REPUTATION IN LONG-RUN RELATIONSHIPS [1]

Alp E. Atakan and Mehmet Ekmekci

We model a long-run relationship as an infinitely repeated game played by two equally patient agents. In each period, the agents play an extensive-form stage-game of perfect information with either locally nonconflicting interests or strictly conflicting interests. There is incomplete information about the type of player 1, while player 2's type is commonly known. We show that a sufficiently patient player 1 can leverage player 2's uncertainty about his type to secure his highest payoff, compatible with player 2's individual rationality, in any perfect Bayesian equilibrium of the repeated game.

Keywords: Repeated Games, Reputation, Equal Discount Factor, Long-run Players.

JEL Classification Numbers: C73, D83.

Koç University, Department of Economics, Rumeli Feneri Yolu, Sarıyer 34450, Istanbul, Turkey.
e-mail: aatakan@ku.edu.tr.

MEDS, Kellogg School of Management, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208.
e-mail: m-ekmekci@kellogg.northwestern.edu.

[1] We thank Nabil Al-Najjar, Bruno Biais, Martin Cripps, Eddie Dekel, Drew Fudenberg, Johannes Horner, Christoph Kuzmics, George Mailath, Larry Samuelson as well as the anonymous referees for their comments; and Umberto Garfagnini for research assistance. Part of this research was conducted while Mehmet Ekmekci was visiting the Cowles Foundation at Yale University and he thanks the foundation for their hospitality. This paper was previously circulated under the title "Reputation with Long-run Players."

Contents

1. Introduction

Maintaining a reputation can benefit economic agents since it lends credibility to their future commitments, threats or promises. A reputation can help a firm commit to fight potential entrants (Kreps and Wilson (1982), Milgrom and Roberts (1982)) or it can lend credibility to a government's monetary and fiscal policies (Barro (1986), Phelan (2006)). So a patient agent may forego short-run profit in order to cultivate a reputation in anticipation of long-run benefits.

Reputation effects are pronounced if an agent is patient, that is, if the short-run cost of building a reputation is less important to the agent than the long-run benefit (Fudenberg and Levine (1989, 1992)). There is a tension, however, if the agent trying to build a reputation faces an opponent who is equally patient: the opponent may also sacrifice payoff in the short-run in order to extensively test the agent's resolve to go through with his commitments, threats, or promises. This can make it prohibitively expensive to build a reputation in a certain class of repeated games where players move simultaneously (Cripps and Thomas (1997)). To highlight this tension, we focus in this paper on equally patient agents and show that in repeated games where players move *sequentially* reputation effects are nevertheless prominent.

We consider an infinitely repeated game played by two *equally patient* agents. We assume that player 2 is uncertain about the type of player 1, while player 1 is perfectly informed about the type of player 2. In each period, the agents play an extensive-form game of *perfect information*. There are either *locally nonconflicting interests* (**LNCI**) or *strictly conflicting interests* (**SCI**) in the stage game.[1] Within this framework we prove a *reputation result*: a sufficiently patient player 1 can guarantee his highest payoff compatible with player 2's individual rationality, in any perfect Bayesian equilibrium of the repeated game.

To make the discussion more concrete, consider the following strategic situation faced by a husband and wife, two legislators, or two countries: In each period of a long-run relationship, the two players must decide whether to undertake player 1's preferred policy A, player 2's preferred policy B, or neither of the two policies. Unanimity is required for any policy to be chosen. These policies can represent competing treaties in a pollution abatement negotiation between two countries, budget alternatives under consideration by two political rivals, or even weekend plans being bargained over by a married couple. The repeated game where figure 1 is played in each period is a simple representation of this strategic situation.[2]

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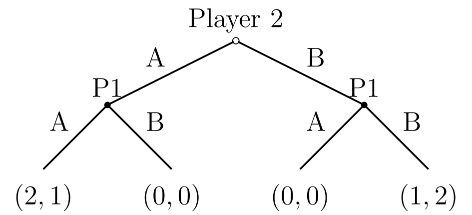
[1] There are LNCI in a game if the unique payoff profile where player 1 receives his highest payoff is strictly individually rational for player 2. Intuitively, there are SCI in a game if the action which is the best for player 1 is the worst for his opponent. See Assumption 1 for precise statements.

[2] The battle-of-the-sexes game is used to model product compatibility in Farrell and Saloner (1988), network externalities in Katz and Shapiro (1985), communication and mediation in Banks and Calvert (1992), and repeated bargaining in Schelling (1960). For the battle-of-the-sexes game applied to pollution abatement negotiations between nations, see Harstad (2007); for an application to negotiations between political rivals,



Note: The first component of the payoff vector is player 1's payoff.

Figure 1: The battle-of-the-sexes game.

Suppose that player 2 (she) believes that player 1 (he) is either fully rational or a Stackelberg type who is committed to choosing A in each period. A rational player 1, cognizant of player 2's uncertainty, has an incentive to mimic the Stackelberg type, i.e., to build a reputation. If player 2 is convinced that player 1 is the Stackelberg type, then she will have no choice but to play A, and policy A will be the outcome in each period. Therefore, a patient player 1 may play A for many periods, even if player 2 plays B (i.e., at the expense of reaching an agreement), in order to convince player 2 that he is indeed the Stackelberg type. However, player 2 knows that player 1 has an incentive to mimic the Stackelberg type. Consequently, an equally patient player 2 may play B (i.e., resist playing A) for many periods, thereby making reputation building particularly costly, especially if she deems it sufficient likely that player 1 is rational and will eventually start playing B.

Given these two opposing forces, can player 1 build a reputation and ensure that policy A is implemented? Or alternatively, will screening by player 2 keep a rational player 1 from building a reputation? These questions are addressed in our main finding: if the players are equally and arbitrarily patient, then policy A is implemented in each period and player 1 receives a payoff equal to two in any perfect Bayesian equilibrium of the repeated game. This outcome is independent of which player moves first and independent of how small the initial uncertainty about player 1's type is.[3]

In the previous example, player 1's reputation allowed him to credibly commit to always choosing the same action. However, we can conceive of other strategic situations where player 1 may want to commit to a more complex strategy that rewards or punishes his opponent in a history-dependent way. For example, player 1 may want to be known for playing tit-for-tat, or for punishing bad behavior consistently. To capture reputation effects more generally, we assume that player 1 is either fully rational or one of many *commitment* types. Each commitment type is programmed to play a certain repeated game strategy. The commitment type central to our analysis is a dynamic *Stackelberg* type. This type plays the repeated game strategy that player 1 would choose if player 1 could publicly commit to any repeated game

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see Alesina and Drazen (1991); and for an application to marital bargaining, see Lundberg and Pollak (1994).

[3] See figure 2d for the battle-of-the-sexes game where player 1 moves first.

strategy. Ideally, player 1 would like to convince his opponent that his future actions will fully conform to the behavior of the Stackelberg type. We show that a sufficiently patient player 1 can use his ability to mimic the Stackelberg type and his opponent's uncertainty about his type to secure his most preferred outcome for the repeated game.

1.1. Related literature. This paper is closely related to the literature on reputation effects in repeated games. Much of the previous literature on reputation considers a patient player 1 who faces a myopic opponent. Most prominently, Fudenberg and Levine (1989, 1992) show that if there is positive probability that player 1 is a type committed to playing the Stackelberg action in every period, then player 1 gets at least his static Stackelberg payoff in any equilibrium of the repeated game.[4] Reputation results have also been established for repeated games where player 1 faces a nonmyopic opponent, but one who is sufficiently less patient than player 1 (see Schmidt (1993), Celentani et al. (1996), Aoyagi (1996), or Evans and Thomas (1997)). However, the repeated games that these papers consider are genuinely long-run only for player 1 and this feature is crucial for the reputation results.

In a game with a nonmyopic opponent, player 1 may achieve a payoff that exceeds his static Stackelberg payoff by using a history-dependent strategy that rewards or punishes player 2. Conversely, future punishments or rewards can induce player 2 to not best respond to a Stackelberg action and thereby force player 1 below his static Stackelberg payoff.[5] These complications render reputation effects fragile in repeated games with equally patient players: A reputation result obtains in a repeated simultaneous-move game only if the stage game is a strictly dominant action game (Chan (2000)), or if there are SCI in the stage game (Cripps et al. (2005)).[6] For other repeated simultaneous-move games, any individually rational payoff can be sustained in a perfect equilibrium if the players are sufficiently patient (see the folk theorem of Cripps and Thomas (1997)).[7]

1.2. Contribution to the literature. We make three main contributions to the literature on reputation effects in repeated games with equally patient players. First, we provide

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[4] The static Stackelberg payoff for player 1 is the highest payoff he can guarantee in the stage game through public commitment to a stage game action (a Stackelberg action). See Mailath and Samuelson (2006, page 465), for a formal definition.

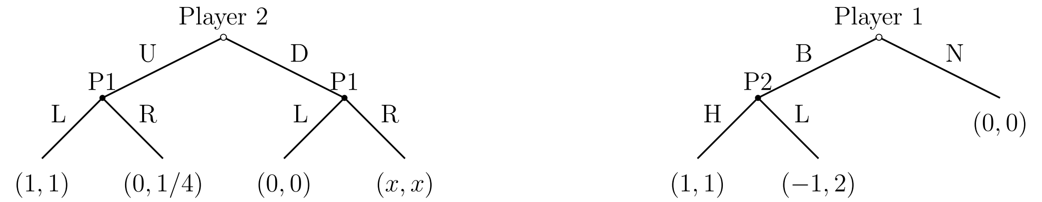
[5] Player 2 may expect punishments or rewards either from the rational type of player 1 after he chooses a move that would not be chosen by the Stackelberg type (Celentani et al. (1996, Section 5) or Cripps and Thomas (1997)), or from a commitment type other than the Stackelberg type (Schmidt (1993) or Celentani et al. (1996)).

[6] A stage-game is a strictly dominant action game if player 1's static Stackelberg payoff is equal to his highest payoff compatible with player 2's individual rationality, and if the Stackelberg action is strictly dominant, see Mailath and Samuelson (2006, Page 540) for a formal definition.

[7] Also, see Cripps and Thomas (1995) for a model of equally patient agents which uses the limit of means criteria instead of equal discounting.

a reputation result for a new class of games: repeated extensive-form games of perfect information. Second, we highlight the distinct role that perfect information plays for a reputation result with equally patient agents. Third, we introduce novel methods, inspired by the bargaining literature (Myerson, 1991, Chapter 8.8), to analyze reputation effects in repeated games.

Previous reputation results for equally patient agents are for certain repeated simultaneous-move games (i.e., Chan (2000) for strictly dominant action stage-games and Cripps et al. (2005) for stage-games with SCI). In contrast, we focus on repeated extensive-form games of perfect information, and as our first main contribution, we establish a reputation result for stage-games with LNCI or SCI.[8] Games that are commonly used in economic applications, such as the examples depicted in figures 1 and 2, are included in the class of games that we cover in our reputation result.[9,10]



(a) Common-interest game ($x < 1$).

(b) Principal-agent game (or the product choice game).

(c) Chain-store game.

(d) The battle-of-the-sexes game.

Figure 2: Sequential-move games with LNCI (2a and 2b) or SCI (2c and 2d).

Games with LNCI have a common-value component whereas games with SCI entail conflict between the two players. A game has LNCI if the unique payoff profile where player 1 receives

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[8] If a game in the class that we consider (i.e., a game with LNCI or SCI) is played under complete information, then the folk theorem of Fudenberg and Maskin (1986) applies under a full dimensionality condition (see Wen (2002)).

[9] Examples of common interest games include the coordination game and the hawk-dove game (for applications, see Morris and Shin (1998) and Baliga and Sjöström (2004)). For an application of the repeated principal-agent game, see Laffont and Martimort (2002, Chapter 9). For an application of the chain-store game in industrial organization, see Tirole (1988, Chapter 9).

[10] A game falls outside of the class of games with LNCI or SCI if the profile where player 1 receives his highest payoff is not strictly individually rational for player 2, and the game does not have SCI. Examples of such games include the prisoner's dilemma game and the principal-agent game in figure 2b if player 2 is the player that is building a reputation instead of player 1. See section 4.2 for a more extensive discussion.

his highest stage-game payoff is strictly individually rational for player 2. The battle-of-the-sexes game where player 2 moves first (figure 1), the common interest game (figure 2a), and the principal-agent game (figure 2b) have LNCI. These games have LNCI because player 1 receives his highest payoff in the payoff profile $(2, 1)$, $(1, 1)$, and $(1, 1)$, in figures 1, 2a, and 2b, respectively. Moreover, in each of these profiles, player 2's payoff strictly exceeds her minimax, which is equal to zero, $1/4$, and zero, in figures 1, 2a, and 2b, respectively.

A game has SCI if player 1 has an action (a Stackelberg action) such that any best reply to this action yields player 1 his highest payoff compatible with player 2's individual rationality and yields player 2 her minimax payoff. The chain-store game (figure 2c) and the battle-of-the-sexes game where player 2 moves second (figure 2d) have SCI. The chain-store game has SCI because, if player 1 commits to action F and player 2 best responds to F, then player 1 receives a payoff equal to four, his highest payoff; and player 2 receives a payoff equal to zero, her minimax payoff. Similarly in the battle-of-the-sexes game where player 2 moves second, if player 1 plays action A and player 2 best responds, then player 1 receives a payoff equal to two, his highest payoff; and player 2 receives a payoff equal to one, her minimax payoff.

Our second main contribution pinpoints why reputation effects are particularly salient in repeated games with LNCI and perfect information, whereas reputation effects are absent in certain repeated simultaneous-move games with LNCI. For example, our reputation result implies that there is a unique equilibrium payoff profile in the repeated sequential-move battle-of-the-sexes game (figure 1 or figure 2d). In contrast, if a simultaneous move game with LNCI, such as the battle-of-the-sexes game (figure 3b), is played in each period, then a folk theorem obtains. For a more striking example, consider the repeated simultaneous-move common interest game (figure 3a), where player 1 is potentially a Stackelberg type who always plays U. This game appears to be a strong candidate for reputation effects to arise. It is costless for player 1 to mimic the Stackelberg type and build a reputation. Also, player 2 unambiguously benefits if player 1 is able to build a reputation and concentrate play on (U, L). Surprisingly, any individually rational payoff profile can be sustained in a perfect Bayesian equilibrium if the players are arbitrarily patient (Cripps and Thomas (1997)). In contrast, we show that in the repeated sequential-move game, the players receive a payoff equal to one, in any perfect Bayesian equilibrium.[11]

Our third main contribution is the novel method that we use to establish our reputation result. A new approach is required because the technique of Fudenberg and Levine (1989, 1992), which is commonly used to establish reputation results, is not applicable with two equally patient players. Our method hinges on having those information sets where player 1's normal type reveals rationality be singletons (perfect information). Sequential rationality,

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[11] For a more detailed discussion, see section 4.1.

(a) Common-interest game ($x < 1$).

(b) The battle-of-the-sexes game.

Figure 3: Simultaneous-move games with LNCI.

coupled with perfect information, imposes tight bounds on player 1's continuation payoffs at these nodes. Moreover, for the class of games that we consider, if there is a tight bound on player 1's continuation payoff, then there is also a tight bound on player 2's continuation payoffs. These bounds preclude the possibility of player 1 building a reputation slowly and punishing player 2 for best responding to the Stackelberg strategy.

2. THE MODEL

In the infinitely repeated game, a stage game Γ is played by players 1 and 2 in periods $t \in \{0, 1, 2, ...\}$ and the players discount their payoffs using a common discount factor $\delta \in [0, 1)$. *The stage game Γ is a two-player finite game of perfect information, that is, all the information sets of Γ are singletons* **(perfect information)**.

The set of nodes (decision nodes and terminal nodes) of the stage game Γ is denoted by $\mathcal{D}$, d is a typical element of $\mathcal{D}$, $Y \subset \mathcal{D}$ is the set of terminal nodes, and y is a typical element of Y. The payoff function of player i is $g_i : Y \to \mathbb{R}$. The finite set of pure stage game actions for player i is A_i, and the set of mixed stage game actions is $\mathcal{A}_i$.[12] For any action profile $a = (a_1, a_2) \in A_1 \times A_2$, there is a unique terminal history $y(a) \in Y$ under the path of play induced by a. Slightly abusing notation we let $g_i(a) = g_i(y(a))$ for any $a \in A_1 \times A_2$, and we let $g_i(\alpha)$ denote the payoff to mixed action profile $\alpha \in \mathcal{A}_1 \times \mathcal{A}_2$.

The minimax payoff for player i is $\hat{g}_i = \min_{\alpha_j \in \mathcal{A}_j} \max_{\alpha_i \in \mathcal{A}_i} g_i(\alpha_i, \alpha_j)$. For games that satisfy perfect information, there exists $a_1^p \in A_1$ such that $g_2(a_1^p, a_2) \leq \hat{g}_2$ for all $a_2 \in A_2$.[13] The set of feasible payoffs F is the convex hull of the set $\{g_1(a_1, a_2), g_2(a_1, a_2) : (a_1, a_2) \in A_1 \times A_2\}$; and the set of feasible and individually rational payoffs is $G = F \cap \{(g_1, g_2) : g_1 \geq \hat{g}_1, g_2 \geq \hat{g}_2\}$. Let $\bar{g}_1 = \max\{g_1 : (g_1, g_2) \in G\}$ and let $M = \max\{\max\{|g_1|, |g_2|\} : (g_1, g_2) \in F\}$.[14]

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[12] An action $a_i \in A_i$ is a contingent plan that specifies a move, from the set of feasible moves for player i, at any decision node d where player i is called upon to move.

[13] Consider the zero-sum game where player 1's payoff is equal to $-g_2(a_1, a_2)$. The minimax of this game is $(-\hat{g}_2, \hat{g}_2)$ by definition. Perfect information and Zermelo's lemma imply that this game has a pure strategy Nash equilibrium $(a_1^p, a_2) \in A_1 \times A_2$. Because the game is a zero sum game and the minimax value of the game is equal to $(-\hat{g}_2, \hat{g}_2)$ we have that $g_2(a_1^p, a_2) = \hat{g}_2$.

[14] Note that with a slight abuse of notation g_i denotes both the payoff function as well as the payoff level for player i.

In the repeated game, players have perfect recall and can observe past outcomes. The set of period $t \geq 0$ public histories is denoted $Y^t \times D$ and $h = (y^0, y^1, ..., y^{t-1}, d)$ is a typical element. The set of period $t \geq 0$ public histories of terminal nodes is denoted $H^t \equiv Y^t$, a typical element is $h^t = (y^0, y^1, ..., y^{t-1})$, and we define $h^0 = \varnothing$. At the end of a period t, player i observes neither player j's stage-game mixed action α_j^t in period t, nor player j's pure action a_j^t. Rather, player i observes the terminal node y^t and consequently the unique sequence of moves at the decision nodes that led to the particular terminal node y^t.[15]

2.1. Types and Strategies.

Before time zero, nature selects player 1's type ω from a countable set of types Ω according to a common-knowledge prior μ. Player 2 is known with certainty to be a normal type that maximizes expected discounted utility. The set of types Ω contains a normal type for player 1 that we denote by N. Slightly abusing notation, we denote player 2's belief over player 1's types after any period t public history by $\mu : \bigcup_{t=0}^{\infty} Y^t \times D \to \Delta(\Omega)$.

A behavior strategy for player i is a function $\sigma_i : \bigcup_{t=0}^{\infty} H^t \to \mathcal{A}_i$, and Σ_i is the set of all behavior strategies. A behavior strategy chooses a mixed stage game action given any period t public history of terminal nodes.[16] Each type $\omega \in \Omega \setminus \{N\}$ is committed to playing a particular repeated game behavior strategy $\sigma_1(\omega)$. A strategy profile $\sigma = (\{\sigma_1(\omega)\}_{\omega\in\Omega}, \sigma_2)$ lists the behavior strategies of all the types of player 1 and player 2. For any period t public history h^t and $\sigma_i \in \Sigma_i$, the continuation strategy induced by h^t is $\sigma_i|_{h^t}$. For $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$, the probability measure over the set of (infinite) public histories induced by (σ_1, σ_2) is $\Pr_{(\sigma_1,\sigma_2)}$.

In what follows we assume that Ω contains a certain *Stackelberg type* S. We elaborate on the Stackelberg type in section 2.5. Also, we denote the set of *other commitment types* by $\Omega_- = \Omega \backslash \{S, N\}$. In words, Ω_- is the set of types other than the Stackelberg type and the normal type.

2.2. The repeated game and payoffs.

A player's repeated game payoff is the normalized discounted sum of the stage game payoffs. For any infinite public history $h^\infty = (y^0, y^1, ...)$, let $u_i(h^\infty, \delta) = (1-\delta)\sum_{k=0}^{\infty} \delta^k g_i(y^k)$; and let $u_i(h^{-t}, \delta) = (1-\delta)\sum_{k=t}^{\infty} \delta^{k-t} g_i(y^k)$ where $h^{-t} = (y^t, y^{t+1}, ...)$. Following a period t public history, player 1 and player 2's expected continuation payoff, under strategy profile σ, are given by the following two equations, respectively:

$$U_1(\sigma, \delta|h^t) = U_1(\sigma_1(N), \sigma_2, \delta|h^t),$$

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[15] See Fudenberg and Levine (1992, Page 564) for more on this particular type of imperfect monitoring inherent in extensive-form games.

[16] Abusing notation, we will use σ_i to also denote mixed repeated game strategies for player i.

and

$$U_2(\sigma, \delta | h^t) = \sum_{\omega \in \Omega} \mu(\omega | h^t) U_2(\sigma_1(\omega), \sigma_2, \delta | h^t),$$

where $U_i(\sigma_1(\omega), \sigma_2, \delta | h^t) = \mathbb{E}_{(\sigma_1(\omega), \sigma_2)}[u_i(h^{-t}, \delta) | h^t]$ is the expectation over continuation histories h^{-t} with respect to $\Pr_{(\sigma_1(\omega)|_{h^t}, \sigma_2|_{h^t})}$. Also, $U_i(\sigma, \delta) = U_i(\sigma, \delta | h^0)$.

The repeated game of complete information (that is, the repeated game without any commitment types) with discount factor equal to $\delta \in [0, 1)$, is denoted by $\Gamma^\infty(\delta)$. The repeated game of incomplete information, with the prior over the set of commitment types given by $\mu \in \Delta(\Omega)$ and the discount factor equal to $\delta \in [0, 1)$, is denoted by $\Gamma^\infty(\mu, \delta)$.

2.3. Dynamic Stackelberg payoff and strategy. We define the *commitment payoff* of player 1's repeated game strategy σ_1 as

$$U_1^c(\sigma_1, \delta) = \min_{\sigma_2 \in BR(\sigma_1, \delta)} U_1(\sigma_1, \sigma_2, \delta),$$

where the set $BR(\sigma_1, \delta)$ denotes player 2's best responses to σ_1 in the repeated game $\Gamma^\infty(\delta)$. Also, we define the *dynamic Stackelberg payoff* as $U_1^s(\delta) = \sup_{\sigma_1 \in \Sigma_1} U_1^c(\sigma_1, \delta)$; and we define a *dynamic Stackelberg strategy* as any strategy, σ_1^*, that satisfies $U_1^c(\sigma_1^*, \delta) = U_1^s(\delta)$, if such a strategy exists.[17] Player 1's dynamic Stackelberg payoff is the highest payoff that player 1 can secure in the repeated game through public commitment to a repeated game strategy. A dynamic Stackelberg strategy for player 1 is a repeated game strategy such that any best response to this strategy gives player 1 at least his dynamic Stackelberg payoff. In other words, a dynamic Stackelberg strategy's commitment payoff is equal to the dynamic Stackelberg payoff.

2.4. Class of stage games. We assume that the stage game satisfies Assumption 1 as stated below:

Assumption 1 *The stage game Γ satisfies either of the following:*

(i) *Locally Nonconflicting Interests (LNCI): For any $g \in G$ and $g' \in G$, if $g_1 = g'_1 = \bar{g}_1$, then $g_2 = g'_2 > \hat{g}_2$; or*

(ii) *Strictly Conflicting Interests (SCI): There exists $a_1 \in A_1$ such that any best response to a_1 yields payoffs $(\bar{g}_1, \hat{g}_2)$. Also, $g_2 = \hat{g}_2$ for all $(\bar{g}_1, g_2) \in G$.*[18]

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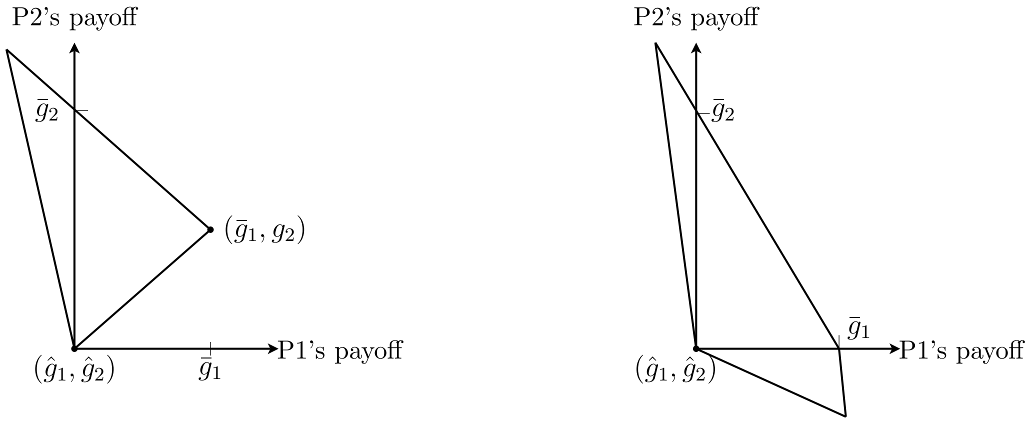
[17] This terminology follows Aoyagi (1996) and Evans and Thomas (1997).

[18] See Cripps et al. (2005), or Mailath and Samuelson (2006, page 541).

Both Assumption 1 item (*i*) and (*ii*) require that there is a unique payoff profile where player 1's payoff is equal to $\bar{g}_1$ (for example, this is true if the game Γ is a *generic* extensive form game). However, the set of games with LNCI and the set of games with SCI are mutually exclusive. Games with LNCI, have a common value component: in the payoff profile where player 1 receives his highest payoff player 2 receives a payoff that strictly exceeds her minimax value. To see that games with LNCI have a common value component, notice that in figure 4a the boundary of the set of feasible payoffs is increasing in a neighborhood of the point $(\bar{g}_1, g_2)$. Some examples of games with LNCI are the battle-of-the-sexes game where player 1 moves second (figure 1), the common interest game (figure 2a) and the principal-agent game (figure 2b). In contrast, a game has SCI if player 1 has an action (a Stackelberg action) such that any best response to this action yields player 1 his highest payoff compatible with player 2's individual rationality and yields player 2 her minimax payoff.[19] Some examples of games with SCI are the chain-store game (figure 2c) and the battle-of-the-sexes game where player 1 moves first (figure 2d). An example of the set of feasible payoffs for a game with SCI is shown in figure 4b. Some games that do not satisfy Assumption 1 are discussed in section 4.2.



(a) LNCI: the set F is bounded above and below by the lines that go through $(\bar{g}_1, g_2)$.

(b) SCI: the set F is bounded above by the downward sloping line that connects $(\bar{g}_1, \hat{g}_2)$ to $(\hat{g}_1, \bar{g}_2)$.

Figure 4: Typical set of feasible payoffs for a game with LNCI (4a) or SCI (4b).

There are two main implications of Assumption 1 that are central for the analysis that follows:

First, if Γ satisfies Assumption 1, then player 1's dynamic Stackelberg payoff is equal to player 1's high payoff that is compatible with the individual rationality of player 2 (i.e., $\bar{g}_1$)

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[19] See the product choice game (figure 10b in section 4.2) for an example where a Stackelberg action does not exist.

and there exists a particular strategy, $\sigma_1(S)$, such that the commitment payoff to $\sigma_1(S)$ is equal to $\bar{g}_1$, in the repeated game $\Gamma^\infty(\delta)$, for all δ that exceed a cutoff $\delta^* \in [0,1)$. We establish this in section 2.5 below by constructing $\sigma_1(S)$ for games that satisfy Assumption 1.[20]

Second, if Γ satisfies Assumption 1, then there are *linear bounds* on the feasible payoffs for player 2 that pass through the point $(\bar{g}_1, g_2)$; and hence, player 2's payoff are in a narrow range if player 1's payoff is close to $\bar{g}_1$ (see figure 4; or for a precise statement see inequalities (2) and (3) in section 3.2.1). This is because Assumption 1 requires that there is a unique payoff profile where player 1's payoff is equal to $\bar{g}_1$.[21]

These two main implications of Assumption 1 together establish the following (when the discount factor exceeds a cutoff $\delta^* \in [0,1)$): *if player 1's repeated game payoff is close to the commitment payoff of $\sigma_1(S)$ (which is equal to player 1's highest payoff compatible with player 2's individual rationality, i.e., $\bar{g}_1$), then player 2's feasible and individually rational repeated game payoffs are in a narrow range determined by the linear bounds introduced in the previous paragraph.*

2.5. The Stackelberg type. For an arbitrary stage game Γ that satisfies Assumption 1, we now construct the strategy $\sigma_1(S)$ such that $U_1^c(\sigma_1(S), \delta) = \bar{g}_1$ for all δ that exceed a cutoff δ^*.[22] We term the commitment type S who plays strategy $\sigma_1(S)$ the *Stackelberg type*.

First, some preliminary definitions: If there is an action for player 1, $a_1 \in A_1$, and a best response for player 2 to action a_1, $a_2 \in A_2$, such that $g_1(a_1, a_2) = \bar{g}_1$, then define $a_1^s = a_1$ and $a_2^b = a_2$.[23] Otherwise, define $(a_1^s, a_2^b) \in A_1 \times A_2$ as a particular action profile such that $g_1(a_1^s, a_2^b) = \bar{g}_1$. Assumption 1 implies that there exists an action profile $(a_1^s, a_2^b) \in A_1 \times A_2$ such that $g_1(a_1^s, a_2^b) = \bar{g}_1$.[24]

Description of $\sigma_1(S)$. The strategy $\sigma_1(S)$ has a profit phase and a punishment phase. In the *profit phase* the strategy plays a_1^s, and in the *punishment phase* the strategy plays a_1^p. The strategy begins the game in the profit phase. The strategy remains in the profit phase in period t, if it was in the profit phase in period $t-1$ and if $g_1(y^{t-1}) = \bar{g}_1$. The strategy moves to the punishment phase in period t, if it was in the profit phase in period $t-1$ and if $g_1(y^{t-1}) \neq \bar{g}_1$. If the strategy moves to the punishment phase in period t, then it remains

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[20] For an example in which our construction of $\sigma_1(S)$ does not work because Assumption 1 is violated, see the product choice game depicted in figure 10b, section 4.2.

[21] For an example that does not satisfy this requirement of Assumption 1, see the non-generic common-interest game depicted in figure 10a, section 4.2.

[22] For games that satisfy Assumption 1, there are typically multiple dynamic Stackelberg strategies. We discuss our particular choice of $\sigma_1(S)$ and other possible dynamic Stackelberg strategies in section 4.3.

[23] If there is more than one action profile that satisfies our definition, then we pick (a_1^s, a_2^b) arbitrarily as any one of these action profiles.

[24] If Γ has SCI, then a_2^b is a best response to a_1^s. If Γ has LNCI, then a_2^b is not necessarily a best response to a_1^s. For an example that satisfies Assumption 1 but where a_2^b is not a best response to a_1^s, see figure 5.

in the punishment phase for $n^p - 1$ periods and then moves to the profit phase. Intuitively, $\sigma_1(S)$ punishes player 2 by minimaxing her for the next $n^p - 1$ periods if she does not allow player 1 to obtain a payoff of $\bar{g}_1$. The number of punishment periods $n^p - 1$ is the *smallest integer* such that

$$(1) \qquad g_2(a_1^s, a_2) + (n^p - 1)\hat{g}_2 < n^p g_2(a_1^s, a_2^b)$$

for any $a_2 \in A_2$ such that $g_1(a_1^s, a_2) < g_1(a_1^s, a_2^b) = \bar{g}_1$.

Assumption 1 implies that $n^p \geq 1$ exists. The number of punishment periods is chosen to ensure that it is a best response for a sufficiently patient player 2 to play a_2^b in every period against $\sigma_1(S)$. More precisely, if $\sigma_2 \in BR(\sigma_1(S), \delta)$, then $U_1(\sigma_1(S), \sigma_2, \delta) = \bar{g}_1$, for all δ that exceed a cutoff δ^*. Consequently, $\sigma_1(S)$ is a dynamic Stackelberg strategy for all δ that exceed a cutoff δ^*. For more detail, see Lemma A.1 and Remark A.1 in the appendix.

If $n^p = 1$, then the strategy $\sigma_1(S)$ does not have a punishment phase, that is, S is a simple type who plays the same stage-game action, a_1^s, in each period of the repeated game. Moreover, player 2's best response to $\sigma_1(S)$ entails playing a_2^b in each period for any discount factor.[25] Thus, if $n^p = 1$, then the static Stackelberg payoff coincides with the dynamic Stackelberg payoff for any discount factor (for example, see figure 1). If $n^p > 1$, then the dynamic Stackelberg payoff strictly exceeds the static Stackelberg payoff for a sufficiently high discount factor (see figure 5).[26]

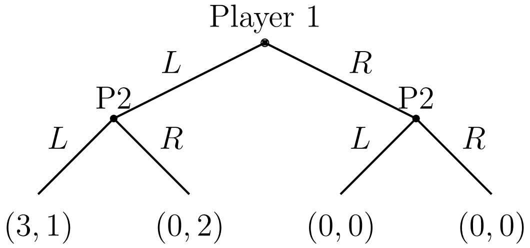

Figure 5: For this stage game $a_1^s = L$, $a_1^p = R$, and we pick a_2^b as the action that always chooses move L. Hence, S plays L in the profit phase, plays R in the two period punishment phase, and $n^p = 3$. Notice that a_2^b is not a best response to a_1^s in this example. However, for sufficiently high δ, player 2's best response to $\sigma_1(S)$ is to play a_2^b in each period of the repeated game. This is because playing a_2^b (instead of playing R after L) avoids the two period punishment phase.

2.6. Equilibrium and beliefs. The analysis in the paper focuses on the perfect Bayesian equilibria (**PBE**) of the game of incomplete information $\Gamma^\infty(\mu, \delta)$.[27] In equilibrium, beliefs

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[25] Notice that if $n^p = 1$ for a stage-game Γ, then by rewriting inequality (1) with $n^p = 1$ we obtain $g_2(a_1^s, a_2) < g_2(a_1^s, a_2^b)$ for any $a_2 \in A_2$ such that $g_1(a_1^s, a_2) < g_1(a_1^s, a_2^b) = \bar{g}_1$, i.e., if $g_1(a_1^s, a_2) < \bar{g}_1$, then a_2 in not a best response to a_1^s.

[26] For a definition of the static Stackelberg payoff, see Mailath and Samuelson (2006, chap. 15).

[27] For a precise statement of PBE, see Fudenberg and Tirole (1991, Definition 8.2).

are obtained, where possible, using Bayes' rule given $\mu(\cdot|h^0) = \mu(\cdot)$ and conditioning on players' equilibrium strategies.

In what follows, *we say that player 1 deviated from $\sigma_1(S)$ in the t^{th} period of a period k public history h* if there exists a decision node d within period $t \leq k$ that is visited in the public history h such that the move of player 1 in public history h at node d differs from the move that strategy $\sigma_1(S)$ would have chosen at node d. Notice that if $\mu(S) > 0$, then the belief $\mu(\cdot|h)$ is well defined after any period k public history h in which player 1 has not deviated from $\sigma_1(S)$.

3. The reputation result

Our main reputation result, Theorem 1, restricts attention to stage games of perfect information that satisfy Assumption 1 and considers a repeated game $\Gamma^\infty(\mu, \delta)$ where $\mu(S) > 0$. Under these assumptions, the theorem provides a lower bound on player 1's payoff in any PBE. Its formal statement is given below.

Theorem 1 *Assume perfect information and Assumption 1. For any $\delta \in [0, 1)$, any $\mu \in \Delta(\Omega)$ such that $\mu(S) > 0$, and any PBE strategy profile σ of $\Gamma^\infty(\mu, \delta)$, we have*

$$U_1(\sigma, \delta) \geq \bar{g}_1 - f(\underline{z}) \max\{1 - \delta, \phi\},$$

where $\underline{z} = \mu(S)$, $\phi = \mu(\Omega_-)/\mu(S)$, and f is the decreasing, positive valued function defined in equation (10) in the appendix.

Proof: The proof is in the appendix. □

The theorem implies that as δ goes to one and $\mu(\Omega_-)$ (the probability of other commitment types) goes to zero, player 1's payoff converges to $\bar{g}_1$, his highest payoff. Consequently, a normal type for player 1 can secure a payoff arbitrarily close to $\bar{g}_1$, his dynamic Stackelberg payoff, in any PBE of the repeated game, for a sufficiently high discount factor and for sufficiently low probability mass on other commitment types. Player 1 can attain the bound given in the theorem by simply mimicking the Stackelberg type. Notice that the bound given in the theorem is not particularly sharp, if the probability of other commitment types, $\mu(\Omega_-)$, is substantial. However, under certain assumptions, player 1 can receive a payoff arbitrarily close to $\bar{g}_1$, with no restrictions on the probability of other commitment types. We discuss such issues that relate to other commitment types in section 4.4.

In order to demonstrate the implications of Theorem 1 and to make the intuition more transparently, we restate our reputation result for the example depicted in figure 6 as Corollary 1; a detailed argument for Corollary 1 appears in section 3.2.2. In this example, the

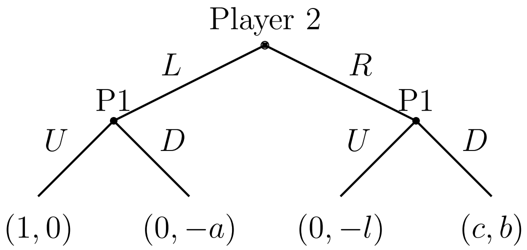

Figure 6: A game of perfect information with LNCI. Assume that $l \in (0,1]$, $a \in (0,1]$, $b \in [-1,1/2]$ and $c \in [0,1/2]$. If $l = a = 1$ and $b = c = 1/2$, then this is a battle-of-the-sexes game. If $l = 1$, $a = 3/4$, $b = -1$ and $c = 0$, then this is a common interest game.

Stackelberg type S plays U at each decision node of player 1, and player 1's highest stage game payoff is equal to one. Our reputation result, for this particular example, is as follows:

Corollary 1 *Suppose that the stage game* Γ *is given by figure 6 and assume that* $\mu(\Omega_-) = 0$. *For any reputation level* $\mu(S) = \underline{z} > 0$, *we have* $\lim_{\delta \to 1} U_1(\sigma(\delta), \delta) = 1$, *where* $\sigma(\delta)$ *is a PBE strategy profile for the repeated game* $\Gamma^\infty(\mu, \delta)$.

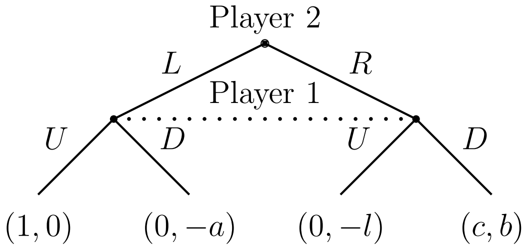

Figure 7: Assume that $l \in (0,1]$, $a \in (0,1]$, $b \in [-1,1/2]$, and $c \in [0,1/2]$. This simultaneous-move version of figure 6 is a game with LNCI.

Remark 1 *If the stage game* Γ *is given by figure 7 instead of figure 6, then the reputation result stated in Corollary 1 fails. We discuss this point further in Remark 2 and section 4.1.*

3.1. The intuition for the reputation result. We now use figure 6 to convey the main intuition driving our reputation result. Our result shows that a sufficiently patient player 1 can receive a payoff approximately equal to one in any PBE by mimicking type S, i.e., by playing U in each period of the repeated game. Equivalently, player 2 plays R, in only a payoff-insignificant number of periods against an opponent who repeatedly plays U.

There are two main incentives that may induce player 2 to play R after observing U in all previous periods. The first is a myopic incentive: she may expect player 1 to play D with high probability in that period. The second is a nonmyopic incentive: she may expect her continuation payoff after R to be sufficiently more attractive than her continuation payoff after L. We show that neither myopic nor nonmyopic incentives are sufficiently strong to induce player 2 to play R against type S for a payoff-significant number of periods. Myopic incentives are insufficient, as in Fudenberg and Levine (1989, 1992), since if player 1 is expected to reveal rationality with high probability, then he can instead mimic type S, thereby increasing his reputation significantly and obtaining a payoff close to one in the continuation game.

Nonmyopic incentives: For player 2 to play R in a period where player 1 plays D with small probability, she must expect a punishment for playing L (or a reward for playing R) in the continuation game. Type S always plays U; hence, any punishment (or reward) for player 2 must occur after player 1 reveals rationality by playing D. Because player 1 moves after observing player 2's move (perfect information), he can continue to mimic type S instead of punishing player 2 after observing L (or rewarding her after R). Hence, his payoff while punishing (or rewarding) player 2 cannot differ significantly from his payoff from mimicking type S. In other words, punishing (or rewarding) player 2 cannot be costly for player 1. For the class of games that we consider, the commitment payoff of type S is equal to the highest payoff of player 1. Moreover, for this class of games, if player 1's payoff is close to his highest payoff, then player 2's payoffs are in a narrow range (also see figure 4a). Therefore, if punishments (or rewards) are not costly for player 1, then player 2's feasible continuation payoffs lie in a narrow range. Thus, the scope for nonmyopic incentives is also limited.

3.2. The argument for the reputation result.

3.2.1. *Preliminaries.* Recall that $(a_1^s, a_2^b) \in A_1 \times A_2$ is an action profile such that $g_1(a_1^s, a_2^b)$ is equal to player 1's highest stage game payoff compatible with individual rationality. For figure 6, the stage game action a_1^s plays U after either L or R; and a_2^b is the best response to a_1^s, that is, $a_2^b = L$. Also, for this game $n^p = 1$, i.e., the static and dynamic Stackelberg payoffs coincide and are equal to one, for any discount factor.

If Γ satisfies Assumption 1 (i), then there exists a finite constant $\rho \geq 0$ such that

$$(2) \qquad \left|g_2 - g_2(a_1^s, a_2^b)\right| \leq \rho \left|\bar{g}_1 - g_1\right|, \text{ for any } (g_1, g_2) \in F.$$

For example in figure 6, any feasible payoff profile (g_1, g_2) satisfies inequality (2) for $\rho = 1$. Also, see figure 4a for a depiction of inequality (2). The set of feasible payoffs in the repeated game is equal to the set of feasible stage game payoffs. Therefore, if Γ satisfies Assumption 1 (i), then inequality (2) implies that

$$\left|U_2(\sigma_1, \sigma_2, \delta) - g_2(a_1^s, a_2^b)\right| \leq \rho \left|\bar{g}_1 - U_1(\sigma_1, \sigma_2, \delta)\right|, \text{ for any pair } (\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2.$$

If Γ satisfies Assumption 1 (*ii*), then there exists a finite constant $\rho \geq 0$ such that

$$g_2 - g_2(a_1^s, a_2^b) \leq \rho(\bar{g}_1 - g_1), \text{ for any } (g_1, g_2) \in F. \tag{3}$$

Also, see figure 4b for a depiction of inequality (3). If Γ satisfies Assumption 1 (*ii*), then inequality (3) implies that

$$U_2(\sigma_1, \sigma_2, \delta) - g_2(a_1^s, a_2^b) \leq \rho(\bar{g}_1 - U_1(\sigma_1, \sigma_2, \delta)), \text{ for any pair } (\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2.$$

We now introduce the **resistance function**, $R(\mu, \delta)$, which is central to the analysis that follows. As a preliminary, we define the resistance of the strategy σ_2 for player 2 as follows:

$$r(\sigma_2, \delta) = \bar{g}_1 - U_1(\sigma_1(S), \sigma_2, \delta).$$

For the example in figure 6, the resistance of strategy σ_2 is equal to the expected discounted number of periods in which (R, U) is played under the strategy profile $(\sigma_1(S), \sigma_2)$. That is, the resistance of strategy σ_2 is equal to the expected number of times a nonbest response is played by strategy σ_2 against the Stackleberg type. Notice that if player 2 uses strategy σ_2 and her opponent uses strategy $\sigma_1(S)$, then player 2's payoff, $U_2(\sigma_1(S), \sigma_2, \delta)$, is equal to $-lr(\sigma_2, \delta)$. This is because either (R, U) or (L, U) is played in each period; and $g_2(R, U) = -l$ and $g_2(L, U) = 0$.

The resistance function, $R(\mu, \delta)$, provides an upper bound on how much player 2 can resist (or hurt) type S in *any PBE* of $\Gamma^\infty(\mu, \delta)$. It is defined as follows:

Definition 1 (Resistance function) *For any measure $\mu \in \Delta(\Omega)$ and $\delta \in [0, 1)$ let*

$$R(\mu, \delta) = \sup\{r(\sigma_2, \delta) : \sigma_2 \textit{ is part of a PBE profile } \sigma \textit{ of } \Gamma^\infty(\mu, \delta)\}.$$

3.2.2. *The argument for Corollary 1.* In this subsection we prove the reputation result given in Corollary 1. At the end of the section we discuss the main argument for Theorem 1 that is given in the Appendix. In what follows, because $\mu(\Omega_-) = 0$, we use $z \in [0, 1]$ to represent the measure μ. One should understand this to mean $\mu(S) = z$ and $\mu(N) = 1 - z$.

In this section we work under the hypothesis that the resistance function $R(z, \delta)$ *is a nonincreasing function of z for each $\delta \in [0, 1)$*. We do this for expositional convenience only, as it allows us to convey the main argument without the more technical details.[28]

At the start of any period t, if player 1's reputation level is at least $z > 0$, then player

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[28] In the Appendix we instead work with the maximal resistance function $\bar{R}(z, \delta) = \sup\{R(z', \delta) : z' \geq z\}$, which is nonincreasing by definition.

1 can guarantee a continuation payoff of at least $1 - R(z, \delta)$ by playing according to the Stackelberg strategy $\sigma_1(S)$. This follows from the definition of R, sequential rationality, and our assumption that R is non-increasing. We will argue that $\lim_{\delta \to 1} R(\underline{z}, \delta) = 0$, for any $\underline{z} > 0$.

Consider a PBE σ of the repeated game $\Gamma^\infty(z, \delta)$. Suppose that the players are at a history in which player 1 has played U in each period before t, and player 2 has played $a_2 \in \{L, R\}$ in period t. Suppose further that player 1 plays D with positive probability at this decision node, i.e., player 1 reveals that he is not the Stackelberg type, with positive probability. Also, let player 1's reputation level be $z' > 0$ at the start of period $t + 1$, if he plays U instead of D. In the next lemma we bound the continuation payoff of player 2 by a linear function of $R(z', \delta)$ at any such decision node. The argument for the lemma is as follows: If player 1 is playing D with positive probability, then the payoff from playing D must be at least as large as the payoff from playing U. However, if player 1 plays U, he gets at worst zero for the period, ensures that his reputation is z' at the start of the subsequent period, and thus guarantees $1 - R(z', \delta)$ at the start of period $t + 1$. Given this lower bound on player 1's continuation payoff, the linear bound on player 2's continuation payoff follows from inequality (2).

Lemma 1 *Suppose $z > 0$. Pick any PBE σ of $\Gamma^\infty(z, \delta)$ and any period t public history of terminal nodes h^t where player 1 has played U in each period; and suppose player 1 plays D in period t given history (h^t, a_2) with positive probability, where $a_2 \in \{L, R\}$. Let $z' = \mu(S|h^t, a_2, U)$; then we have*

$$|U_2(\sigma_1(N), \sigma_2, \delta|h^t, a_2, D)| \leq R(z', \delta) + (1 - \delta)/\delta.$$

Proof: If player 1 plays U in period t, then his reputation level is $z' = \mu(S|h^t, a_2, U)$ and he can guarantee a continuation payoff equal to $1 - R(z', \delta)$, by using $\sigma_1(S)$. Also, player 1 can get at worst zero in period t by playing U. Consequently, his payoff from playing U is at least $\delta U_1(\sigma, \delta|h^t, a_2, U) \geq \delta(1 - R(z', \delta))$. If player 1 plays D instead, then he can get at most c for the current period and $\delta U_1(\sigma, \delta|h^t, a_2, D)$ as his continuation payoff. Because player 1 is willing to play D instead of U, we have $(1 - \delta)c + \delta U_1(\sigma, \delta|h^t, a_2, D) \geq \delta U_1(\sigma, \delta|h^t, a_2, U)$. Hence, $U_1(\sigma, \delta|h^t, a_2, D) \geq 1 - R(z', \delta) - (1 - \delta)c/\delta \geq 1 - R(z', \delta) - (1 - \delta)/\delta$. The bound on player 2's payoff follows from inequality (2) because the payoff profile $(U_1(\sigma, \delta|h^t, a_2, D), U_2(\sigma_1(N), \sigma_2, \delta|h^t, a_2, D))$ is an element of F and because the constant ρ is at most one for this particular game. Also see figure 8. $\square$

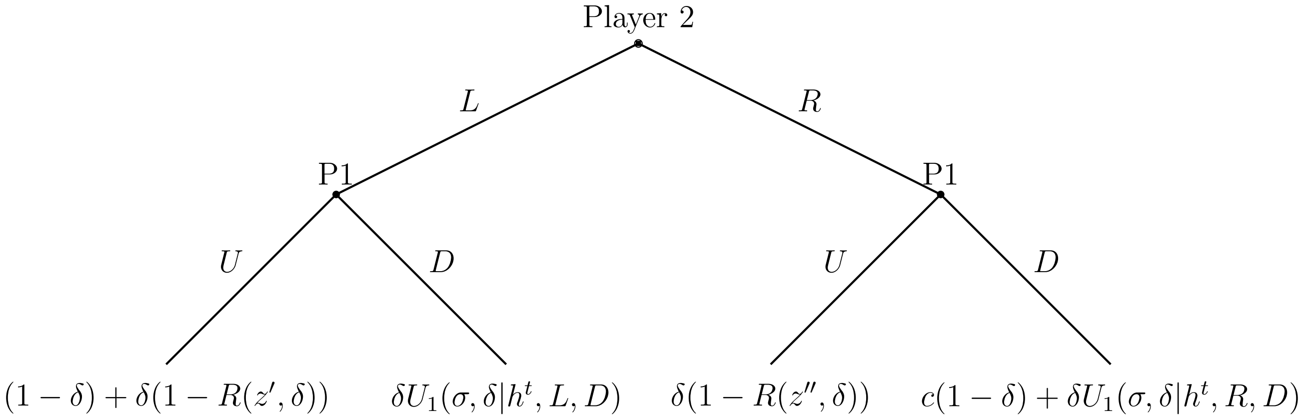

Figure 8: This figure depicts the payoff player 1 can guarantee by playing U and his payoff if he plays D instead. In the figure, $z' = \mu(S|h^t, L, U)$ and $z'' = \mu(S|h^t, R, U)$. If player 1 is to play D with positive probability after R, then $(1-\delta)c + \delta U_1(\sigma, \delta|h^t, R, D) \geq \delta(1 - R(z'', \delta))$. Consequently, $U_1(\sigma, \delta|h^t, R, D) \geq 1 - R(z'', \delta) - (1-\delta)c/\delta$, inequality (2), and $\rho \leq 1$ together imply that $|U_2(\sigma, \delta|h^t, R, D)| \leq R(z'', \delta) + (1-\delta)c/\delta$. Similarly, if player 1 is to play D after L, then $|U_2(\sigma, \delta|h^t, L, D)| \leq R(z', \delta) - (1-\delta)/\delta$.

REMARK 2 *Lemma 1 puts a bound on* $U_2(\sigma_1(N), \sigma_2, \delta|h^t, a_2, D)$*, not only for player 2's equilibrium choice of* a_2*, but for any* $a_2 \in \{L, R\}$*. Instead of the game of perfect information in figure 6, suppose that the stage game* Γ *is given by the simultaneous-move game in figure 7. Further suppose, in a given PBE* σ *after history* h^t*, player 2 plays* R *with probability one and player 1 plays* D *with positive probability. Then, as in Lemma 1, player 1's ex-ante incentive constraint implies that* $U_1(\sigma, \delta|h^t, R, D) \geq 1 - R(z', \delta) - (1-\delta)/\delta$*. However, in contrast to Lemma 1, it is no longer possible to assert that player 1's ex-ante incentives require* $U_1(\sigma, \delta|h^t, L, D) \geq 1 - R(z', \delta) - (1-\delta)/\delta$*; and as a consequence, it is not possible to assert that* $|U_2(\sigma_1(N), \sigma_2, \delta|h^t, L, D)| \leq \rho(R(z', \delta) + (1-\delta)/\delta)$*. This is because player 1 chooses to play* D *before seeing player 2's move (i.e., perfect information is violated) and expects player 2 to play* L *with probability zero when making his choice; therefore player 1's continuation payoff after* (h^t, L, D) *does not affect his ex-ante incentives. We discuss this point further in section 4.1.*

We now use Lemma 1 to *sketch the argument for Corollary 1*. Suppose that player 1's reputation level is z. Consider a PBE $\sigma = (\sigma_1(N), \sigma_1(S), \sigma_2)$ where player 2 resists the Stackelberg type by approximately $R(z, \delta)$. In this PBE player 2 loses approximately $lR(z, \delta)$ in the event that player 1 is the Stackelberg type. We compare player 2's payoff in this PBE with her payoff if she uses an *alternative strategy* that plays L until player 1 plays D for the first time, and then reverts back to the equilibrium strategy σ_2. If player 2 uses the alternative strategy, then she avoids losing $lR(z, \delta)$ in the event that player 1 is the Stackelberg type.

We then use the fact that the PBE strategy σ_2 must give player 2 a payoff that is at least as great as the payoff from using the alternative strategy. This establishes a bound on $R(z, \delta)$, for any z sufficiently close to 1.

Upper bound on player 2's equilibrium payoff. Suppose that player 1 plays D for the first time in some period t. In each period up to period t, player 2 receives at most zero; in period t, she receives at most $1-\delta$; and after period t, she receives at most $R(z, \delta) + (1-\delta)/\delta$ as a continuation payoff, by Lemma 1 and because R is nonincreasing. Consequently, player 2 gets at most $\delta^t(1-\delta) + \delta^{t+1}(R(z, \delta) + (1-\delta)/\delta) \leq R(z, \delta) + 2(1-\delta)$, if player 1 plays D for the first time in period t. Alternatively, if player 1 plays U in each period, then player 2 receives at most $-lR(z, \delta)$. Player 1 will play U in every period with probability at least z because type S always plays U. So, player 1 will play D in some period t, with probability at most $1-z$. Thus, player 2's payoff in the PBE σ is at most $(1-z)(R(z, \delta) + 2(1-\delta)) - zlR(z, \delta)$. This line of reasoning is formalized by the upper bound that we establish in Lemma 2 further below.

Lower bound on player 2's equilibrium payoff. Suppose that player 2 uses the alternative strategy and player 1 plays D for the first time in some period k. Player 2 receives at least $-R(z, \delta) - (1-\delta)/\delta$ as a continuation payoff after period k, by Lemma 1 and because R is nonincreasing. Also, she receives zero in each period up to period k, because she plays L and player 1 plays U. In period k she receives $-a(1-\delta) \geq -(1-\delta)$, because she plays L and player 1 plays D, and because $a \in (0, 1]$. Alternatively, if player 1 plays U in every period, then player 2 receives zero. Player 1 will play D in some period k, with probability at most $1-z$. Consequently, if player 2 uses the alternative strategy, then her payoff is at least $-(1-z)(\delta^k(1-\delta) + \delta^{k+1}(R(z, \delta) + (1-\delta)/\delta)) \geq -(1-z)(R(z, \delta) + 2(1-\delta))$. This line of reasoning is formalized by the lower bound that we establish in Lemma 3 further below.

Bounding resistance. The payoff that player 2 gets from the equilibrium strategy σ_2 must be at least as great as the payoff she receives from the alternative strategy. So, $-(1-z)(R(z, \delta) + 2(1-\delta)) \leq (1-z)(R(z, \delta) + 2(1-\delta)) - zlR(z, \delta)$. Rearranging, $R(z, \delta) \leq 4(1-z)(1-\delta)/(lz - 2(1-z)) \leq 4(1-\delta)/(lz - 2(1-z))$. Thus, for z sufficiently close to one, i.e., if $1-z \leq \underline{q} \equiv l\underline{z}/4$, then $R(z, \delta) \leq C(1-\delta)$ where $C = 16/\underline{z}l$.[29] Therefore, the resistance at reputation level z is very close to zero, if δ is close to one.

More generally, the argument for Corollary 1 shows that for any two reputation levels $z'' > z' \geq \underline{z}$ such that $z'/z'' \geq 1 - \underline{q}$, the resistance function satisfies the following *functional inequality*: $R(z', \delta) \leq CR(z'', \delta) + C(1-\delta)$. That is, decreasing player 1's reputation level by a factor of $1 - \underline{q}$ increases resistance by at most a factor of C plus an additive term

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[29] To be precise, if $z \geq \underline{z}$ and $1 - z \leq l\underline{z}/4$, then $lz - 2(1-z) \geq l\underline{z}/2$. Hence, $R(z, \delta) \leq 4(1-\delta)/l\underline{z}/2 = 8(1-\delta)/l\underline{z} \leq 16(1-\delta)/l\underline{z}$.

equal to $C(1-\delta)$. Observe that we can recover the inequality that we established in the previous paragraph by substituting $z' = 1-\underline{q}$ and $z'' = 1$ into the functional inequality and by using the fact that $R(1,\delta) = 0$. figure 9 depicts the upper bound on resistance implied by this functional inequality. Notice that for any reputation level $z \geq \underline{z}$, the upper bound on resistance depicted in figure 9, converges to zero, as δ converges to one.

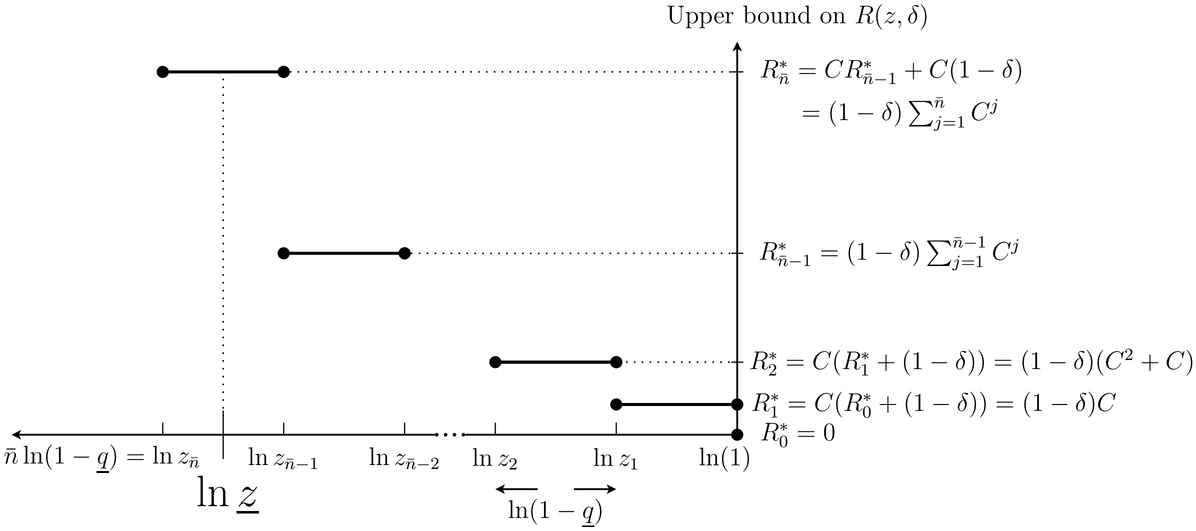

Figure 9: Resistance's upper bound as a function of $\ln(z)$. This upper bound is implied by the functional inequality $R(z',\delta) \leq CR(z'',\delta) + C(1-\delta)$ that holds for any $z'' > z' \geq \underline{z}$ such that $z'/z'' \geq 1-\underline{q}$. The reputation levels z_n, that are shown on the x-axis, are such that $\ln z_n - \ln z_{n+1} = \ln(1-\underline{q})$ for each n.

We now proceed with the formal proof of Corollary 1 by establishing an upper and a lower bound (Lemmata 2 and 3) for player 2's PBE payoffs. Definition 2 below introduces a stopping time which we use in constructing the upper and the lower bound.

DEFINITION 2 (Stopping time) *For an integer k, let $E_{[0,k]}$ denote the event (set of infinite public histories) where player 1 plays D for the first time in period t for some $t \in \{0,...,k\}$. For any strategy profile $\sigma = (\sigma_1, \sigma_1(S), \sigma_2)$, and for reputation levels $z > 0$ and $z' > z$, let*

$$T(\sigma, z, z') = \min\{k \in \{0,1,2,...\} : z/(1-\pi(k)) \geq z'\},$$

where $\pi(k) = (1-z)\Pr_{(\sigma_1,\sigma_2)}[E_{[0,k]}]$; and let $T(\sigma,z,z') = \infty$ if the set is empty. Notice that $\Pr_{(\sigma_1,\sigma_2)}[E_{[0,k]}]$ is the probability that player 1 plays D for the first time in period t for some $t \in \{0,...,k\}$, if player 1 is using strategy σ_1 and player 2 is using strategy σ_2.

Intuitively, the stopping time $T(\sigma,z,z')$ gives the first period in which player 1's reputation level exceeds z', if his initial reputation level is z and if the players use strategy profile σ.

The specific implications of Definition 2, that we use in Lemmata 2 and 3, are summarized in the following remark:

Remark 3 *Suppose that player 1's initial reputation level is z and suppose that $z' > z$. Let $\sigma^* = (\sigma_1(N), \sigma_1(S), \sigma_2^*)$ be any strategy profile where σ_2^* is a pure strategy. Let $T = T(\sigma^*, z, z')$. By definition, the total probability that player 1 plays D for the first time in any period $t \leq T-1$ (i.e., $\pi(T-1)$ in the terminology used in Definition 2) is at most $1 - z/z'$. Also, because both $\sigma_1(S)$ and σ_2^* are pure strategies, we have the following:*

(i) There is a unique public history of terminal nodes h^{T+1} consistent with strategies $\sigma_1(S)$ and σ_2^*.

(ii) If h^{T+1} is the unique history consistent with $\sigma_1(S)$ and σ_2^* (i.e., player 1 has always played U in all periods up to and including period T), then Bayes' rule implies that $\mu(S|h^{T+1}) \geq z'$.

Lemma 2 (Upper bound) *Suppose $0 \leq z < z' \leq 1$. Let $\sigma = (\sigma_1(N), \sigma_1(S), \sigma_2)$ denote a PBE of $\Gamma^\infty(z, \delta)$ where player 2's resistance is at least $R(z, \delta) - \epsilon$ and $\epsilon > 0$. Then,*

$$(4) \qquad U_2(\sigma, \delta) \leq q(R(z, \delta) + 2(1-\delta)) + R(z', \delta) + 2(1-\delta) - zl(R(z, \delta) - \epsilon)$$

where $q = 1 - z/z'$.

Proof: Let σ_2^* denote a pure strategy in the support of σ_2 such that the resistance of σ_2^* is at least $R(z, \delta) - \epsilon$. Since the resistance of σ_2 is at least $R(z, \delta) - \epsilon$, there must be a pure strategy in the support of σ_2 that has resistance of at least $R(z, \delta) - \epsilon$. Let profile $\sigma^* = (\sigma_1(N), \sigma_1(S), \sigma_2^*)$ and let $T = T(\sigma^*, z, z')$. As we argued in Remark 3, player 1's reputation exceeds z' at the end of period T if player 1 plays U and if player 2 plays according to σ_2^*, in all periods up to and including T.

We bound player 2's payoffs from σ_2^* in the following three events: (i) The event where player 1 plays D for the first time in some period $t < T$; the probability of this event is at most $q = 1 - z/z'$ by Remark 3. (ii) The event where player 1 plays D for the first time in some period $t \geq T$; the probability of this event is at most 1. (iii) The event where player 1 never plays D; the probability of this event is at least z, because S never plays D. These three events are exhaustive.

In a period where player 1 plays U, player 2 receives at most zero. Consequently, player 2's total payoff in all the periods until player 1 plays D for the first time is at most zero.

If event (i) occurs and player 1 plays D for the first time in some period t, then player 2 receives zero until period t, receives at most $(1-\delta)$ in period t,[30] and knows with certainty that she faces the normal type N. Hence, she receives a continuation payoff of at most $R(z,\delta)+(1-\delta)/\delta$ by Lemma 1 and because R is nonincreasing. So if event (i) occurs, then player 2's payoff is at most $R(z,\delta)+2(1-\delta)$ because

$$R(z,\delta)+2(1-\delta) \geq \delta^t(1-\delta)+\delta^{t+1}(R(z,\delta)+(1-\delta)/\delta) \text{ for any period } t.$$

If event (ii) occurs and player 1 plays D for the first time in some period t, then player 2 receives zero until period t, receives at most $(1-\delta)$ in period t, and receives a continuation payoff of at most $R(z',\delta)+(1-\delta)/\delta$, by Lemma 1 and because R is nonincreasing. So if event (ii) occurs, then player 2's payoff is at most $R(z',\delta)+2(1-\delta)$ because

$$R(z',\delta)+2(1-\delta) \geq \delta^t(1-\delta)+\delta^{t+1}(R(z',\delta)+(1-\delta)/\delta) \text{ for any period } t.$$

If event (iii) occurs, then player 1 plays U in each period; player 2's payoff in this event is at most $-l(R(z,\delta)-\epsilon)$. This is because the resistance of σ_2^* is at least $R(z,\delta)-\epsilon$.

Putting the bounds on player 2's payoffs in the three events together, we obtain:

$$U_2(\sigma^*,\delta) \leq q(R(z,\delta)+2(1-\delta))+R(z',\delta)+2(1-\delta)-zl(R(z,\delta)-\epsilon).$$

Recall that σ_2^* is in the support of PBE strategy σ_2. Consequently, we have $U_2(\sigma^*,\delta)=U_2(\sigma,\delta)$ which implies the following:

$$U_2(\sigma,\delta) \leq q(R(z,\delta)+2(1-\delta))+R(z',\delta)+2(1-\delta)-zl(R(z,\delta)-\epsilon).$$

□

LEMMA 3 (Lower bound) *Suppose* $0 \leq z < z' \leq 1$. *In any PBE* σ *of* $\Gamma^\infty(z,\delta)$, *we have*

$$U_2(\sigma,\delta) \geq -q(R(z,\delta)+2(1-\delta))-R(z',\delta)-2(1-\delta), \tag{5}$$

where $q = 1 - z/z'$.

PROOF: Pick any PBE σ of $\Gamma^\infty(z,\delta)$. Let σ_2^* denote a pure strategy that moves according to a_2^b after any period k public history h^k that is consistent with $\sigma_1(S)$; and that coincides with a pure strategy in the support of the PBE strategy σ_2 if h^k is not consistent with $\sigma_1(S)$. Let

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[30] Player 2's highest stage game payoff is one in this game.

strategy profile $\sigma^* = (\sigma_1(N), \sigma_1(S), \sigma_2^*)$, and let $T = T(\sigma^*, z, z')$. As we argued in Remark 3, player 1's reputation exceeds z' at the end of period T if player 1 plays U and if player 2 plays according to σ_2^*, in all periods up to and including T.

Once again we bound player 2's payoff from strategy σ_2^* in the following three events: (i) The event where player 1 plays D for the first time in some period $t < T$; the probability of this event is at most q, by Remark 3. (ii) The event where player 1 plays D for the first time in some period $t \geq T$; the probability of this event is at most 1. (iii) The event that player 1 never plays D.

Player 2's payoff until player 1 plays D for the first time is at most zero. If event (i) occurs and player 1 plays D for the first time in some period t, then player 2 receives zero until period t, receives at worst $-a(1-\delta) \geq -(1-\delta)$ in period t, and receives a continuation payoff of at worst $-R(z,\delta) - (1-\delta)/\delta$, by Lemma 1 and because R is nonincreasing. Consequently, player 2's payoff is at least $-R(z,\delta) - 2(1-\delta)$ because

$$-\delta^t(1-\delta) - \delta^{t+1}(R(z,\delta) + (1-\delta)/\delta) \geq -R(z,\delta) - 2(1-\delta).$$

If event (ii) occurs and player 1 plays D for the first time in some period t, then player 2 receives zero until period t, receives at worst $-(1-\delta)$ in period t, and receives a continuation payoff of at worst $-R(z',\delta) - (1-\delta)/\delta$, by Lemma 1 and because R is non-increasing. Consequently, player 2's payoff is at least $-R(z',\delta) - 2(1-\delta)$ because

$$-\delta^t(1-\delta) - \delta^{t+1}(R(z',\delta) + (1-\delta)/\delta) \geq -R(z',\delta) - 2(1-\delta).$$

If event (iii) occurs, then player 1 never plays D and consequently player 2 receives zero. Putting the bounds on player 2's payoffs in the three events together implies that:

$$U_2(\sigma^*,\delta) \geq -q(R(z,\delta) + 2(1-\delta)) - R(z',\delta) - 2(1-\delta).$$

Because σ_2 is player 2's PBE strategy we have $U_2(\sigma,\delta) \geq U_2(\sigma^*,\delta)$. Consequently,

$$U_2(\sigma,\delta) \geq -q(R(z,\delta) + 2(1-\delta)) - R(z',\delta) - 2(1-\delta).$$

□

Below we use the fact that the upper bound provided in Lemma 2 must exceed the lower bound given in Lemma 3 to obtain a functional inequality that relates maximal resistance at any two reputation levels. We then use this functional inequality to complete our proof.

LEMMA 4 (Functional Inequality) *For any* $z \in [\underline{z}, 1]$ *and* $z < z' \leq 1$*, we have*

$$(6) \qquad R(z,\delta)(\underline{z}l - 2q) \leq 2R(z',\delta) + 8(1-\delta),$$

where $q = 1 - z/z'$.

PROOF: For any $\epsilon > 0$, there exists a PBE σ of $\Gamma^{\infty}(z,\delta)$ where player 2's resistance is at least $R(z,\delta) - \epsilon$, by the definition of the resistance function. By Lemma 2, inequality (4) holds for any $\epsilon > 0$ and any PBE σ where player 2's resistance is at least $R(z,\delta) - \epsilon$. Also, the upper bound given by inequality (4) must exceed the lower bound given by inequality (5) for any PBE σ. Combining (4) and (5), taking the limit as ϵ goes to zero, and substituting $\underline{z}$ for z together imply that $R(z,\delta)(\underline{z}l - 2q) \leq 4R(z',\delta) + 4(1+q)(1-\delta)$. Using $q \leq 1$ then delivers inequality (6). □

PROOF OF COROLLARY 1 UNDER THE HYPOTHESIS THAT R IS NONINCREASING: Let $\underline{q} = \underline{z}l/4$ and let $\bar{n}$ be the smallest integer such that $(1-\underline{q})^{\bar{n}} \leq \underline{z}$. We will show that $R(\underline{z},\delta) \leq (1-\delta)\sum_{j=1}^{\bar{n}} C^j$ where $C = 16/\underline{z}l$ and hence $0 \leq \lim_{\delta\to 1} R(\underline{z},\delta) \leq \lim_{\delta\to 1}(1-\delta)\sum_{j=1}^{\bar{n}} C^j = 0$.

If $z, z' \in [\underline{z}, 1]$ and $z \in [z'(1-\underline{q}), z']$, then $1 - z/z' \leq \underline{q}$. Hence, substituting $\underline{q}$ for $q = 1 - z/z'$ in inequality (6) delivers the following:

$$R(z,\delta)(\underline{z}l - 2\underline{q}) \leq 2R(z',\delta) + 8(1-\delta).$$

Substituting $\underline{z}l/4$ for $\underline{q}$ in the previous inequality and rearranging we obtain the following:

$$R(z,\delta) \leq \frac{4}{\underline{z}l}(R(z',\delta) + 4(1-\delta)).$$

Substituting C for $16/\underline{z}l$ in the previous inequality and using the fact that $R(z',\delta) \geq 0$ we obtain the following:

$$(7) \qquad R(z,\delta) \leq CR(z',\delta) + C(1-\delta).$$

If $z \geq \underline{z}$ and $z \in [1-\underline{q}, 1]$, then substituting $z' = 1$ into inequality (7) we obtain the following:

$$R(z,\delta) \leq CR(1,\delta) + C(1-\delta),$$

for $z, z' \in [\underline{z}, 1]$ such that $z \in [z'(1-\underline{q}), z']$. Notice that $R(1,\delta) = 0$. Consequently, if $z \geq \underline{z}$ and $z \in [1-\underline{q}, 1]$, then $R(z,\delta) \leq C(1-\delta)$.

More generally, we will show that if $z \geq \underline{z}$ and if $z \in [(1-\underline{q})^n, (1-\underline{q})^{n-1}]$, then $R(z,\delta) \leq (1-\delta)\sum_{j=1}^{n} C^j$ by using induction on n. We make the inductive hypothesis that if $z \geq \underline{z}$

and if $z \in [(1-\underline{q})^{k-1}, (1-\underline{q})^{k-2}]$, then we have $R(z,\delta) \leq (1-\delta)\sum_{j=1}^{k-1} C^j$.

If $z \geq \underline{z}$ and $z \in [(1-\underline{q})^{k}, (1-\underline{q})^{k-1}]$, then substituting $(1-\underline{q})^{k-1}$ for z' in inequality (7) gives us the following:

$$(8) \qquad R(z,\delta) \leq CR((1-\underline{q})^{k-1},\delta) + C(1-\delta).$$

However, inequality (8) and the inductive hypothesis together show that if $z \geq \underline{z}$ and $z \in [(1-\underline{q})^{k}, (1-\underline{q})^{k-1}]$, then $R(z,\delta) \leq (1-\delta)\sum_{j=1}^{k} C^j$, completing the induction.

The definition of $\bar{n}$ implies that $\underline{z} \in [(1-\underline{q})^{\bar{n}}, (1-\underline{q})^{\bar{n}-1}]$, and consequently, $R(\underline{z},\delta) \leq (1-\delta)\sum_{j=1}^{\bar{n}} C^j$. See figure 9 for a depiction of this argument. □

3.2.3. *Description of the proof of Theorem 1.* Our discussion up to this point established a reputation result for the game depicted in figure 6. Here we describe the additional arguments we use to prove Theorem 1. In particular, we sketch the steps involved in allowing for the Stackelberg type that uses punishments (i.e., $n^p > 1$) and allowing for other commitment types (i.e., $\mu(\Omega_-) > 0$).

In order to accommodate the Stackelberg type who punishes player 2, i.e., the case where $n^p > 1$, we prove Lemmata A.1 and A.2. Lemma A.1 shows that player 2 faces an average per-period cost, $l > 0$, for not best responding to the Stackelberg type. Lemma A.2 is an analog of Lemma 1 that accounts for punishment phases. This lemma is needed because at any node where player 1 deviates from $\sigma_1(S)$ under equilibrium play, if he instead plays according to $\sigma_1(S)$ in order to maintain his reputation, then he may have to carry-out an $n_p - 1$ period punishment phase.

Allowing for $\mu(\Omega_-) > 0$ requires accounting for the event where player 2 faces another commitment type in the lower and upper bound calculations. In particular, we show that the effect of the other commitment types is at most $\pm M\mu(\Omega_-)$ on the lower bound and the upper bound. This is because player 1 is another commitment type with probability $\mu(\Omega_-)$, and because player 2 can gain or lose at most M against any type. Consequently, if $\mu(\Omega_-)$ is small, then the effect of other commitment types on the functional inequality is also small.

4. DISCUSSION

4.1. Without perfect information, the reputation result can fail. For example, without perfect information a folk theorem applies to the simultaneous-move common interest game in figure 3a (Cripps and Thomas, 1997), which has LNCI. For a reputation result in repeated games with SCI, the perfect information assumption is not required (Cripps et al. (2005) or section 4.6 in this paper).

Corollary 1 provides a reputation result for the repeated sequential common interest game.[31] Lemma 1 is central for establishing Corollary 1 and the perfect-information assumption is required for Lemma 1. In order to flesh out the intuition of why perfect information is necessary, we construct a PBE for the repeated simultaneous-move common interest game given in figures 3a (where we take $x = 0$), where there is no analog of Lemma 1. In this PBE, the players' payoffs are low if z is close to zero and δ is close to one.[32] That is, the failure of Lemma 1 also leads to the failure of the reputation result.

Suppose player 2 plays R and player 1 uses a mixed strategy that plays D with small probability for the first K periods. After the first K periods, (L, U) is played forever. In this construction $U_1(\sigma) = U_2(\sigma) = \delta^K$. Also, the continuation payoff for the players, after (R, D) or (R, U), is equal to δ^{K-t} in any period $t \in \{0, .., K-1\}$. To ensure that player 2 has an incentive to play R, she is punished in the event that she plays L and player 1 plays D (thus revealing rationality). Punishment entails a continuation payoff for player 2 that is close to zero.[33] Player 1 is willing to mix between U and D in the first K periods since player 2 only plays R on the equilibrium path.

In this construction, player 2 is deterred from playing L, even if player 1 reveals rationality with a small probability in each period, because her continuation payoff is close to zero at (L, D). However, if the probability that player 1 reveals rationality is small in each period, then it takes many periods for player 1 to build a reputation and K can be chosen large to ensure low payoffs for both players.

This argument hinges on choosing low continuation payoffs for player 2 after terminal node (L, D), during the first K periods. This does not conflict with player 1's incentive to play D instead of U, even if low continuation payoffs for player 2 also implies low continuation payoffs for player 1, after node (L, D). This is because, in the first K periods, when player 1 makes his move, he expects player 2 to play L with probability zero and the terminal node (L, D) is reached with probability zero. Thus, *payoffs at node* (L, D) *have no effect on player 1's ex-ante incentive to play D and consequently player 1's incentive constraint puts no restrictions on player 2's continuation payoff at* (L, D). In contrast, if player 1 moves after observing player 2, as in figure 2a, then as shown in Lemma 1, player 1's incentive constraint implies that player 2's continuation payoff after (L, D) is at least $1 - R(z', \delta) - (1-\delta)/\delta$, i.e., player 1's incentive to play D instead of U imposes a bound on the amount of punishment that player 2 can expect after choosing L.

For our reputation result we make extensive use of Lemma 1 in establishing the upper and lower bounds for player 2's payoffs (Lemma 2 and Lemma 3). In Lemma 2, player 2's payoff

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[31] This is because figure 6 is a normalized sequential common interest game if $a = 1$ and $b = -1$.

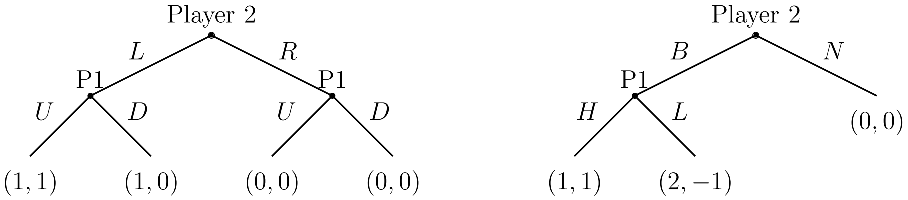
[32] This construction follows Cripps and Thomas (1997).

[33] After (L, D) or (R, D), the continuation game is a repeated game of complete information and any payoff in $[0, 1]$ can be supported in equilibrium.

is bounded along the equilibrium path. Consequently, in this lemma the perfect information assumption is not required. Consider again the equilibrium described for the simultaneous-move game. The bound in Lemma 1 applies without alteration to the simultaneous-move game at node (R, D) (the node of interest for Lemma 2), because player 1 believes that player 2 plays R with probability one on the equilibrium path.

In contrast to Lemma 2, perfect information is essential for Lemma 3. In Lemma 3 we consider a strategy for player 2 that plays L until player 1 deviates from U, and we give a lower bound for player 2's payoff after (L, D). Lemma 1 provides a lower bound on player 2's payoff after (L, D) in the case of perfect information. However, there is no analog to Lemma 1 that provides a tight bound on player 2's payoff after (L, D) for the simultaneous-move game. For example, in the PBE we construct we can put no restrictions on payoffs after node (L, D) beyond individual rationality and feasibility. This is because player 1 expects to reach node (L, D) with probability zero.

4.2. Without Assumption 1, the reputation result can fail. Assumption 1 can fail in two ways. First, Assumption 1 fails if the payoff profile where player 1 receives $\bar{g}_1$ is not unique in G (for example, if Γ is nongeneric). Such a failure is depicted in figure 10a. Second, Assumption 1 fails if, $(\bar{g}_1, \hat{g}_2) \in G$, but Γ is not a strictly conflicting interests game. Such a failure is depicted in figure 10b. Below we demonstrate that a reputation result can fail to obtain in these examples.



(a) A nongeneric common interest game. (b) The product choice game.

Figure 10: Games that fail to satisfy Assumption 1.

In the nongeneric common interest game depicted in figure 10a, suppose that the Stackelberg type of player 1 always plays U and $\mu(S) < 1/2$. We describe a PBE where player 1 receives a payoff strictly lower than one. Suppose on the equilibrium path (R, U) is played in the first K periods and (L, U) is played thereafter. Player 1 does not build a reputation in this PBE. Choose K such that both players receive a payoff equal to 1/2. Suppose that if player 2 deviates from equilibrium by playing L, then player 1's normal type reveals rationality by playing D, and the stage-game equilibrium (L, D) is played thereafter. Consequently, player 2 receives $\mu(S)$ if she deviates from the equilibrium strategy which is less than her equilibrium payoff 1/2.

In the product choice game depicted in figure 10b, player 1's dynamic Stackelberg payoff is 1.5 and player 2's minimax value is zero.[34] Although a dynamic Stackelberg strategy does not exist in this game, there are strategies that deliver a payoff arbitrarily close to the dynamic Stackelberg payoff. Suppose that player 1's mixed actions are observed at the end of each period. One might conjecture that player 1 can obtain a payoff arbitrarily close to the dynamic Stackelberg payoff by mimicking a type, ω^*, that plays H with probability $1/2+\epsilon$. However, this is not the case: Suppose that on the equilibrium path player 1 plays H with probability $1/2+\epsilon$, in each period. Player 2 plays N for the first K periods and plays B thereafter. Choose K such that $\delta^K = 1/2$. Consequently, no reputation is built on the equilibrium path and equilibrium payoffs are $((1.5-\epsilon)/2, \epsilon/2)$. If player 1 deviates from equilibrium and reveals rationality, then player 2 plays N forever. If player 2 deviates from equilibrium and plays B, then player 1 reveals rationality by playing L. In the subsequent complete-information game an equilibrium with payoffs $(1.5, 0)$ is played.[35] This construction is a PBE for any choice of ϵ, if $\mu(\omega^*) < 1/2$: If player 2 deviates and plays B, then she is facing ω^* with probability $\mu(\omega^*)$ and receives payoff equal to ϵ. Alternatively, she is facing the normal type with probability $1-\mu(\omega^*)$ and receives payoff equal to zero. However, $\mu(\omega^*)\epsilon < \epsilon/2$.

4.3. The Stackelberg type. In the repeated games that we consider here, the dynamic Stackelberg strategy is not necessarily unique. For example in the game depicted in figure 5, the grim-trigger strategy is also a dynamic Stackelberg strategy. Mimicking the grim-trigger strategy would, however, not give player 1 a high payoff. This is because the punishment phase is also very costly for player 1. In contrast, the particular Stackelberg type that we choose is not very costly to mimic since the punishment phase is short, i.e., n^p is chosen minimally. If we had chosen any other finite length $n > n^p$ for the punishment phase instead of n^p, our reputation result would still hold.

4.4. Other commitment types. As noted previously by Schmidt (1993), Celentani et al. (1996), or Evans and Thomas (1997), if there is a chance that player 1 is a commitment type other than the Stackelberg type, then player 1 may be unable to build a reputation. Previous work has addressed this issue by assuming that types are learned due to exogenous noise

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[34] In this stage-game, the Stackelberg payoff is also equal to 1.5 because, for any $\epsilon > 0$, player 1 can guarantee a payoff equal to $1.5-\epsilon$ by playing H with probability $1/2+\epsilon$. Yet a Stackelberg action does not exist. The unique action profile that yields player 1 a payoff exactly equal to 1.5 has player one mixing between H and L with equal probability and player two playing B. However, both B and N are best responses to player 1's equal mixture and if player 2 best responds by playing N instead of B, then player 1's payoff is equal zero. Therefore, player 1 cannot guarantee 1.5 by committing to this mixed action, i.e., a Stackelberg action does not exist.

[35] Playing (N, L) in each period is a PBE of the complete information repeated game. Consequently, the threat of switching to (N, L) can incentivize a patient player 1 to play H with probability $1/2$ in each period.

(Celentani et al. (1996) or Aoyagi (1996)), by restricting the class of games (Schmidt, 1993), or by considering more complicated types (Evans and Thomas, 1997).

In the environment we consider, the presence of commitment types can also hinder player 1 from building a reputation. A patient player 2 may resist the Stackelberg type because she fears punishment or expects a reward for not best responding, either from another commitment type or from player 1's normal type. Accordingly, our reputation result holds because, as we show, punishments or rewards cannot come from player 1's normal type; and because we assume that the probability of another commitment type is small compared to the probability of the Stackelberg type.

The restriction on the relative likelihood of other commitment types can be relaxed if the other commitment types are *uniformly learnable*. A uniformly learnable type reveals itself not to be the Stackelberg type at a rate that is bounded away from zero, uniformly across all histories. If the other commitment types are uniformly learnable, then player 1 can play according to $\sigma_1(S)$, thereby ensuring that player 2's posterior belief that player 1 is a type in Ω_- is arbitrarily small in finitely many periods. If player 2's posterior belief that player 1 is a type in Ω_- is small, then Theorem 1 implies that player 1's payoff is close to one for sufficiently large discount factors. However, the restriction to uniformly learnable types is a nontrivial assumption. For example, it rules out the "perverse" type (see Schmidt, 1993) who plays like the Stackelberg type on the equilibrium path but responds to deviations in a history-dependent way.

In previous work, Schmidt (1993) and Celentani et al. (1996) establish reputation results with a nonmyopic player 2, even when the set of commitment types is arbitrary. Celentani et al. (1996) assume that player 2's moves are imperfectly observed with full support.[36] This assumption ensures that all relevant histories are sampled with positive probability without any experimentation by player 2. If player 2's moves are imperfectly observed, then a rich set of commitment types is uniformly learnable. A similar assumption would also enable us to allow for a rich set of commitment types in the framework that we consider here.[37]

The reputation result of Schmidt (1993) obtains if there are conflicting interests in the stage game, player 2's discount factor is fixed, and player 1 is arbitrarily more patient. Conflicting interests imply that the punishment that player 2 can expect from any other commitment type (her minimax payoff) is no worse than best responding to the Stackelberg type and receiving her minimax payoff. A commitment type may also reward player 2 for not best

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[36] Also, see Aoyagi (1996) for a similar assumption.

[37] See Atakan and Ekmekci (2008a), which assumes player 2's moves are imperfectly observed with full support; under this assumption it shows that the set of other types can be taken as the set of all finite automata and the perfect information assumption can be dropped.

responding to the Stackelberg type. But since player 2's discount factor is fixed, a reward for player 2 must entail behavior, that differs from the Stackelberg type and that occurs in a bounded number of periods T. If player 1 is sufficiently patient, then he can mimic the Stackelberg type for these T periods, depriving player 2 of the reward and thus building a reputation. However, rewards for an equally patient player 2 need not accrue in a bounded number of periods. A commitment type that rewards player 2 for resisting the Stackelberg type, in a history-dependent manner can hinder player 1 from building a reputation against an equally patient opponent, even with strictly conflicting interests.

4.5. Two-sided incomplete information. The reputation results in games with asymmetric discounting are robust to the introduction of two-sided uncertainty, while the reputation result that we present in this paper is not. In order to obtain our one-sided reputation result we allow for only one-sided uncertainty. In other words, we replace asymmetric discount factors as in Fudenberg and Levine (1989, 1992), or Celentani et al. (1996), with one-sided asymmetric information.

In a related paper, Atakan and Ekmekci (2008b), we consider a repeated game of perfect information with equally patient agents, two-sided LNCI or SCI, and two-sided uncertainty. In this related paper, we show two results: First, the repeated game has a unique equilibrium if the players are sufficiently patient. Second, under certain additional conditions, in the unique equilibrium of the repeated game, a war-of-attrition (similar to Abreu and Gul (2000)) is played prior to one player revealing herself to be the normal type, and once this has occurred, an equilibrium of the game of one-sided incomplete information, as characterized in Theorem 1, is played.

4.6. Simultaneous-move games with SCI. Cripps et al. (2005) obtain a reputation result for the Bayes-Nash equilibria of repeated simultaneous-move games with SCI. A similar result can be obtained using the method we develop here. In particular, redefine $R(z, \delta)$ using Bayes-Nash equilibrium instead of PBE. The upper bound established in Lemma A.3 remains valid for Bayes-Nash equilibria. This is because all the arguments were constructed on the equilibrium path without any appeal to perfect information or sequential rationality. Also, $U_2(\sigma) \geq \hat{g}_2 = 0$ in any Bayes-Nash equilibrium. Consequently, functional inequality (6) holds, and a reputation result follows.

4.7. Reputation in dynamic games. We do not know whether our reputation result extends to more general dynamic games where a different stage game is played in each period. However, in the following restricted class of dynamic games our reputation result also holds: any one of a finite number of stage games of perfect information is played in each period. All these stage games satisfy Assumption 1. The stage game which is played in a particular period is determined by a transition function, the transitions are stationary, and the transitions depend only on which game was played in the previous period, but not on the outcome of the game played in the previous period. For example, if the battle-of-the-sexes game in figure 1 is played in the odd periods and if the battle-of-the-sexes game in figure 2d is played in the even periods, then our reputation result would hold.

A. PROOF OF THEOREM 1

Normalize payoffs, without loss of generality, such that

$$(9) \qquad \bar{g}_1 = 1;\; g_1(a_1, a_2) \geq 0 \text{ for all } a \in A; \text{ and } g_2(a_1^s, a_2^b) = 0.$$

Recall that $M = \max\{\max\{|g_1|, |g_2|\} : (g_1, g_2) \in F\}$, hence $M \geq 1$.

For any $z \in (0,1]$, let $K(z) = \max\{\frac{4\rho}{zl}, \frac{8M}{zl}(\rho n^p + 2), 2\}$. For any $z \in (0,1]$ let

$$(10) \qquad f(z) = K(z)^{\bar{n}(z)},$$

where $\bar{n}(z)$ is the smallest positive integer j such that $(1 - zl/4\rho)^{j-1} < z$. Note that both K and $\bar{n}$ are decreasing, positive valued functions of z. Hence, $f : (0,1] \to \mathbb{R}^{++}$ is a decreasing, positive valued function.

In what follows, ***we fix constant*** $\underline{z} > 0$, and ***we fix constants***

$$(11) \qquad K = K(\underline{z}), \text{ and } \bar{n} = \bar{n}(\underline{z}).$$

Also, ***we fix constant*** $\phi \in [0,1)$. We show that for any $\mu \in \Delta(\Omega)$ such that $\mu(S) \geq \underline{z}$ and $\mu(\Omega_-)/\mu(S) \leq \phi$, and for any PBE strategy profile σ of $\Gamma^\infty(\mu, \delta)$, the following inequality holds

$$U_1(\sigma, \delta) \geq 1 - f(\underline{z}) \max\{1 - \delta, \phi\} \geq 1 - K^{\bar{n}} \max\{1 - \delta, \mu(\Omega_-)/\mu(S)\}.$$

LEMMA A.1 *Posit perfect information and Assumption 1. There exists* $\delta^* \in [0,1)$ *and* $l > 0$ *such that for any* $r \geq 0$, *if* $U_1(\sigma_1(S), \sigma_2, \delta) = 1 - r$, *then* $U_2(\sigma_1(S), \sigma_2, \delta) \leq -lr$, *for all* $\delta > \delta^*$.

PROOF: The definition of n^p given in inequality (1) implies that there exists a $\delta^* < 1$ and $l > 0$ such that, for all $\delta > \delta^*$, and for any $a_2 \in A_2$ such that $g_1(a_1^s, a_2) < 1$ and any $a_2' \in A_2$, we have

$$(12) \qquad g_2(a_1^s, a_2) + \sum_{k=1}^{n^p - 1} \delta^k g_2(a_1^p, a_2') < -l n^p.$$

For public history $h^t = (y^0, y^1, ..., y^t)$, let $i(h^t) = 1$, if $g_1(y^t) < 1$ and $\sigma_1(S, h^t) = a_1^s$; and $i(h^t) = 0$, otherwise. Player 1 receives at least zero in any period t where $i(h^t) = 1$ and also receives at least zero in the subsequent $n^p - 1$ period punishment phase. In all other periods player 1 receives one. Consequently,

$$U_1(\sigma_1(S), \sigma_2, \delta) \geq 1 - n^p(1-\delta)\mathbb{E}_{(\sigma_1(S),\sigma_2)}\left[\sum_{t=0}^{\infty} \delta^t i(h^t)\right],$$

and $(1-\delta)\mathbb{E}_{(\sigma_1(S),\sigma_2)}\left[\sum_{t=0}^{\infty} \delta^t i(h^t)\right] \geq r/n^p$.[38] If $i(h^t) = 1$, then player 2 receives a total discounted payoff of at most $-n^p l(1-\delta)$ for periods t through $t+n^p-1$, if $\delta > \delta^*$ by inequality (12). In any period where a_1^s is played and $i(h^t)$=0, player 2 receives zero. Consequently,

$$U_2(\sigma_1(S), \sigma_2) \leq -n^p l(1-\delta)\mathbb{E}_{(\sigma_1(S),\sigma_2)}\left[\sum_{t=0}^{\infty} \delta^t i(h^t)\right] \leq -lr,$$

if $\delta > \delta^*$. □

Remark A.1 *We argue that $U_1^C(\sigma_1(S), \delta) = 1$, i.e., $\sigma_1(S)$ is a dynamic Stackelberg strategy, for all $\delta > \delta^*$. Lemma A.1 implies that if $U_1(\sigma_1(S), \sigma_2, \delta) < 1$, then $U_2(\sigma_1(S), \sigma_2, \delta) < 0$, for all $\delta > \delta^*$. Thus, if $U_2(\sigma_1(S), \sigma_2, \delta) \geq 0$, then $U_1(\sigma_1(S), \sigma_2, \delta) \geq 1$, for all $\delta > \delta^*$. If player 2 plays a_2^b in each period of the repeated game against $\sigma_1(S)$, then player 2's payoff is equal to zero. Therefore, if $\sigma_2 \in BR(\sigma_1(S), \delta)$, then $U_2(\sigma_1(S), \sigma_2, \delta) \geq 0$ and as a consequence $U_1(\sigma_1(S), \sigma_2, \delta) \geq 1$, for all $\delta > \delta^*$. Also, if $\sigma_2 \in BR(\sigma_1(S), \delta)$, then $U_2(\sigma_1(S), \sigma_2, \delta)$ is at least as large as player 2's minimax. Hence, if $\sigma_2 \in BR(\sigma_1(S), \delta)$, then $U_1(\sigma_1(S), \sigma_2, \delta) = 1$, for all $\delta > \delta^*$. This follows because player 1's highest payoff compatible with player 2's individual rationality is equal to one.*

In what follows, ***we assume that*** $\delta > \delta^*$, where δ^* is the cutoff established in Lemma A.1.

Definition A.1 *For any $z \in (0, 1]$, define the maximal resistance function as follows:*

$$\bar{R}(z, \delta) = \sup\{R(\mu, \delta) : \mu \in \Delta,\ \mu(S) \geq z,\ \mu(\Omega_-)/\mu(S) \leq \phi\}^+,$$

where Δ is the set of all measures over $\Sigma_1 \cup \{N\}$ with countable support, each commitment type is identified by the strategy that it plays, and Ω is the support of μ.[39]

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[38] The bound on player 1's payoff is crude, especially for low δ.

[39] For any $a \in \mathbb{R}$, $a^+ = \max\{a, 0\}$.

Remark A.2 *Definition A.1 implies that* $\bar{R}(\cdot, \delta) : (0, 1] \to [0, 1]$ *is a nonincreasing function.*

Lemma A.2 *Suppose that* $\mu(\Omega_-)/\mu(S) \leq \phi$*. Pick any PBE* σ *of* $\Gamma^\infty(\mu, \delta)$*, and any period* t *public history* $h = (h^t, d_0)$*; and suppose player 1 deviates from* $\sigma_1(S)$ *at node* d_0 *with positive probability. Let* h^{t+1} *be any public history of terminal nodes that is reached with positive probability under* $\Pr_{(\sigma_1(N)|_h, \sigma_2|_h)}$*; and let* $h' = (h^t, d')$ *be the public history that is reached immediately (with positive probability under* $\Pr_{(\sigma_1(S)|_h, \sigma_2|_h)}$*) if* $\sigma_1(S)$ *is used at* d*. For any* $z' > 0$*, if* $\mu(S|h') \geq z'$*, then*

$$|U_2(\sigma_1(N), \sigma_2, \delta|h^{t+1})| \leq \rho(\bar{R}(z', \delta) + n^p M(1 - \delta)/\delta), \text{ if } \Gamma \text{ satisfies Ass. 1 (i); and}$$
$$U_2(\sigma_1(N), \sigma_2, \delta|h^{t+1}) \leq \rho(\bar{R}(z', \delta) + n^p M(1 - \delta)/\delta), \text{ if } \Gamma \text{ satisfies Ass. 1 (ii).}$$

Proof: Note that player 1's reputation level $\mu(S|h') \geq z'$ and $\mu(\Omega_-|h')/\mu(S|h') \leq \phi$. Therefore, if a history (h^k, d'') is consistent with $\sigma_1(S)$ and if the node specified by the history (h^k, d'') comes after the node specified by the history (h^t, d'), then player 1's reputation level $\mu(S|h^k, d'') \geq z'$ and $\mu(\Omega_-|h^k, d'')/\mu(S|h^k, d'') \leq \phi$. If player 1 plays according to $\sigma_1(S)$ at d_0 and through the remaining nodes of period t, then he obtains at least zero for the period and an $n^p - 1$ period punishment phase may ensue. His payoff is at least zero in these periods. Consequently, if he plays according to $\sigma_1(S)$, his payoff is at least:

$$0 \times (1 - \delta^{n^p}) + \delta^{n^p}(1 - \bar{R}(z', \delta)) = \delta^{n^p}(1 - \bar{R}(z', \delta)),$$

because $\bar{R}$ is nonincreasing. Alternatively, if he chooses a move that differs from the move that $\sigma_1(S)$ would have chosen, then he receives at most $M(1-\delta)$ for the period, and $U_1(\sigma, \delta|h^{t+1})$ as his continuation payoff. Therefore, $M(1 - \delta) + \delta U_1(\sigma, \delta|h^{t+1}) \geq \delta^{n^p}(1 - \bar{R}(z', \delta))$. This implies:

$$U_1(\sigma, \delta|h^{t+1}) \geq \delta^{n^p - 1}(1 - \bar{R}(z', \delta)) - M(1 - \delta)/\delta \geq 1 - \bar{R}(z', \delta) - n^p M(1 - \delta)/\delta,$$

where the last inequality follows because $M \geq 1$, by definition. The bounds on player 2's payoff follow from inequalities (2) and (3), and from the fact that the payoff profile $(U_1(\sigma, \delta|h^{t+1}), U_2(\sigma_1(N), \sigma_2, \delta|h^{t+1}))$ is an element of the set F. □

Definition A.2 (Stopping time) *For any integer* k*,* $E_{[0,k]}$ *denotes the event (set of infinite public histories) where player 1 deviates from* $\sigma_1(S)$ *for the first time in period* t *for some* $0 \leq t \leq k$*. For any strategy profile* $\sigma = (\{\sigma_1(\omega)\}_{\omega \in \Omega}, \sigma_2)$*, any measure* $\mu \in \Delta$*, and any*

$z' \in (0,1]$, *let*

$$T(\sigma,\mu,z') = \min\{k : \mu(S) \geq z'(1-\pi(k))\},$$

where $\pi(k) = \sum_{\omega\in\Omega} \mu(\omega) \Pr_{(\sigma_1(\omega),\sigma_2)} \left[E_{[0,k]}\right]$*; and let* $T(\sigma,\mu,q)=\infty$ *if the set is empty.*

Suppose that player 1's initial reputation level $\mu(S) = z$ and $\mu(\Omega_-)/\mu(S) \leq \phi$; pick $z' > 0$ and pick a strategy profile $\sigma^* = (\{\sigma_1(\omega)\}_{\omega\in\Omega}, \sigma_2^*)$. Let $T = T(\sigma^*,\mu,z')$. Further suppose that σ_2^* is a pure strategy. Because both $\sigma_1(S)$ and σ_2^* are pure strategies, there is a unique path of play that is induced by $\sigma_1(S)$ and σ_2^*. Suppose that $T < \infty$ and let h^T and h^{T+1} denote the unique public histories of terminal nodes consistent with $(\sigma_1(S),\sigma_2^*)$. If $z < z'$, then the stopping time definition and Bayes' rule implies that that $\mu(S|h^T) < z'$ and $\mu(S|h^{T+1}) \geq z'$. Therefore there exists a unique public history (h^T, d^*) consistent with $(\sigma_1(S),\sigma_2^*)$ such that $\mu(S|h^T,d^*) < z'$ and $\mu(S|h') \geq z'$ where $h' = (h^T,d')$ is the public history that is reached immediately after d^* if $\sigma_1(S)$ is used at node d^* in period T. Also, by Bayes' rule, the total probability that player 1 deviates from the Stackelberg strategy at any decision node (in periods zero through T) up to but excluding (h^T,d^*) is at most $1 - z/z'$.

Lemma A.3 *Posit perfect information and Assumption 1. For any* $\mu \in \Delta$ *such that* $\mu(S) = z > 0$ *and* $\mu(\Omega_-)/\mu(S) \leq \phi$*, pick a PBE* σ *of* $\Gamma^\infty(\mu,\delta)$ *such that* $r(\delta,\sigma_2) \geq R(\mu,\delta) - \xi$.[40] *For the chosen PBE* σ *and any* $z' > 0$,

$$U_2(\sigma,\delta) \leq \rho(q(z,z')\bar{R}(z,\delta) + \bar{R}(z',\delta) + 2n^pM\epsilon) + 5M\epsilon - (R(\mu,\delta)-\xi)zl, \tag{13}$$

where $\epsilon = \max\{\phi, 1-\delta\}$ *and* $q(z,z') = \max\{1 - z/z', 0\}$.

Proof: Choose a pure strategy σ_2^* in the support of the possibly mixed strategy σ_2 such that $r(\sigma_2^*,\delta) \geq R(\mu,\delta) - \xi$. Such a pure strategy exists because the mixed strategy σ_2 has resistance of at least $R(\mu,\delta)-\xi$. Let profile $\sigma^* = (\{\sigma_1(\omega)\}_{\omega\in\Omega},\sigma_2^*)$ and let $T = T(\sigma^*,\mu,z')$. If $z < z'$ and $T < \infty$, then let (h^T,d^*) denote the unique public history consistent with $(\sigma_1(S),\sigma_2^*)$ such that $\mu(S|(h^T,d^*)) < z'$ and $\mu(S|h') \geq z'$ where $h' = (h^T,d')$ is the public history that is reached immediately after (h^T,d^*) if $\sigma_1(S)$ is used at node d^*. If $z' \leq z$, then $T = 0$ and we let d^* denote the initial node of the game. If $T = \infty$, then we say $d^* = \infty$ which means that there are no decision nodes that come after d^*.

Given that $\mu(S) = z$ and $\mu(\Omega_-)/\mu(S) \leq \phi$, if $h = (h^t,d'')$ is a public history that is consistent with $(\sigma_1(S),\sigma_2^*)$, then $\mu(\Omega_-|h)/\mu(S|h) \leq \phi$ and $\mu(S|h) \geq z$; and moreover if the

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[40] For each $\xi > 0$, such a PBE of $\Gamma^\infty(\mu,\delta)$ exists because the resistance function R is defined as the supremum over the set $\{r(\delta,\sigma_2) : \sigma_2 \text{ is part of a PBE of } \Gamma^\infty(\mu,\delta)\}$.

decision node (h^t, d'') comes after d^*, then $\mu(S|h) \geq z'$.

Let E_1 denote the event, i.e., set of infinite histories, where player 1 deviates from $\sigma_1(S)$ in a decision node before (and excluding) the decision node d^* of period T. Also, let E_2 denote the event where player 1 deviates from $\sigma_1(S)$ in a decision node after (and including) the decision node d^* of period T.[41] We will bound player 2's payoff from σ^* in the following five events: $\omega = N$ and E_1; $\omega = N$ and E_2; $\omega = N$ and player 1 never deviates from $\sigma_1(S)$; $\omega = S$; and $\omega \in \Omega_-$.

Before proceeding to bound player 2's payoff in the five events, as a preliminary step, we argue that player 2's payoff until the period t where player 1 deviates from $\sigma_1(S)$ for the first time is at most $(1-\delta)M \leq \epsilon M$. To see why, consider the following three possibilities: First, if player 2 plays a_2^b in each period until time t, then her payoff is zero. Second, if player 2 deviates from a_2^b in period $t' \leq t - n^p$, then she receives at most $(1-\delta)M$ in period t' and a punishment phase ensues. Lemma A.1 implies that player 2's discounted payoff, for periods t' through $t' + n^p - 1$, is negative. Third, if player 2 deviates from a_2^b in period $t' < t$ but $t' > t - n^p$, then she receives at most $(1-\delta)M$ in period t', a punishment phase ensues (but is not completed before period t), and she receives at most zero in periods $t'+1$ through $t-1$, i.e., she receives at most zero in each period of the incomplete punishment phase.

We now bound player 2's payoff in the event $\omega = N$ and E_1. Suppose that $h^\infty \in E_1$, then let $h = (h^j, d)$ denote the node in period j in which player 1 deviates from $\sigma_1(S)$ for the first time in the infinite public history h^∞. Player 2's payoff until period j is at most ϵM and player 2's payoff in period j is at most ϵM. Lemma A.2 and the fact that $\epsilon \geq (1-\delta)$ together imply that $U_2(\sigma_1(N), \sigma_2^*, \delta | h^{j+1}) \leq \rho(\bar{R}(z,\delta) + \epsilon M n^p/\delta)$. Hence, for any such period j, player 2's repeated game payoff is at most

$$M\epsilon + \delta^j M\epsilon + \delta^{j+1}\rho(\bar{R}(z,\delta) + \epsilon M n^p/\delta) \leq 2M\epsilon + \rho(\bar{R}(z,\delta) + n^p M\epsilon).$$

We therefore obtain:

$$U_2(\sigma_1(N), \sigma_2^*, \delta | E_1) \leq 2M\epsilon + \rho(\bar{R}(z,\delta) + n^p M\epsilon). \tag{14}$$

We bound player 2's payoff in the event $\omega = N$ and E_2. Suppose that $h^\infty \in E_2$, then let $h = (h^j, d)$ denote the node at which player 1 deviates from $\sigma_1(S)$ for the first time in the infinite public history h^∞. Player 1's reputation is at least z' if he plays according to $\sigma_1(S)$ at the decision node d of period j. Consequently, Lemma A.2 implies that $U_2(\sigma_1(S), \sigma_2^*, \delta | h^{j+1}) \leq \rho(\bar{R}(z',\delta) + n^p M\epsilon/\delta)$. As a result, an argument identical to that in the previous paragraph

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[41] Observe that if d^* is the initial node, then $E_1 = \emptyset$. Also, if $d^* = \infty$, then $E_2 = \emptyset$.

implies that

$$(15)\qquad U_2(\sigma_1(N), \sigma_2^*, \delta | E_2) \leq 2M\epsilon + \rho(\bar{R}(z', \delta) + n^p M\epsilon).$$

Player 2's payoff in the event that $\omega = S$ (i.e., the event in which she faces type S) is at most $-(R(\mu,\delta) - \xi)l$, and the probability of this event is equal to z. This is because player 2's resistance is at least $R(\mu,\delta) - \xi$ for the strategy σ_2^*; hence, she loses at least $(R(\mu,\delta)-\xi)l$ against S, by Lemma A.1. Player 2's payoff in the event that $\omega \in \Omega_-$ (i.e., the event in which she faces any other commitment type) is at most M, and the probability of this event is at most $\phi z \leq \phi \leq \epsilon$. A bound on player 2's payoff in the event that $\omega = N$ and E_1 is given by inequality (14), and the probability of this event is at most $q(z, z')$. A bound on player 2's payoff in the event that $\omega = N$ and E_2 is given by inequality (15), and the probability of this event is at most one. Player 2's payoff in the event that $\omega = N$ and player 1 never deviates from $\sigma_1(S)$ is at most zero. Consequently,

$$U_2(\sigma^*, \delta) \leq q(z, z')\rho\bar{R}(z, \delta) + \rho\bar{R}(z', \delta) - z(R(\mu, \delta) - \xi)l + 2\rho n^p M\epsilon + 5M\epsilon.$$

Since σ_2^* is in the support of PBE strategy σ_2, we have $U_2(\sigma, \delta) = U_2(\sigma^*, \delta)$. Hence,

$$U_2(\sigma, \delta) \leq q(z, z')\rho\bar{R}(z, \delta) + \rho\bar{R}(z', \delta) - z(R(\mu, \delta) - \xi)l + 2\rho n^p M\epsilon + 5M\epsilon.$$

□

Lemma A.4 *Posit perfect information and Assumption 1 (i). Suppose that* $\mu(S) = z > 0$ *and* $\mu(\Omega_-)/\mu(S) \leq \phi$. *In any PBE* σ *of* $\Gamma^\infty(\mu, \delta)$ *and for any* $z' > 0$, *we have*

$$(16)\qquad U_2(\sigma, \delta) \geq -\rho(q(z, z')\bar{R}(z, \delta) + \bar{R}(z', \delta) + 2n^p M\epsilon) - 3M\epsilon,$$

where $\epsilon = \max\{\phi, 1-\delta\}$ *and* $q(z, z') = \max\{1 - z/z', 0\}$.

Proof: Fix a PBE profile σ of $\Gamma^\infty(\mu, \delta)$. Let σ_2^* denote a pure strategy which moves according to a_2^b after any public history h that is consistent with $\sigma_1(S)$; and coincides with a pure strategy in the support of the PBE strategy σ_2 if public history h is not consistent with $\sigma_1(S)$. Let profile $\sigma^* = (\{\sigma_1(\omega)\}_{\omega\in\Omega}, \sigma_2^*)$ and let $T = T(\sigma^*, \mu, z')$. If $z < z'$ and $T < \infty$, then let (h^T, d^*) denote the unique public history consistent with $(\sigma_1(S), \sigma_2^*)$ such that $\mu(S|(h^T, d^*)) < z'$ and $\mu(S|h') \geq z'$ where $h' = (h^T, d')$ is the public history that is reached immediately after (h^T, d^*) if $\sigma_1(S)$ is used at node d^*. If $z' \leq z$, then $T = 0$ and we let d^* denote the initial node of the game. If $T = \infty$, then we say $d^* = \infty$ which means that there are no decision nodes that come after d^*.

Because (a_1^s, a_2^b) is played in each period under $(\sigma_1(S), \sigma_2^*)$, player 2 receives zero in each period until player 1 deviates from $\sigma_1(S)$. Also, player 2's payoff in the period in which player 1 deviates from $\sigma_1(S)$ is at least $-M\epsilon$. Using the reasoning in Lemma A.3 and applying Lemma A.2 we obtain

$$U_2(\sigma_1(N), \sigma_2^*, \delta | E_1) \geq -\rho(\bar{R}(z,\delta) + n^p M\epsilon) - M\epsilon,$$

and

$$U_2(\sigma_1(N), \sigma_2^*, \delta | E_2) \geq -\rho(\bar{R}(z',\delta) + n^p M\epsilon) - M\epsilon,$$

where E_1 and E_2 are the events defined in Lemma A.3.

If player 1 never deviates from $\sigma_1(S)$, then player 2 receives zero. Player 2 can get at least $-M$ against any other commitment type, whom she faces with probability of at most $\phi \leq \epsilon$; she gets zero against type S, whom she faces with probability z. Following the same reasoning as in Lemma A.3 and because σ_2 is part of the PBE σ, we obtain

$$U_2(\sigma, \delta) \geq U_2(\sigma^*, \delta) \geq -\rho q(z,z')\bar{R}(z,\delta) - \rho\bar{R}(z',\delta) - 2\rho n^p M\epsilon - 3M\epsilon.$$

□

COMPLETING THE PROOF OF THEOREM 1 BY USING LEMMA A.3 AND LEMMA A.4: If Γ satisfies Assumption 1 and perfect information, then inequality (13) is satisfied, by Lemma A.3. If Γ satisfies Assumption 1 (i) and perfect information, then inequality (16) is satisfied, by Lemma A.4. Also, if Γ satisfies Assumption 1 (ii), then $U_2(\sigma,\delta) \geq \hat{g}_2 = 0$, and inequality (16) is trivially satisfied because the right-hand side of the inequality is negative. By combining the upper and lower bounds for $U_2(\sigma,\delta)$, given by inequalities (13) and (16), and using the fact that $\xi > 0$ can be chosen arbitrarily, we obtain

$$zlR(\mu,\delta) \leq 2\rho(q(z,z')\bar{R}(z,\delta) + \bar{R}(z',\delta) + 2n^p M\epsilon) + 8M\epsilon, \tag{17}$$

for any $\mu \in \Delta$ such that $\mu(S) = z$ and $\mu(\Omega_-)/\mu(S) \leq \phi$, and for any $z' \in (0,1]$. Pick another measure $\mu' \in \Delta$ such that $\mu'(S) \geq z$ and $\mu'(\Omega_-)/\mu'(S) \leq \phi$. By rewriting inequality (17) for μ' and $z' \in (0,1]$ and by rearranging, we obtain the following inequality:

$$R(\mu',\delta) \leq \left(2\rho(q(\mu'(S),z')\bar{R}(\mu'(S),\delta) + \bar{R}(z',\delta) + 2n^p M\epsilon) + 8M\epsilon\right)/\mu'(S)l. \tag{18}$$

However, $q(z,z') \geq q(\mu'(S),z')$, because $\mu'(S) \geq z$; and $\bar{R}(z,\delta) \geq \bar{R}(\mu'(S),\delta) \geq 0$, because $\bar{R}$ is nonnegative and nonincreasing in z. Substituting z for $\mu'(S)$, $q(z,z')$ for $q(\mu'(S),z')$, and $\bar{R}(z,\delta)$ for $\bar{R}(\mu'(S),\delta)$ on the right-hand side of inequality (18) delivers the following:

$$(19) \qquad zlR(\mu',\delta) \leq 2\rho(q(z,z')\bar{R}(z,\delta) + \bar{R}(z',\delta) + 2n^p M\epsilon) + 8M\epsilon,$$

for all $\mu' \in \Delta$ such that $\mu'(S) \geq z$ and $\mu'(\Omega_-)/\mu'(S) \leq \phi$. Because $\bar{R}(z,\delta)$ is the supremum over the set $\{R(\mu',\delta) : \mu \in \Delta, \mu'(S) \geq z \text{ and } \mu'(\Omega_-)/\mu'(S) \leq \phi\}$, and because each $R(\mu',\delta)$ in this set satisfies inequality (19), we obtain the following:

$$(20) \qquad zl\bar{R}(z,\delta) \leq 2\rho(q(z,z')\bar{R}(z,\delta) + \bar{R}(z',\delta) + 2n^p M\epsilon) + 8M\epsilon.$$

For any $z \geq \underline{z}$, substituting $\underline{z}$ for z in inequality (20) and rearranging gives the following functional inequality:

$$(21) \qquad \bar{R}(z,\delta)(\underline{z}l - 2\rho q(z,z')) \leq 2\rho\bar{R}(z',\delta) + 4M(\rho n^p + 2)\epsilon.$$

Let $\underline{q} = \underline{z}l/4\rho$. If $z, z' \in [\underline{z}, 1]$ and $z \in [z'(1-\underline{q}), z']$, then $q(z,z') \leq \underline{q}$. Hence, substituting $\underline{q}$ for $q(z,z')$ in inequality (21) we obtain the following:

$$\bar{R}(z,\delta)(\underline{z}l - 2\rho\underline{q}) \leq 2\rho\bar{R}(z',\delta) + 4M(\rho n^p + 2)\epsilon.$$

Substituting $\underline{z}l/4\rho$ for $\underline{q}$ in the previous inequality and rearranging, we obtain the following:

$$\bar{R}(z,\delta) \leq \frac{4\rho}{\underline{z}l}\bar{R}(z',\delta) + \frac{8M}{\underline{z}l}(\rho n^p + 2)\epsilon.$$

Using the fact that $\bar{R}(z',\delta) \geq 0$, and substituting $K = \max\{\frac{4\rho}{\underline{z}l}, \frac{8M}{\underline{z}l}(\rho n^p + 2), 2\}$ for $\frac{4\rho}{\underline{z}l}$ and $\frac{8M}{\underline{z}l}(\rho n^p + 2)$ in the previous inequality, we obtain the following:

$$(22) \qquad \bar{R}(z,\delta) \leq K\bar{R}(z',\delta) + K\epsilon.$$

However, the functional inequality (22) is identical to inequality (7) (since K and ϵ in inequality (22) serve the same roles as C and $1-\delta$ in inequality (7)). Also, $\bar{R}(1,\delta) = 0$. Consequently, an argument identical to the one used to establish Corollary 1 implies that $\bar{R}(\underline{z},\delta) \leq \sum_{j=1}^{\bar{n}-1} K^j\epsilon$, where $\bar{n}$ is the smallest integer j such that $(1-\underline{q})^{j-1} < \underline{z}$. Because $K \geq 2$ we have $\bar{R}(\underline{z},\delta) \leq \sum_{j=1}^{\bar{n}-1} K^j\epsilon \leq K^{\bar{n}}\epsilon = K^{\bar{n}}\max\{1-\delta,\phi\}$. For any μ such that $\mu(S) \geq \underline{z}$ and $\mu(\Omega_-)/\mu(S) \leq \phi$, and for any PBE strategy σ of $\Gamma^\infty(\mu,\delta)$, we have $U_1(\sigma,\delta) \geq 1 - R(\mu,\delta)$ and $R(\mu,\delta) \leq \bar{R}(\underline{z},\delta)$. Consequently, $U_1(\sigma,\delta) \geq 1 - K^{\bar{n}}\max\{1-\delta,\phi\}$. $\square$

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