bar{g}$. But for this player i waiting to see whether player j quits at time t and then quitting immediately afterwards does strictly better than quitting at time t. Consequently, both players cannot concede with positive probability at any time t; and in particular, at most one of the players can concede with positive probability at time zero.

Player i cannot concede with probability one at any time. If player i were to concede with certainty at time t, then by not conceding player i would ensure that player j believes that player i is the Stackelberg type with probability one. But this would induce player j to concede immediately improving i's payoff. Consequently, condition (ii) implies that the hazard rate λ_i must leave j indifferent between conceding immediately and waiting for an additional $\triangle$ units of time and then conceding. Conceding immediately guarantees player j zero. By waiting for $\triangle$ units of time player j incurs cost $l_j(1 - e^{-r\triangle})$, but receives $\bar{g}$ if i quits which happens with probability $\triangle\ \lambda_i$. Consequently, $0 = \lim_{\triangle \to 0} \triangle\ \lambda_i \bar{g} - l_j(1 - e^{-r\triangle})$ and so $\lambda_i = \lim_{\triangle \to 0} \frac{(1-e^{-r\triangle})l_j}{\bar{g}\triangle} = \frac{rl_j}{\bar{g}}$.

Once one of the players' normal type has finished conceding and the player is known as the Stackelberg type with certainty, then the normal type of the other player should also concede immediately. Because a Stackelberg type never concedes, the normal player has no incentive to insist. Consequently, condition (iii) holds and both players complete conceding by the same finite time T. Conditions (i) through (iii) imply the following lemma:

Lemma 1. *Let $F_i(t)$ denote the cumulative probability that player i concedes by time t, that is, $F_i(t)$ is $1 - z_i$ multiplied by the probability that the normal type of player i quits by time t. Let $\lambda_i = \frac{rl_j}{\bar{g}}$, $T_i = \frac{-\ln z_i}{\lambda_i}$, $T = \min\{T_1, T_2\}$ and $c_i \in [0,1]$, then*

$$F_i(t) = 1 - c_i e^{-\lambda_i t} \text{ for all } t \leq T < \infty, \text{ and } F_i(T) = 1 - z_i,$$

where $1 - c_i$ is the probability that player i concedes at time 0, and $1 - c_i > 0$ if and only if $T_i > T_j$. The unique PBE of the war of attrition is (F_1, F_2). Also, the unique PBE payoff vector for the war of attrition is $((1 - c_2)\bar{g}, (1 - c_1)\bar{g})$.

Proof. Observe if F_1 jumps at time t, then F_2 does not jump at time t. This follows from the argument provided for condition (i). The rest of the argument in Abreu and Gul (2000) applies verbatim. Thus F_1 and F_2 comprise the unique equilibrium for the war of attrition.

Suppose player 1 concedes with positive probability at time zero. Since player 1 concedes with positive probability at time zero, he is indifferent between conceding immediately and receiving a payoff equal to zero and continuing. Consequently, player 1's equilibrium payoff must equal zero. Player 2 is also indifferent between quitting and conceding at any time after time zero. This implies that player 2's expected payoff at time $t > 0$, conditional on neither player conceding by time t, is equal to zero. So, player 2's equilibrium payoff is equal to $(1 - c_1)\bar{g}$. $\square$

3.2. **The Two Sided Reputation Result.** Let $G_1^n(t)$ denote the cumulative probability that player 1 reveals rationality by t, if he is playing against the Stackelberg type of player 2, in any PBE profile σ of $\Gamma^\infty(z, \Delta^n)$. In the argument for Theorem 2 we demonstrate that the distributions (G_1^n, G_2^n) have a unique limit and we prove that the limiting distributions solve the war of attrition and is thus are equal to (F_1, F_2). We then show $(G_1^n, G_2^n) \to (F_1, F_2)$ implies that the equilibrium payoffs in the repeated game also converge to the unique equilibrium payoff of the war of attrition. We prove the theorem under the assumption that $\sigma_1(S_1)$ is not a best response to $\sigma_2(S_2)$.[7] If $\sigma_1(S_1)$ is a best response to $\sigma_2(S_2)$, then Theorem 1 immediately implies that $(U_1(\sigma^n, \Delta^n), U_2(\sigma^n), \Delta^n)$ converges to the unique payoff profile $(\bar{g}, \bar{g})$ for any sequence of PBE profiles σ^n of $\Gamma^\infty(z, \Delta^n)$.

Theorem 2. *Suppose that* Γ *is a game of perfect information and satisfies Assumption 1; or alternatively, suppose that* Γ *satisfies Assumption 1* (ii) *for both player. Assume that* $\sigma_1(S_1)$ *is not a best response to* $\sigma_2(S_2)$*. For any* $z = (z_1 > 0, z_2 > 0)$ *and any* $\epsilon > 0$*, there exists* $\Delta^* > 0$ *such that, for any* $\Delta < \Delta^*$*, any PBE profile* σ *of* $\Gamma^\infty(z, \Delta)$*,* $|U_1(\sigma, \Delta) - (1 - c_2)\bar{g}| < \epsilon$ *and* $|U_2(\sigma, \Delta) - (1 - c_1)\bar{g}| < \epsilon$*.*

Proof. Suppose in partial history h^t player i has played according to $\sigma_i(S_i)$ and player j has deviated from $\sigma_j(S_j)$, then $z_i(h^t) > z_i$ and $z_j(h^t) = 0$. Consequently, Theorem 1 implies that $U_i(\sigma|h_t) \geq \bar{g} - K(\Delta)$ where $K(\Delta) = (1 - e^{-r\Delta})K(z_i) + n^p(1 - e^{-r\Delta})M$.

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[7] For games that satisfy Assumption 1, $\sigma_1(S_1)$ is a best response to $\sigma_2(S_2)$ if and only if $\sigma_2(S_2)$ is a best response to $\sigma_1(S_1)$.

Also, let $M(\Delta) = 2Mn^p(1 - e^{-r\Delta n^p})$ which converges to zero when Δ goes to zero. Take Δ sufficiently small so that $\rho K(\Delta) + 2M(\Delta) < l_i$ and $\bar{g} - K(\Delta) > 0$.

Step 1. Let the set $R_i(n)$ denote all pure repeated game strategies σ_i such that: if σ_i is played against $\sigma_j(S_j)$, then in all periods $l < n$ the strategy plays according to $\sigma_i(S_i)$; in period n, the strategy picks a move that differs from $\sigma_i(S_i)$ at some decision node of period n.

The sets $R_i(n)$ are disjoint and their union $\bigcup_n R_i(n)$ gives all pure repeated game strategies excluding the set of strategies N_i that never deviate from $\sigma_i(S_i)$, if played against $\sigma_j(S_j)$. Let

$$G_i(t) = (1 - z_i) \sum_{\Delta n \leq t} \sigma_i(R_i(n)),$$

where, for mixed repeated game strategy σ_i, $\sigma_i(R)$ denotes the probability that a pure strategy in the set R is played. $G_i(t)$ is the probability that player i will reveal rationality by playing an action incompatible with the Stackelberg type by time t. Step 4 given below shows that for all equilibria σ of the repeated game $\Gamma^\infty(z, \Delta)$, there exists a time T such that $G_i(T) = 1 - z_i$, that is, every normal player reveals rationality by time T, if faced with a sufficiently long history of play compatible with the Stackelberg type. Consequently, for any equilibrium σ, $\sigma_i(N_i) = 0$.

If $\sigma_i(R_i(n)) > 0$, then let $U_i(\sigma | R_i(n))$ denote the expected repeated game payoff for player i, under mixed strategy profile σ, conditional on player i having picked a strategy in the set $R_i(n)$. If $\sigma_i(R_i(n)) = 0$, then let $U_i(\sigma, \Delta | R_i(n)) = \sup_{\sigma_i' \in R_i(n)} U_i(\sigma_i', \sigma_j, \Delta)$. Also, for any real number t, let $U_i(\sigma, \Delta | R_i(t)) = U_i(\sigma, \Delta | R_i(\bar{n}))$ where $\bar{n} = \max_n \{\Delta n \leq t\}$. Observe that for any equilibrium mixed strategy profile σ,

$$\begin{aligned} U_i(\sigma, \Delta) &= \sum_{n=0}^{\infty} \sigma_i(R_i(n)) U_i(\sigma, \Delta | R_i(n)) + \sigma_i(N_i) U_i(\sigma, \Delta | N_i) \\ &= \frac{1}{1 - z_i} \int_{t=0}^{\infty} U_i(\sigma, \Delta | R_i(t)) dG_i(t). \end{aligned}$$

Step 2. Define

$$\begin{aligned} \bar{U}_i(t, k) &= e^{-r \min\{t,k\}} \bar{g} - l_i(1 - e^{-r \min\{t,k\}}) & \text{if } t \geq k \\ &= -l_i(1 - e^{-r \min\{t,k\}}) & \text{if } t < k, \end{aligned}$$

where $t, k \in \mathbb{R}_+$. For any PBE profile σ^n of $\Gamma^\infty(z, \Delta^n)$

$$U_i(\sigma^n, \Delta^n) \leq \frac{1}{1-z_i} \int_t \int_k \bar{U}_i(t,k) dG_j^n(k) dG_i^n(t) + \rho K(\Delta^n) + M(\Delta^n).$$

Proof. Fix PBE profile σ. Pick t such that $\Delta n = t$ for some n. We bound $U_i(\sigma, \Delta | R_i(t))$. If j does not reveal rationality in any period $k \leq t$, then player i will reveal rationality. Consequently, the continuation utility for player j will be at least $\bar{g} - K(\Delta)$. This implies that player i's continuation utility after period t is at most $\rho K(\Delta)$. Also, player i will incur $-(1 - e^{-rt}) l_i$ since both players will play according to the Stackelberg action up to period t. This event occurs with probability $1 - G_j(t)$. If player j reveals rationality at any time $\Delta m = k \leq t$, then player 1 will receive payoff at most $\bar{g}$ from that period onwards and will incur $-(1 - e^{-rk}) l_i$ up to time k. Consequently,

$$U_i(\sigma, \Delta | R_i(t)) \leq \int_{\{t \geq k\}} (e^{-rk}\bar{g} - l_i(1 - e^{-rk})) dG_j(k) + (1 - G_j(t))(\rho K(\Delta) - l_i(1 - e^{-rt})) + M(\Delta)$$

$$U_i(\sigma, \Delta | R_i(t)) \leq \int \bar{U}_i(t,k) dG_j(k) + \rho K(\Delta) + M(\Delta)$$

where the factor $M(\Delta)$ corrects for revelations that occur during punishment phases as well as the inaccuracy of using l_i as the cost of resisting the Stackelberg strategy. Hence,

$$U_i(\sigma, \Delta) \leq \frac{1}{1-z_i} \int \int \bar{U}_i(t,k) dG_j(k) dG_i(t) + \rho K(\Delta) + M(\Delta)$$

□

Step 3. Define

$$\begin{aligned} \underline{U}_i(t,k) &= e^{-r\min\{t,k\}}\bar{g} - l_i(1 - e^{-r\min\{t,k\}}) && \text{if } t > k \\ &= -l_i(1 - e^{-r\min\{t,k\}}) && \text{if } t \leq k \end{aligned}$$

where $t, k \in \mathbb{R}_+$. For any PBE profile σ^n of $\Gamma^\infty(z, \Delta^n)$

$$U_i(\sigma^n, \Delta^n) \geq \frac{1}{1-z_i} \int \int \underline{U}_i(t,k) dG_j^n(k) dG_i^n(t) - K(\Delta^n)\rho - M(\Delta^n)$$

Proof. Fix PBE profile σ and suppose that j behaves according to σ_j. Pick t such that $\Delta n = t$ for some n. We bound $U_i(\sigma, \Delta | R_i(t))$. If j reveals rationality in any period $\Delta m = k < t$, then player i incurs $-l_i$ up to that time, and receives continuation payoff $\bar{g} - K(\Delta)$. This exceeds $-(1 - e^{-rk}) l_i + e^{-rk}(\bar{g} - K(\Delta))$. If player i reveals rationality first in period t, then she receives as a continuation $-\rho K(\Delta) \leq 0$. If player j reveals first in period t, then player i receives in continuation $\bar{g} - K(\Delta) > 0$. Consequently,

$$U_i(\sigma, \Delta | R_i(t)) \geq \int_{k<t} (e^{-rk}\bar{g} - (1-e^{-rk})l_i) dG_j(k) - (1-G_j(t^-))(1-e^{-rt})l_i - (1+\rho)K(\Delta) - M(\Delta)$$

where $1 - G_j(t^-)$ denotes the probability that player j reveals at a time $k \geq t$. This implies that

$$U_i(\sigma, \Delta | R_i(t)) \geq \int \underline{U}_i(t,k) dG_j(k) - (1+\rho)K(\Delta) - M(\Delta)$$

Hence,

$$U_i(\sigma, \Delta) \geq \frac{1}{1-z_i} \int \int \underline{U}_i(t,k) dG_j(k) dG_i(t) - (1+\rho)K(\Delta) - M(\Delta)$$

proving the result. □

Step 4. There exists a T such that $G_i^n(T) = 1 - z_i$.

Proof. Fix $s \in \mathbb{R}_+$. Suppose that $y^k(\sigma_1(S_1), \sigma_2(S_2))$ has been realized in each period k such that $\Delta k < s$. Consider the strategy for i that continues to play $\sigma_i(S_i)$ for all periods n such that $\Delta n \in [s, s+t]$, given that $y(\sigma_1(S_1), \sigma_2(S_2))$ has been realized in all prior periods. For this strategy to be considered, it must do better for player i than revealing rationality which guarantees her $-\rho K(\Delta)$ by loosing at most $M(\Delta)$. So,

$$-\rho K(\Delta) - M(\Delta) \leq \mathbb{P}_j([s, s+t])(\bar{g}e^{-rt} - l_i(1-e^{-rt}) + M(\Delta)) + (1 - \mathbb{P}_j([s, s+t]))\bar{g}$$

where $\mathbb{P}_j([s, s+t])$ denotes the probability that player j does not deviate from $\sigma_j(S_j)$ in $[s, s+t]$ given that i plays strategy $\sigma_i(S_i)$. Consequently,

$$\mathbb{P}_j([s, s+t]) \leq \frac{\bar{g} + \rho K(\Delta) + M(\Delta)}{(\bar{g} + l_i)(1 - e^{-rt})}.$$

Observe that for t large, $\frac{\bar{g}+\rho K(\Delta)+M(\Delta)}{(\bar{g}+l_i)(1-e^{-rt})-M(\Delta)} < 1$. This implies that for i to be willing to play $\sigma_i(S_i)$ for all $\Delta n \in [0, tk]$

$$\begin{aligned} z_j \leq \mathbb{P}_j([0, tk]) &= \prod_{s=0}^{k} \mathbb{P}_j([s, s+t]) \\ &\leq \left(\frac{\bar{g} + \rho K(\Delta) + M(\Delta)}{(\bar{g} + l_i)(1 - e^{-rt})} \right)^k \end{aligned}$$

However for t and k sufficiently large this is not possible. $\square$

Step 5. There exists a subsequence $\{n_k\} \subset \{n\}$ such that $(G_1^{n_k}(t), G_2^{n_k}(t)) \to (\hat{G}_1(t), \hat{G}_2(t))$.

Proof of Step 5. Since G_1^n and G_2^n are distribution functions, by Helly's theorem they have a (possibly) subsequential limit $(\hat{G}_1(t), \hat{G}_2(t))$. Also, since the support of the G_1^n and G_2^n's is uniformly bounded, the limiting functions $\hat{G}_1(t)$ and $\hat{G}_2(t)$ are also distribution functions. $\square$

Step 6. The distribution functions $\hat{G}_1(t)$ and $\hat{G}_2(t)$ do not have any common points of discontinuity.

Proof. Assume that $\hat{G}_1$ and $\hat{G}_2$ have a common point t where they are both discontinuous. Let $J_1 = \hat{G}_1(t) - \lim_{s \nearrow t} \hat{G}_1(s)$ and let $J_2 = \hat{G}_2(t) - \lim_{s \nearrow t} \hat{G}_2(s)$. We can pick ζ, arbitrarily close to t, such that both $\hat{G}_1$ and $\hat{G}_2$ are continuous at $t+\zeta$ and $t-\zeta$. This implies that for each $\epsilon > 0$, there is a N such that the game is played at least once in each interval of length 2ζ and $G_i^n[t-\zeta, t+\zeta] = G_i^n(t+\zeta) - G_i^n(t-\zeta) \geq J_i - \epsilon > 0$, for all $n \geq N$. In words, the probability that 1 plays an action different than $a_i(s)$ in the interval $[t-\zeta, t+\zeta]$ is greater than $J_1 - \epsilon$. Also, pick N such that the value to any player after she has played an action other than the commitment action is less than ϵ, the payoff to any player who has not played an action different than the commitment action against an opponent known to be rational is greater than $\bar{g} - \epsilon$, and $M(\Delta^n) < \epsilon$. Pick the first period k such that $\Delta^n k \in [t-\zeta, t+\zeta]$ and $\mathbb{P}\{a_1^k \neq \sigma_1(S_1)\} > 0$ or $\mathbb{P}\{a_2^k \neq \sigma_2(S_2)\} > 0$. Since $G_i^n[t-\zeta, t+\zeta] > 0$ such a period exists for all $n > N$.

Without loss of generality assume that $\mathbb{P}\{a_1^k \neq \sigma_1(S_1)\} > 0$. The payoff that player 1 receives from deviating from $\sigma_1(S_1)$ in period k must be at least as large as playing $\sigma_1(S_1)$ throughout the interval $[t-\zeta, t+\zeta]$. Let U_i denote the payoff that player i receives in equilibrium conditional on both players not playing $\sigma_i(S_i)$ in period k. Consequently,

$$\mathbb{P}\{a_2^k \neq \sigma_2(s)\}U_1 + (1 - \mathbb{P}\{a_2^k \neq \sigma_2(s)\})\epsilon \geq e^{-r4\zeta}(\bar{g} - \epsilon)G_2^n[t-\zeta, t+\zeta] - l(1 - e^{-r4\zeta}) + \epsilon$$

Redefine $\epsilon' = \epsilon + \frac{\epsilon}{e^{-r4\zeta}(J_i - \epsilon)} + \frac{l(1-e^{-r4\zeta}) - \epsilon}{e^{-r4\zeta}(J_i - \epsilon)}$ and rewrite the above equation as follows:

$$\mathbb{P}\{a_2^k \neq \sigma_2(s)\}U_1 \geq e^{-r4\zeta}(\bar{g}-\epsilon')G_2^n[t-\zeta, t+\zeta].$$

For ϵ and ζ sufficiently small, the right hand side of the equation is approximately $J_2\bar{g}$ and the left hand side is $\mathbb{P}\{a_2^k \neq \sigma_2(S_2)\}U_1$. Consequently, for this inequality to hold, $\mathbb{P}\{a_2^k \neq \sigma_2(S_2)\} > 0$. Also, by definition, $\mathbb{P}\{a_2^k \neq \sigma_2(S_2)\} \leq G_2^n[t-\zeta, t+\zeta]$, and consequently, $U_1 \geq e^{-r(4\zeta)}(\bar{g}-\epsilon')$.

$\mathbb{P}\{a_2^k \neq \sigma_2(S_2)\} > 0$ implies, by a symmetric argument as in the case of player 1, that

$$\mathbb{P}\{a_1^k \neq \sigma_1(s)\}U_2 \geq e^{-r4\zeta}(\bar{g}-\epsilon')G_1^n[t-\zeta, t+\zeta]$$

Consequently, $U_2 \geq e^{-r4\zeta}(\bar{g}-\epsilon')$. However, $U_1 \geq e^{-r4\zeta}(\bar{g}-\epsilon')$, implies that $U_2 \leq \rho(\bar{g}-U_1) = \rho(\bar{g}(1-e^{-r4\zeta}) + \epsilon' e^{-r4\zeta})$. So,

$$\rho(\bar{g}(1-e^{-r4\zeta}) + \epsilon' e^{-r4\zeta}) \geq e^{-r4\zeta}(\bar{g}-\epsilon')$$

However, taking $\epsilon' \to 0$ and $\zeta \to 0$ implies $0 \geq \bar{g}$ which is a contradiction. □

Step 7. If $(G_1^n(t), G_2^n(t)) \to (\hat{G}_1(t), \hat{G}_2(t))$, then

$$\lim U_i(\sigma^n)(1-z_i) = \int\int \bar{U}_i(t,k)d\hat{G}_j(k)d\hat{G}_i(t) = \int\int \underline{U}_i(t,k)d\hat{G}_j(k)d\hat{G}_i(t).$$

Proof. If G_1^n converges to $\hat{G}_1$ and G_2^n converges to $\hat{G}_2$, then the product measure $G_1^n \times G_2^n$ converges to $\hat{G}_1 \times \hat{G}_2$, see Billingsley (1995), Page 386, Exercise 29.2. Observe that the functions $\bar{U}_1(t,k)$ and $\underline{U}_1(t,k)$ are continuous at all points except on the set $\{t=k\}$. By the previous lemma, $\int_{\mathbb{R}^2} 1_{\{t=k\}}d(\hat{G}_1 \times \hat{G}_2) = 0$. Consequently, the $\hat{G}_1 \times \hat{G}_2$ measure of the points of discontinuity of $\bar{U}_1(t,k)$ and $\underline{U}_1(t,k)$ is zero. Billingsley (1995), Theorem 29.2, shows that if the set of discontinuities of a measurable function h, D_h, has μ measure zero, i.e., $\mu(D_h)=0$ and $\mu_n \to \mu$, then $\int h d\mu_n \to \int h d\mu$. So,

$$\lim_n \int_{t_1}\left(\int_{t_2} \bar{U}_1(t_1,t_2)dG_2^n(t_2)\right)dG_1^n(t_1) = \lim_n \int_{\mathbb{R}^2} \bar{U}_1 d(G_1^n \times G_2^n) = \int_{\mathbb{R}^2} \bar{U}_1 d(\hat{G}_1 \times \hat{G}_2)$$

and similarly for $\underline{U}_1$. Also, since $\bar{U}_1$ and $\underline{U}_1$ differ only on a set of zero measure, $\int_{\mathbb{R}^2} \bar{U}_1 d(\hat{G}_1 \times \hat{G}_2) = \int_{\mathbb{R}^2} \underline{U}_1 d(\hat{G}_1 \times \hat{G}_2)$. □

Step 8. The distribution functions $(\hat{G}_1(t), \hat{G}_2(t))$ solve the war of attrition and consequently $(\hat{G}_1(t), \hat{G}_2(t)) = (F_1(t), F_2(t))$.

Proof. In the continuous time war of attrition, if player 1 is behaving according to $\hat{G}_1$, then for each ϵ, there is a N such that for all $n > N$, G_2^n is an ϵ best response to $\hat{G}_1$ and consequently, since ϵ is arbitrary $\hat{G}_2$ is a best response to $\hat{G}_1$. Also, the symmetric argument is true for player 2 showing that $\hat{G}_1$ is a best response to $\hat{G}_2$. Proving that $\hat{G}_1$ and $\hat{G}_2$ form an equilibrium for the continuous time war of attrition. Since the war of attrition has a unique equilibrium $\hat{G}_1 = F_1$ and $\hat{G}_2 = F_2$. This argument is identical to the proof of Abreu and Gul (2000) Proposition 4, on page 114. A more detailed argument can be found there. □

Step 9. Observe that $\lim U_i(\sigma^n, \Delta^n) = \frac{1}{1-z_i} \int \int \bar{U}_i(t,k) dF_j(k) dF_i(t)$. However,

$$\frac{1}{1-z_i} \int \int \bar{U}_i(t,k) dF_j(k) dF_i(t) = (1-c_j)\bar{g}$$

is just the expected payoff of player i in the war of attrition.

□

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