Political Economy in a Changing World*

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Abstract

We provide a general framework for the analysis of the dynamics of institutional change (e.g., democratization, extension of political rights, or repression of different groups), and how these dynamics interact with (anticipated and unanticipated) changes in the distribution of political power and in economic structure. We focus on Markov Voting Equilibria, which require that economic and political changes should take place if there exists a subset of players with the power to implement such changes and who will obtain higher expected discounted utility by doing so. Assuming that economic and political institutions as well as individual types can be ordered, and preferences and the distribution of political power satisfy natural "single crossing" (increasing differences) conditions, we prove the existence of a pure-strategy equilibrium, provide conditions for its uniqueness, and present a number of comparative static results that apply at this level of generality. We then use this framework to study the dynamics of political rights and repression in the presence of radical groups that can stochastically grab power and the dynamics of collective experimentation over institutions.

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1 Introduction

Political change often takes place in the midst of uncertainty and turmoil, which sometimes brings to power the most radical factions, such as the militant Jacobins during the Reign of Terror in the French Revolution or the Nazis during the crisis of the Weimar Republic. The events leading up to the October Revolution of 1917 in Russia illustrate both how an extremist fringe group can ascend to power, and the dynamics of repression partly motivated by the desire of ruling elites to prevent the empowerment of extremist — and sometimes also of moderate — elements.

Russia entered the 20$^{\text{th}}$ century as an absolute monarchy, but started a process of limited political reforms in response to labor strikes and civilian unrest in the aftermath of its defeat in the Russo-Japanese war of 1904-1905. Despite the formation of political parties (for the first time in Russian history) and an election with a wide franchise, the tsar still retained control, in part relying on repression against the leftist groups, his veto power, the right to dissolve the parliament (the Duma), full control of the military and cabinet appointments, and his ability to rule by decree when the Duma was not in session (Pipes, 1995). This may have been partly motivated by the fear of further strengthening the two major leftist parties, Social Revolutionaries and Social Democrats (i.e., communists, consisting in turn of the Bolsheviks and the Mensheviks), which together controlled about 2/5 of the 1906 Duma and explicitly targeted a revolution.[1]

World War I created the opening that the Bolsheviks had been looking for, bringing to power the Provisional Government in the February Revolution of 1917, and then later, the moderate Social Revolutionary Alexander Kerensky. Additional military defeats of the Russian army in the summer of 1917, the destruction of the military chain of command by Bolshevik-led soldier committees, and Kerensky's willingness to enter into an alliance with Social Democrats to defeat the attempted coup by the army during the Kornilov affair strengthened the Bolsheviks further. Though in the elections to the Constituent Assembly in November 1917, they had only a small fraction of the vote, the Bolsheviks successfully exploited their control of Petrograd Soviets to

[1] Lenin, the leader of the Bolshevik wing of the Social Democrats, recognized that a revolution was possible only by exploiting turmoil. In the context of the 1906 Duma, he stated: "Our task is [...] to use the conflicts within this Duma, or connected with it, for choosing the right moment to attack the enemy, the right moment for an insurrection against the autocracy." Later, he argued: "[...] the Duma should be used for the purposes of the revolution, should be used mainly for promulgating the Party's political and socialist views and not for legislative 'reforms,' which, in any case, would mean supporting the counter-revolution and curtailing democracy in every way."

outmaneuver the more popular Social Revolutionaries, first entering into an alliance with so-called Left Social Revolutionaries, and then coercing them to leave the government so as to form their own one-party dictatorship.

This episode illustrates both the possibility of a series of transitions bringing to power some of the most radical groups and the potential implications of the concerns of moderate political transitions further empowering radical groups. Despite a growing literature on political transitions, the issues we have just emphasized in the context of the Bolshevik Revolution cannot be studied with existing models, because they necessitate a dynamic stochastic model where several groups can form temporary coalitions, potentially leading to a sequence of political transitions away from current powerholders. Such a model, if tractably developed, could also shed further light on key questions in the literature on regime transitions, including those concerning political transitions with several heterogeneous groups, gradual enfranchisement or disenfranchisement, and the interactions between regime dynamics and coalition formation.[2] In this paper, we develop a framework for the study of dynamic political economy in the presence of stochastic shocks and changing environments, which we then apply to an analysis of the implications of potential shifts of power to radical groups during tumultuous times and to the problem of institutional experimentation. The next example provides a first glimpse of the type of abstraction we will utilize.

Example 1 Consider a society consisting of n groups, spanning from $-l < 0$ (left-wing) to $r > 0$ (right-wing), with group 0 normalized to contain the median voter. For example, with $n = 3$, we can think of the rightmost player as corresponding to the Russian tsar, the middle player to moderate groups, and the leftmost group to Bolsheviks. The stage payoff of each group depends on current policies, which are determined by the politically powerful coalition in the current "political state". Suppose that there are $2n - 1$ political states, each specifying which of the "extreme" players are repressed and excluded from political decision-making. With $n = 3$, the five states are $s = 2$ (both moderates and Bolsheviks are repressed and the tsar is the dictator), 1 (Bolsheviks are repressed), 0 (nobody is repressed and power lies with moderates), -1 (the tsar is repressed or eliminated), and finally -2 (the tsar and moderates are repressed, i.e. a Bolshevik dictatorship). Since current policies depend on the (political) state, we can

[2] These types of political dynamics are not confined to episodes in which extreme left groups might come to power. The power struggles between secularists and religious groups in Turkey and more recently in the Middle East and North Africa are also partly motivated by concerns on both sides that political power will irrevocably — or at least persistently — shift to the other side.

directly define stage payoffs as a function of the current state for each player, $u_i(s)$ (which is inclusive of repression costs, if any). Suppose that starting in any state $s \neq -2$, a stochastic shock can bring the Bolsheviks to power and this shock is more likely when s is lower.

In addition to proving the existence and characterizing the structure of pure-strategy equilibria, this framework enables us to establish the following types of results. First, in the absence of stochastic shocks bringing Bolsheviks to power, $s = 0$ (no repression or democracy) is stable in the sense that moderates would not like to initiate repression, but $s > 0$ may also be stable, because the tsar may prefer to incur the costs of repression to implement policies more in line with his preferences. Second, and more interestingly, moderates may also initiate repression starting with $s = 0$ if there is the possibility of a switch of power to Bolsheviks. Third, and paradoxically, the tsar may be more willing to grant political rights to moderates when Bolsheviks are stronger, because this might make a coalition between the latter two groups less likely (this is an illustration of what we refer to as "*slippery slope*" considerations and shows the general non-monotonicities in our model: when Bolsheviks are stronger, the tsar has less to fear from the slippery slope).[3] Finally, there is strategic complementarity in repression: the anticipation of repression by Bolsheviks encourages repression by moderates and the tsar.[4]

Though stylized, this example communicates the rich strategic interactions involved in dynamic political transitions in the presence of stochastic shocks and changing environments. Against this background, the framework we develop will show that, under natural assumptions, we can characterize the equilibria of this class of situations fairly tightly and perform comparative statics, shedding light on these and a variety of other dynamic strategic interactions.

Formally, we consider a generalization of the environment discussed in the example. Society consists of $i = 1, 2, ..., n$ players (groups or individuals) and $s = 1, 2, ..., m$ *states*, which represent both different economic arrangements with varying payoffs for different players, and different political arrangements and institutional choices. Stochastic shocks are modeled as (stochastic) changes in *environments*, which encode information on preferences of all players over states and

[3] By "slippery slope" considerations we refer to the following type of situation: a "winning coalition" (a sufficiently powerful group of players) does not move to a state z starting from s even though all of its members would obtain strictly greater state utility in z than s. This happens because a move from s is expected to shift power to another winning coalition which will then start a move to another sequence of states which are less preferred by some members of the initial winning coalition.

[4] This result also provides a new perspective on why repression may differ markedly across societies. For example, Russia before the Bolshevik Revolution repressed the leftists, and thereafter, the rightists and centrists, while the extent of repression of either extreme has been more limited in the United Kingdom. Such differences are often ascribed to differences in "political culture". Our result instead suggests that (small) differences in economic interests or political costs of repression can lead to significantly different repression outcomes.

the distribution of political power within states. This approach is general enough to capture a rich set of permanent and transitory (as well as both anticipated or unanticipated) stochastic shocks depending on the current state and environment. Players care about the expected discounted sums of their utilities, and based on their political power, they make joint choices among feasible political transitions. Our key assumption is that both preferences and the distribution of political power satisfy a natural *single crossing* (increasing differences) property: we assume that players and states are "ordered," and higher-indexed players relatively prefer higher-indexed states and also tend to have greater political power in such states. (Changes in environments shift these preferences and distribution of political power, but maintain increasing differences).[5]

Our notion of equilibrium is *Markov Voting Equilibrium (MVE)*, which comprises two natural requirements: (1) that changes in states should take place if there exists a subset of players with the power to implement them and who will obtain higher continuation utility (along the equilibrium path) by doing so; (2) that strategies and continuation utilities should only depend on payoff-relevant variables and states. Under these assumptions, we establish the existence of pure-strategy equilibria. Furthermore, we show that the stochastic path of states in any MVE is *monotone between shocks*: so long as there is no exogenous shock, the path of states remains monotone (Theorem 8). Though this result does not imply that the institutional path is monotone everywhere, it does imply that the direction of society's institutional path changes only when shocks arrive. Coupled with our assumption that there is a finite number of shocks, this result also ensures that a limit state exists, though this limit state (and thus the long-run equilibrium institution that the society eventually converges to) depends on the exact timing and realizations of shocks (Theorems 1 and 3). Although MVE are not always unique, we also provide sufficient conditions that ensure uniqueness (Theorems 2 and 4). We further demonstrate a close correspondence between these MVEs and the pure-strategy Markov Perfect Equilibria of our environment (Theorem 5).

Despite the generality of the framework described here and the potential countervailing forces highlighted by Example 1, we also establish some comparative static results. Consider, for example, a change in environment which leaves preferences or the allocation of political power in any of the states $s = 1, \ldots, s'$ unchanged, but potentially changes them in states

[5] Formally, we assume "increasing differences" rather than single crossing, but in the informal discussion, we use the two terms interchangeably.

$s = s' + 1, \ldots, m$. The result is that if the steady state of equilibrium dynamics described above, x, did not experience change (i.e., $x \leq s'$), then the new steady state emerging after the change in environment can be no smaller than this steady state (Theorem 6). Intuitively, before the change, a transition to any of the smaller states $s \leq x$ could have been chosen, but was not. Now, given that preferences and political power did not change for these states, they have not become more attractive.[6] An interesting and novel implication of this result is that in some environments, there may exist *critical states*, such as a "sufficiently democratic constitution," and if these critical states are reached before the arrival of certain major shocks or changes (which might have otherwise led to their collapse), there will be no turning back (see Corollary 1). This result provides a different interpretation of the durability of certain democratic regimes than the approaches based on "democratic capital" (e.g., Persson and Tabellini, 2009): a democracy will survive forever if it is not shocked or challenged severely while still progressing towards the "sufficiently democratic constitution/state", but will be reversed if there is a shock before this state is reached.

The second part of the paper applies our framework to two new and substantive applications. The first is the emergence and implications of radical politics, in the context of which we establish the results mentioned at the end of Example 1 above. The second is a model of collective experimentation over different sets of institutions. Neither application can be studied without the tools developed in this paper.

Our paper is related to several literatures. First, our previous work, in particular Acemoglu, Egorov, and Sonin (2012), took one step in this direction by introducing a model for the analysis of the dynamics and stability of different political rules and constitutions. However, that approach not only heavily relied on the absence of shocks (thus ruling out stochastic changes in political power or preferences), but also focused on environments in which the discount factor is sufficiently close to 1 so that all agents just care about the payoff from a stable state (that will emerge and persists) if such a state exists. Here, in contrast, it is crucial that political change and choices are motivated by the entire path of payoffs.[7]

[6] In contrast, some of the higher-ranked states may have become more attractive, which may induce a transition to a higher state. In fact, somewhat surprisingly, transition to a state $s \geq s' + 1$ can take place even if all states $s = s' + 1, \ldots, m$ become *less* attractive for all agents in society.

[7] In Acemoglu, Egorov, and Sonin (2010), we studied political selection and government formation in a population with heterogeneous abilities and allowed stochastic changes in the competencies of politicians. Nevertheless, this was done under two assumptions, which significantly simplified the analysis and made it much less applicable: stochastic shocks were assumed to be very infrequent and the discount factor was taken to be close to 1. Acemoglu, Egorov and Sonin (2011) took a first step towards introducing stochastic shocks, but only confined to the exogenous emergence of new extreme states (and without any of the general characterization or comparative

Second, several papers on dynamic political economy and on dynamics of clubs emerge as special cases of our paper. Among these, Roberts (1999) deserves special mention as an important precursor to our analysis. Roberts studies a dynamic model of club formation in which current members of the club vote on whether to admit new members or exclude some of the existing ones; members' preferences satisfy single crossing type assumptions (see also Barberà, Maschler, and Shalev, 2001). Our setup and results generalize, extend, and strengthen Robert's in several dimensions. First, Roberts focuses on a stationary model without shocks, whereas we allow for nonstationary elements and rich stochastic shocks. Second, we allow for fairly general distributions of political power across states, which is crucial for our focus, while Roberts assumes majority rule for every club. Third, we prove existence of pure-strategy equilibria and provide conditions for uniqueness — results that do not have equivalents in Roberts. Fourth, we provide a general characterization of the structure of MVE, which in turn paves the way for our comparative static results — again results that have no equivalents and Roberts. Fifth, we show the relationship between this equilibrium concept and MPE of a fully specified dynamic game. Finally, we show how our framework can be applied to a political economy problem, providing new and interesting insights in this instance. Gomes and Jehiel's (2005) paper, which studies dynamics in a related environment with side transfers, is also noteworthy, yet does not include stochastic elements or similar general characterization results either.

Third, our motivation is also related to the literature on political transitions. Acemoglu and Robinson (2000a, 2001) consider environments in which institutional change is partly motivated by a desire to reallocate political power in the future to match the current distribution of power.[8] Acemoglu and Robinson's analysis is simplified by focusing on a society consisting of two social groups (and in Acemoglu and Robinson, 2006, with three social groups). In Acemoglu and Robinson (2001), Fearon (2004), Padro i Miquel (2007), Powell (2006), Hirshleifer, Boldrin, and Levine (2009), and Acemoglu, Ticchi, and Vindigni (2010), anticipation of future changes in political power leads to inefficient policies, civil war, or collapse of democracy. There is a growing literature that demonstrates inefficiencies in environments where current political decisions affect the future allocation of political power or bargaining power (see Besley and Coate, 1998, Fearon, 1996, Powell, 2006, and Acharya and Ortner, 2013).

static results presented here).

[8] Other related contributions here include Alesina, Angeloni, and Etro (2005), Barberà and Jackson (2004), Messner and Polborn (2004), Bourguignon and Verdier (2000), Burkart and Wallner (2000), Jack and Lagunoff (2008), Lagunoff (2006), and Lizzeri and Persico (2004).

Fourth, there is a small literature on strategic use of repression, which includes Acemoglu and Robinson (2000b), Gregory, Schroeder, and Sonin (2011) and Wolitzky (2011). None of the papers discussed in the previous three paragraphs study the issues we focus on or make progress towards a general framework of the sort presented here.

Finally, our approach is related to but quite different from the study of games with strategic complementarities (see Milgrom and Roberts, 1990, Vives, 1990, for early contributions, Echenique, 2004, for the relationship of such games to games with unique equilibria, and Chassang, 2010, for games with strategic complementarities and private information). As in this literature, we impose a joint order over players and strategies and utilize an increasing differences assumption. However, crucially, ours is not a game of strategic complementarities, there are no "monotone" comparative statics (as evidenced by the slippery slope considerations discussed in footnote 3 and the type of results already mentioned in footnote 6 above), and the mathematical arguments underlying our results and their logic are very different.

The rest of the paper is organized as follows. In Section 2, we present our general framework and introduce the concept of MVE. Section 3 contains the analysis of MVE. We start with the stationary case (without shocks), then extend the analysis to the general case where shocks are possible, and then compare the concepts of MVE to Markov Perfect Equilibrium in a properly defined dynamic game. We also establish several comparative static results that hold even at this level of generality; this allows us to study the society's reactions to shocks in applied models. Section 4 applies our framework to the study of radical politics and to the problem of institutional experimentation. Section 5 concludes. Appendix A contains some important lemmas and proofs of the main theorems. Appendix B, which is available online, contains additional proofs, several extensions, and examples.

2 General Framework

Time is discrete and infinite, indexed by $t \geq 1$. The society consists of n *players* (representing individuals or groups), $N = \{1, \ldots, n\}$. The set of players is ordered, and the order reflects the initial distribution of some variable of interest. For example, higher-indexed players may be richer, or more pro-authoritarian, or more right-wing on social issues. In each period, the society is in one of the h *environments* $\mathcal{E} = \{E^1, \ldots, E^h\}$, which determine preferences and the distribution of political power in society (as described below). We model stochastic elements by assuming that, at each date, the society transitions from environment E to environment E'

with probability $\pi(E, E')$. Naturally, $\sum_{E' \in \mathcal{E}} \pi(E, E') = 1$. We assume:

Assumption 1 ***(Ordered Transitions)*** *If $1 \leq x < y \leq h$, then*

$$\pi(E^y, E^x) = 0.$$

Assumption 1 implies that there can only be at most a finite number of shocks. It also stipulates that environments are numbered so that only transitions to higher-numbered environments are possible. This numbering convention is without loss of generality and enables us to use the convention that once the last environment, E^h, has been reached, there will be no further stochastic shocks.[9] In what follows, we will call the pair $\left(\mathcal{E} = \{E^1, \ldots, E^h\}, \{\pi_{E,E'}\}_{E,E' \in \mathcal{E}}\right)$ a *stochastic environment*. In other words, a stochastic environment is a collection of environments and transition probabilities such that Assumption 1 is satisfied.

We model preferences and the distribution of political power by means of *states*, belonging to a finite set $S = \{1, \ldots, m\}$.[10] The set of states is *ordered*: loosely speaking, this will generally imply that higher-indexed states provide both *greater economic payoffs* and *more political power* to higher-indexed players. An example would be a situation in which higher-indexed states correspond to less democratic arrangements, which are both economically and politically better for more "elite" groups. The payoff of player $i \in N$ in state $s \in S$ and environment $E \in \mathcal{E}$ is $u_{E,i}(s)$.

To capture relative preferences and power of players in different states, we will frequently make use of the following definition:

Definition 1 ***(Increasing Differences)*** *Vector $\{w_i(s)\}_{i \in A}^{s \in B}$, where $A \subset N$, $B \subset S$, satisfies the* increasing differences *condition if for any agents $i, j \in A$ such that $i > j$ and any states $x, y \in B$ such that $x > y$,*

$$w_i(x) - w_i(y) \geq w_j(x) - w_j(y).$$

The following is one of our key assumptions:

Assumption 2 ***(Increasing Differences in Payoffs)*** *In every environment $E \in \mathcal{E}$, the vector of (stage) payoffs, $\{u_{E,i}(s)\}_{i \in N}^{s \in S}$, satisfies the increasing differences condition.*

[9] In particular, Assumption 1 does not preclude the possibility that the same environment will recur several times. For example, the possibility of q transitions between E^1 and E^2 can be modeled by setting $E^3 = E^1$, $E^4 = E^2$, etc. It also does not mean that the society must reach E^h on every path: for example, it is permissible to have three environments with $\pi(E^1, E^2) = \pi(E^1, E^3) > 0$, and all other transition probabilities equal to zero.

[10] The implicit assumption that the set of states is the same for all environments is without any loss of generality.

Note that payoffs $\{u_{E,i}(s)\}$ are directly assigned to combinations of states and environments. An alternative would be to assign payoffs to some other actions, e.g., "policies", which are then selected endogenously by the same political process that determines transitions between states. This is what we do in Section 4: under fairly weak conditions, the current state will determine the choice of action (policy), so payoffs will then be indirectly defined over states and environments. Here we are thus reducing notation by directly writing them as $\{u_{E,i}(s)\}$.[11]

Assumption 2 is the first of our two most substantive assumptions. It essentially imposes that we can think of political conflict in this society as taking place over a "single-dimensional" issue over which all agents have well-defined preferences. In particular, if we think of this single-dimensional issue as representing a left-right divide, then this assumption implies that agents can also be ordered in terms of their left vs. right stance, and as we go to more right-wing agents, they obtain increasingly greater additional utility from the implementation of policies further to the right. Though restrictive, this is exactly the type of assumption that is employed in the majority of static models of political economy in order to obtain general existence and characterization results (e.g., Austen-Smith and Banks, 2000). Technically, it is a key input into the following result: despite the fact that agents care not only about a single policy but about the entire future sequence of policies, they can still be ranked from left to right, and any move to a further right state that is preferred by an agent will also be preferred by all agents to his right.

We model the distribution of political power in a state flexibly using the notion of *winning coalitions*. This captures information on which subsets of agents have the (political) power to implement economic or political change, here corresponding to a transition from one state to another. We denote the set of winning coalitions in state s and environment E by $W_{E,s}$, and impose the following standard assumption:

Assumption 3 ***(Winning Coalitions)*** *For environment $E \in \mathcal{E}$ and state $s \in S$, the set of winning coalitions $W_{E,s}$ satisfies:*

1. *(monotonicity) if $X \subset Y \subset N$ and $X \in W_{E,s}$, then $Y \in W_{E,s}$;*

2. *(properness) if $X \in W_{E,s}$, then $N \setminus X \notin W_{E,s}$;*

[11] This in principle allows for a setup where the group in power chooses a different policy than its bliss point because of some (endogenous) constraints, such as the "no revolution constraint" in Acemoglu and Robinson (2000a, 2006). We do not explicitly discuss this possibility to keep the exposition focused.

3. (decisiveness) $W_{E,s} \neq \varnothing$.

The first part of Assumption 3 states that if some coalition has the capacity to implement change, then a larger coalition also does. The second part ensures that if some coalition has the capacity to implement change, then the coalition of the remaining players, its complement, does not (effectively ruling out "submajority rule"). Finally, the third part, in light of the monotonicity property, is equivalent to $N \in W_{E,s}$, and thus states that if all players want to implement a change, they can do so. Several common models of political power are special cases. For example, if a player is a dictator in some state, then the winning coalitions in that state are all those that include him; if we need unanimity for transitions, then the only winning coalition is N; if there is majoritarian voting in some state, then the set of winning coalitions consists of all coalitions with an absolute majority of the players.

Assumption 3 puts minimal and natural restrictions on the set of winning coalitions $W_{E,s}$ in each given state $s \in S$. Our main restriction on the distribution of political power will be, as discussed in the Introduction, the requirement of some "monotonicity" of political power — that higher-indexed players have no less political power in higher-indexed states. We first introduce the notion of a quasi-median voter (see Acemoglu, Egorov, and Sonin, 2012).

Definition 2 ***(Quasi-Median Voter, QMV)*** *Player ranked i is a* quasi-median voter *(QMV) in state s (in environment E) if for any winning coalition $X \in W_{E,s}$,* $\min X \leq i \leq \max X$.

Let $M_{E,s}$ denote the set of QMVs in state s in environment E. Then by Assumption 3, $M_{E,s} \neq \varnothing$ for any $s \in S$ and $E \in \mathcal{E}$; moreover, the set $M_{E,s}$ is connected: whenever $i < j < k$ and $i, k \in M_{E,s}$, $j \in M_{E,s}$. In many cases, the set of QMVs is a singleton, $|M_{E,s}| = 1$. Examples include: one player as a dictator, i.e., $X \in W_{E,s}$ if and only if $i \in X$ (and then $M_{E,s} = \{i\}$), majoritarian voting among sets containing odd numbers of players, or a weighted majority in voting with "generic weights" (see the discussion below and also Section 4 on the meaning of this term). An example where $M_{E,s}$ is not a singleton is the unanimity rule.

The following assumption ensures that the distribution of political power is "monotone" over states.

Assumption 4 ***(Monotone Quasi-Median Voter Property)*** *In any environment $E \in \mathcal{E}$, the sequences $\{\min M_{E,s}\}_{s \in S}$ and $\{\max M_{E,s}\}_{s \in S}$ are nondecreasing in s.*

The essence of Assumption 4 is that political power (weakly) shifts towards higher-indexed players in higher-indexed states. To see this, we can rewrite $\min M_{E,s} = \max_{X \in W_{E,s}} \min_{i \in X} i$. Thus $\min M_{E,s}$ corresponds to the minimal ("critical") left-wing agent whose support is needed to get a winning coalition. Assumption 2 implies that if $\min M_{E,s}$ supports a change from s to some $s' > s$, then all agents in $M_{E,s}$ will also do so. Similarly, $\max M_{E,s} = \min_{X \in W_{E,s}} \max_{i \in X} i$ is the minimal right-wing agent needed for a winning coalition, and if $\max M_{E,s}$ supports a change from s to some $s' < s$, then all others in $M_{E,s}$ will also do so.[12] Clearly, if $M_{E,s}$ is a singleton in every state, this assumption is equivalent to $M_{E,s}$ being nondecreasing (where $M_{E,s}$ is treated as the single element). Therefore, in words, Monotone QMV property says that higher states are more likely to include right-wing players and less likely to include left-wing players in a winning coalition — thus shifting political power towards right-wing players in states that are further to the right.[13]

Assumption 4 is our second key assumption. To see its role, observe that Assumption 2 alone is not sufficient to ensure that dynamic preferences satisfy single crossing (or increasing differences)—that an agent necessarily prefers states further to the right if a more left-wing agent does so. This is because even though her stage payoff is greater in this state, her political power may be lower, leading to a significantly lower continuation utility. Assumption 4 rules this possibility out as it imposes that this right-wing agent may only lose power to agents further to the right —thus implying that agents further to the left will lose even more political power. Hence, Assumptions 2 and 4 jointly ensure that if an agent prefers to move to a state to the right, then all agents to her right will also do so, thus implying that dynamic preferences satisfy increasing differences.

For some applications, one might want to restrict feasible transitions between states that the society may implement; for example, it might be realistic to assume that only transitions to adjacent states are possible. To incorporate these possibilities, we introduce the mapping $F = F_E : S \to 2^S$, which maps every $x \in S$ into the set of states to which society may transition. In other words, $y \in F_E(x)$ means that the society may transition from x to y in environment E. We do not assume that $y \in F_E(x)$ implies $x \in F_E(y)$, so certain transitions may be irreversible. We impose:

Assumption 5 ***(Feasible Transitions)*** *For each environment $E \in \mathcal{E}$, F_E satisfies:*

[12] We thank an anonymous referee for this helpful intuition.

[13] This assumption holds in a variety of applications, including the ones we present in Section 4 and Roberts's (1999) model mentioned in the Introduction.

1. *For any $x \in S$, $x \in F_E(x)$;*

2. *For any states $x, y, z \in S$ such that $x < y < z$ or $x > y > z$: If $z \in F_E(x)$, then $y \in F_E(x)$ and $z \in F_E(y)$.*

The key requirement, encapsulated in the second part, is that if a transition between two states is feasible, then any transition (in the same direction) between intermediate states is also feasible. Special cases of this assumption include: (a) any transition is possible: $F_E(x) = S$ for any x and E; (b) *one-step transitions*: $y \in F_E(x)$ if and only if $|x - y| \leq 1$; (c) one-directional transitions: $y \in F_E(x)$ if and only if $x \leq y$.[14]

Finally, we assume that the discount factor, $\beta \in [0, 1)$, is the same for all players and across all environments. To recap, the full description of each environment $E \in \mathcal{E}$ is given by a tuple $\left(N, S, \beta, \{u_{E,i}(s)\}_{i \in N}^{s \in S}, \{W_{E,s}\}_{s \in S}, \{F_E(s)\}_{s \in S}\right)$.

Each period t starts with environment $E_{t-1} \in \mathcal{E}$ and with state s_{t-1} inherited from the previous period; Nature determines E_t with probability distribution $\pi(E_{t-1}, \cdot)$, and then the players decide on the transition to any feasible s_t as we describe next.[15] We take $E_0 \in \mathcal{E}$ and $s_0 \in S$ as given. At the end of period t, each player receives the stage payoff

$$v_i^t = u_{E_t,i}(s_t). \tag{1}$$

Denoting the expectation at time t by $\mathbb{E}_t$, the expected discounted utility of player i can be written as

$$V_i^t = u_{E_t,i}(s_t) + \mathbb{E}_t \sum\nolimits_{k=1}^{\infty} \beta^k u_{E_{t+k},i}(s_{t+k}).$$

The timing of events within each period is:

1. The environment E_{t-1} and state s_{t-1} are inherited from period $t-1$.

2. There is a change in environment from E_{t-1} to $E_t \in \mathcal{E}$ with probability $\pi(E_{t-1}, E_t)$.

3. Society (collectively) decides on state s_t, subject to $s_t \in F_{E_t}(s_{t-1})$.

4. Each player gets stage payoff v_i^t given by (1).

[14] In an earlier version, we also allowed for costs of transitions between states, which we now omit to simplify the exposition.

[15] Throughout the paper, we use lower indices, e.g., E_t, to denote the period, and upper indices, e.g., $E^1, \ldots, E^h$, to denote different environments.

We omit the exact sequence of moves determining transitions across states (in step 3) as this is not required for the Markov Voting Equilibrium (MVE) concept. The details of the game form will be introduced when we study the noncooperative foundations of MVE.[16]

MVE will be characterized by a collection of transition mappings $\phi = \{\phi_E : S \to S\}_{E \in \mathcal{E}}$. With ϕ, we associate continuation payoffs $V^{\phi}_{E,i}(s)$ for player i in state s and environment E, which are recursively given by

$$V^{\phi}_{E,i}(s) = u_{E,i}(s) + \beta \sum_{E' \in \mathcal{E}} \pi(E, E') V^{\phi}_{E',i}(\phi_{E'}(s)). \tag{2}$$

As $0 \leq \beta < 1$, the values $V^{\phi}_{E,i}(s)$ are uniquely defined by (2).

Definition 3 ***(Markov Voting Equilibrium, MVE)*** *A collection of transition mappings* $\phi = \{\phi_E : S \to S\}_{E \in \mathcal{E}}$ *is a* Markov Voting Equilibrium *if the following three properties hold:*

1. ***(feasibility)*** *for any environment* $E \in \mathcal{E}$ *and for any state* $x \in S$, $\phi_E(x) \in F_E(x)$*;*

2. ***(core)*** *for any environment* $E \in \mathcal{E}$ *and for any states* $x, y \in S$ *such that* $y \in F_E(x)$*,*

$$\left\{i \in N : V^{\phi}_{E,i}(y) > V^{\phi}_{E,i}(\phi_E(x))\right\} \notin W_{E,x}; \tag{3}$$

3. ***(status quo persistence)*** *for any environment* $E \in \mathcal{E}$ *and for any state* $x \in S$*,*

$$\left\{i \in N : V^{\phi}_{E,i}(\phi_E(x)) \geq V^{\phi}_{E,i}(x)\right\} \in W_{E,x}.$$

Property 1 requires that MVE involves only feasible transitions (in the current environment). Property 2 is satisfied if no (feasible) alternative $y \neq \phi(x)$ is supported by a winning coalition in x over $\phi_E(x)$ prescribed by the transition mapping ϕ_E. This is analogous to a "core" property: no alternative should be preferred to the proposed transition by some "sufficiently powerful" coalition of players; otherwise, the proposed transition would be blocked. Of course, in this comparison, players should focus on continuation utilities, which is what (3) imposes. Property 3 requires that it takes a winning coalition to move from any state to some alternative —i.e., to move away from the status quo. This requirement singles out the status quo if there is no alternative strictly preferred by some winning coalition.

[16] In what follows, we use MVE both for the singular (Markov Voting Equilibrium) and plural (Markov Voting Equilibria).

Definition 4 ***(Monotone Transition Mappings)*** *A transition mapping $\phi : S \to S$ is called* monotone *if for all $x, y \in S$ such that $x \geq y$, we have $\phi(x) \geq \phi(y)$. A set of transition mappings $\{\phi_E : S \to S\}_{E \in \mathcal{E}}$ is* monotone *if each mapping ϕ_E is monotone.*

We prove that there always exists a monotone MVE (an MVE with a monotone transition mapping), and we can provide sufficient conditions under which all MVE are monotone. In particular, whenever the MVE is unique (Theorem 2), it is monotone.

In what follows, we refer to any state x such that $\phi_E(x) = x$ as a *steady state* or *stable* in E. With some abuse of notation, we will often suppress the reference to the environment and use, e.g., $u_i(s)$ instead of $u_{E,i}(s)$ or ϕ instead of ϕ_E, when this causes no confusion.

Throughout the paper, we say that a property holds *generically*, if it holds for all parameter values, except possibly for a subset of Lebesgue measure zero (see Halmos, 1974, and Example 3).[17] Loosely speaking, a property holds generically if, whenever it does not hold, "almost all" perturbations of the relevant parameters restore it.

3 Analysis

In this section, we analyze the structure of MVE. We first prove existence of monotone MVE in a stationary (deterministic) environment. We then extend these results to situations in which there are stochastic shocks. After establishing the relationship between MVE and Markov Perfect Equilibria (MPE) of a dynamic game representing the framework of Section 2, we present comparative static results for our general model.

3.1 Nonstochastic environment

We first study the case without any stochastic shocks, or, equivalently, the case of only one environment ($|\mathcal{E}| = 1$) and thus suppress the subscript E.

For any mapping $\phi : S \to S$, the continuation utility of player i after a transition to s has

[17] The key feature that genericity ensures for us is the following: For any agent i and any set of mappings $\{\phi_E : S \to S\}_{E \in \mathcal{E}}$, we have that *generically* the continuation values that solve (2) satisfy $V^{\phi}_{E,i}(x) \neq V^{\phi}_{E,i}(y)$ for any $E \in \mathcal{E}$ and any $x, y \in S$ with $x \neq y$. In other words, $V^{\phi}_{E,i}(x) = V^{\phi}_{E,i}(y)$ can only be true for a non-generic set of parameter values. That this property holds generically is established in the proof of Theorem 2. Here it suffices to note that for any discount factor $\beta > 0$ and any transition probabilities $\{\pi(E, E')\}_{E,E' \in \mathcal{E}}$, $V^{\phi}_{E,i}(x)$ and $V^{\phi}_{E,i}(y)$ are given by different linears combination of the payoffs $\{u_{E,i}(s)\}^{E \in \mathcal{E}}_{s \in S}$. Thus $V^{\phi}_{E,i}(x) = V^{\phi}_{E,i}(y)$ can only hold for a set of parameters given by the union of a finite number of hyperplanes, which has Lebesgue measure zero in the set of feasible payoffs $\{u_{E,i}(s)\}^{E \in \mathcal{E}}_{s \in S}$. This then immediately implies that the set of all parameters, (β, $\{\pi(E, E')\}_{E,E' \in \mathcal{E}}$, $\{u_{E,i}(s)\}^{E \in \mathcal{E}}_{s \in S}$) for which $V^{\phi}_{E,i}(x) = V^{\phi}_{E,i}(y)$ is also of Lebesgue measure zero.

taken place is given by

$$V_i^{\phi}(s) = u_i(s) + \sum\nolimits_{k=1}^{\infty} \beta^k u_i\left(\phi^k(s)\right), \tag{4}$$

where ϕ^k is the k^{th} iteration of ϕ (with $\phi^0(s) = s$).

The critical role of Assumption 2 in our analysis can be seen from a simple but important observation (see Lemma 2 in Appendix A): when Assumption 2 holds and ϕ is monotone, continuation utilities $\left\{V_i^{\phi}(s)\right\}_{i\in N}^{s\in S}$ satisfy increasing differences. This result is at the root of the central role of QMVs in our model. As is well known, median voter type results do not generally apply with multidimensional policy choices. Since our players are effectively voting over infinite dimensional choices, i.e., a sequence of policies, a natural conjecture would have been that such results would not apply in our setting either. The reason they do has a similar intuition to why voting sequentially over two dimensions of policy, over each of which preferences satisfy single crossing or increasing differences, does lead to the median voter-type outcomes. By backward induction, the second vote has a well-defined median voter, and then given this choice, the median voter over the first one can be determined. Loosely speaking, our recursive formulation of today's value enables us to apply this reasoning between the vote today and the vote tomorrow, and the fact that continuation utilities satisfy increasing differences is the critical step in this argument.

The role of Assumption 4, in turn, is related to the monotonicity of ϕ. That political power shifts to the right in states that are further to the right ensures that ϕ is monotone. This together with the observation on continuation utility satisfying increasing differences under the monotonicity of ϕ enables us to establish the following theorem.[18]

Theorem 1 ***(Existence)*** *There exists a monotone MVE. Moreover, in any MVE ϕ the equilibrium path $s_0, s_1 = \phi(s_1), s_2 = \phi(s_2), \ldots$ is monotone, and there exists a limit state $s_\tau = s_{\tau+1} = \cdots = s_\infty$.*

The next theorem provides sufficient conditions for generic uniqueness of monotone MVE. We say that preferences are *single-peaked* if for every $i \in N$ there exists $x \in S$ such that whenever for states $y, z \in S$, $z < y \leq x$ or $z > y \geq x$, $u_i(z) < u_i(y)$.

Theorem 2 ***(Uniqueness)*** *The MVE is (generically) unique if*

[18] The actual technical argument is more involved and makes use of several key lemmas, stated and proved in Appendix A, where the proof of all our main theorems are presented.

1. *for every $s \in S$, M_s is a singleton;* ***or***

2. *only one-step transitions are possible and preferences are single-peaked.*

Though somewhat restrictive, several interesting applied problems satisfy one or the other parts of the conditions of this theorem. Since Theorem 1 established existence of a monotone MVE, under the conditions of Theorem 2, the unique MVE is monotone.

Neither the conditions nor the genericity provision in Theorem 2 can be dispensed with as shown by the next two examples.

Example 2 (Example with two MVE) Suppose that there are three states A, B, C, and two players 1 and 2. The decision-making rule is unanimity in all states. Payoffs are given by

id	A	B	C
1	20	5	10
2	10	5	20

Then, with β sufficiently close to 1 (e.g., $\beta = 0.9$), there are two MVE, both of which are monotone. In one, $\phi_1(A) = \phi_2(B) = A$ and $\phi_1(C) = C$. In another, $\phi_2(A) = A$, $\phi_2(B) = \phi_2(C) = C$.

In view of Theorem 2, multiple equilibria arise here because preferences are not single-peaked, and there is more than one QMV in all states. Example B1 in Appendix B shows that making preferences single-peaked is by itself insufficient to restore uniqueness.

Example 3 (Multiple MVE for non-generic utilities) There are two states A and B and two players 1 and 2. Player 1 is the dictator in both states. Payoffs are given by

id	A	B
1	20	20
2	15	25

For any discount factor β, there exist three equilibria: two monotone MVE (given by $\phi_1(A) = \phi_1(B) = A$ and $\phi_2(A) = \phi_2(B) = B$) and a non-monotone (in fact, cyclic) MVE ϕ_3 given by $\phi_3(A) = B$ and $\phi_3(B) = A$. However, any perturbation of the payoffs of player 1 removes the non-monotone equilibrium and one of the monotone ones, restoring uniqueness.

3.2 Stochastic environments

We now extend our analysis to *stochastic environments*, that is, to the case where there are stochastic shocks closing changes in environments.[19] This will enable us to deal with "non-

[19] Formally, a stochastic environment is a collection of environments and transition probabilities, $\mathcal{E} = (\{E^1, \dots, E^h\}, \{\pi_{E,E'}\}_{E,E' \in \mathcal{E}})$ such that Assumption 1 is satisfied for each environment E^k, $k = 1, \dots, h$.

stationarities" in the economic environment, for example, because the distribution of political power or economic preferences will change in a specific direction in the future. By Assumption 1, environments are ordered as $E^1, E^2, \ldots, E^h$ so that $\pi(E^x, E^y) = 0$ if $x > y$. This means that when (and if) we reach environment E^h, there will be no further shocks, and the analysis from Section 3.1 will apply thereafter.

Our approach uses backward induction from environment E^h to characterize equilibrium transition mappings in lower-indexed environments. Here we outline this argument heuristically. Take an MVE ϕ_{E^h} in environment E^h (its existence is guaranteed by Theorem 1). Suppose that we have characterized an MVE $\{\phi_E\}_{E\in\{E^{k+1},\ldots,E^h\}}$ for some $k = 1, \ldots, h-1$; let us construct ϕ_{E^k} which would make $\{\phi_E\}_{E\in\{E^k,\ldots,E^h\}}$ an MVE in $\{E^k, \ldots, E^h\}$. Continuation utilities in environment E^k are:

$$\begin{aligned} V^{\phi}_{E^k,i}(s) &= u_{E^k,i}(s) + \beta \sum_{E'\in\{E^k,\ldots,E^h\}} \pi\left(E^k, E'\right) V^{\phi}_{E',i}\left(\phi_{E'}(s)\right) \\ &= u_{E^k,i}(s) + \beta \sum_{E'\in\{E^{k+1},\ldots,E^h\}} \pi\left(E^k, E'\right) V^{\phi}_{E',i}\left(\phi_{E'}(s)\right) \\ &\quad + \beta\pi\left(E^k, E^k\right) V^{\phi}_{E^k,i}\left(\phi_{E^k}(s)\right). \end{aligned} \tag{5}$$

By induction, we know $\phi_{E'}$ and $V^{\phi}_{E'}(\phi_{E'}(s))$ for $E' \in \{E^{k+1}, \ldots, E^h\}$. We next show that there exists ϕ_{E^k} that is an MVE given continuation values $\left\{V^{\phi}_{E^k,i}(s)\right\}_{s\in S}$ from (5). Denote

$$\tilde{U}_{E^k,i}(s) = u_{E^k,i}(s) + \beta \sum_{E'\in\{E^{k+1},\ldots,E^h\}} \pi\left(E^k, E'\right) V^{\phi}_{E',j}\left(\phi_{E'}(s)\right), \tag{6}$$

and let $\tilde{\beta} = \beta\pi\left(E^k, E^k\right)$.[20] Then rearranging equation (5), where

$$V^{\phi}_{E^k,i}(s) = \tilde{U}_{E^k,i}(s) + \tilde{\beta} V^{\phi}_{E^k,i}\left(\phi_{E^k}(s)\right).$$

Since $\left\{\tilde{U}_{E^k,i}(s)\right\}^{s\in S}_{i\in N}$ satisfy increasing differences, we can simply apply Theorem 1 to the modified environment $\tilde{E} = \left(N, S, \tilde{\beta}, \left\{\tilde{U}_{E^k,i}(s)\right\}^{s\in S}_{i\in N}, \{W_{E^k,s}\}_{s\in S}, \{F_{E^k}(s)\}_{s\in S}\right)$ to characterize ϕ_{E^k}. Then by definition of MVE, since $\{\phi_E\}_{E\in\{E^{k+1},\ldots,E^h\}}$ was an MVE, we have that

We use the term stationary environment when we wish to stress the distinction from a stochastic environment.

[20] Intuitively, $\tilde{U}_{E^k,i}(s)$ is the expected utility of agent i from staying in state s as long as the environment remains the same, and following the MVE play thereafter (i.e., after a change in environment). The continuation utility from such path is therefore

$$\tilde{V}_i(s) = u_{E^k,i}(s) + \beta \sum_{E'\in\{E^{k+1},\ldots,E^h\}} \pi(E^k, E') V^{\phi}_{E',i}(\phi_{E'}(s)) + \beta\pi(E^k, E^k)\tilde{V}_i(s),$$

and thus $\tilde{U}_{E^k,i}(s) = (1-\tilde{\beta})\tilde{V}_i(s)$.

$\{\phi_E\}_{E\in\{E^k,\ldots,E^h\}}$ is an MVE in $\mathcal{E}^k$, proving the desired result. Proceeding inductively, we characterize an entire MVE $\phi = \{\phi_E\}_{E\in\{E^1,\ldots,E^h\}}$ in $\mathcal{E}^1 = \mathcal{E}$. This argument establishes:[21]

Theorem 3 ***(Existence)*** *There exists an MVE $\phi = \{\phi_E\}_{E\in\mathcal{E}}$. Furthermore, there exists a limit state $s_\tau = s_{\tau+1} = \cdots = s_\infty$ (with probability 1) but this limit state depends on the timing and realization of stochastic shocks and the path to a limit state need not be monotone everywhere.*

This theorem establishes that a limit state exists, and more importantly, this limit state (and the resulting equilibrium path) generally depends on the exact timing and sequence of shocks. The path to the limit state need not be monotone everywhere, but we show below (Theorem 8) that it is monotone between shocks, i.e., it is monotone in any time interval in which there are no shocks. The following theorem provides sufficient conditions for uniqueness in the stochastic case.[22]

Theorem 4 ***(Uniqueness)*** *The MVE is (generically) unique if at least one of the following conditions holds:*

1. *for every environment $E \in \mathcal{E}$ and any state $s \in S$, $M_{E,s}$ is a singleton;*

2. *in each environment, only one-step transitions are possible; each player's preferences are single-peaked; and moreover, for each state s there is a player i such that $i \in M_{E,s}$ for all $E \in \mathcal{E}$ and the peaks (for all $E \in \mathcal{E}$) of i's preferences do not lie on different sides of s.*

The first sufficient condition is the same as in Theorem 2, while the second strengthens the one in Theorem 2: it would be satisfied, for example, if players' bliss points (most preferred state) and the distribution of political power does not change "much" as a result of shocks. Uniqueness of MVE again implies that this MVE is monotone.

3.3 Noncooperative game

We have so far presented the concept of MVE without introducing an explicit noncooperative game. This is partly motivated by the fact that several plausible noncooperative games would underpin the notional MVE. We now provide one plausible and transparent noncooperative

[21] The proof is again in Appendix A. In addition, Example B2 in Appendix B shows that the limits state does depend on the realization of shocks.

[22] The difficulty here is that as shown, for instance, by Example B3 in Appendix B, single-peakedness is not necessarily inherited by continuation utilities.

game and formally establish the relationship between the Markov Perfect Equilibria (MPE) of this game and the set of MVE.

For each environment $E \in \mathcal{E}$ and state $s \in S$, let us introduce a *protocol* $\theta_{E,s}$, which is a finite sequence of all states in $F_s \setminus \{s\}$ capturing the order in which different transitions are considered within the period. Then the exact sequence of events in this noncooperative game is:

1. The environment E_{t-1} and state s_{t-1} are inherited from period $t-1$.

2. Environment transitions are realized: $E_t = E \in \mathcal{E}$ with probability $\pi(E_{t-1}, E)$.

3. The first alternative, $\theta_{E_t,s_{t-1}}(j)$ for $j = 1$, is voted against the status quo s. That is, all players are ordered in a sequence and must support either the "current proposal" $\theta_{E_t,s_{t-1}}(j)$ or the status quo s.[23] If the set of those who supported $\theta_{E_t,s_{t-1}}(j)$ is a winning coalition — i.e., it is in $W_{E_t,s_{t-1}}$— then $s_t = \theta_{E_t,s_{t-1}}(j)$; otherwise, this step repeats for the next j. If all alternatives have been voted and rejected for $j = 1, \ldots, |F_s| - 1$, then the new state is $s_t = s_{t-1}$.

4. Each player gets stage payoff given by (1).

We study (pure-strategy) MPE of this game. Each MPE induces (an equilibrium behavior which can be represented by) a set of transition mappings $\phi = \{\phi_E\}_{E\in\mathcal{E}}$. Here $\phi_E(s)$ is the state to which the equilibrium play transitions starting with state s in environment E.

Theorem 5 ***(MVE vs. MPE)***

1. *For any MVE ϕ, there exists a set of protocols $\{\theta_{E,s}\}_{E\in\mathcal{E}}^{s\in S}$ such that there exists a MPE which induces ϕ.*

2. *Conversely, if for some set of protocols $\{\theta_{E,s}\}_{E\in\mathcal{E}}^{s\in S}$ and some MPE σ, the corresponding transition mapping $\phi = \{\phi_E\}_{E\in\mathcal{E}}$ is monotone, then it is an MVE.*

This theorem thus establishes the close connection between MVE and MPE. Essentially, any MVE corresponds to an MPE (for some protocol), and conversely, any MPE corresponds to an MVE, provided that this MPE induces monotone transitions.

[23] To avoid the usual multiplicity problems with equilibria in voting games, we assume sequential voting for some fixed sequence of players. See Acemoglu, Egorov, and Sonin (2009) for a solution concept which would refine out unnatural equilibria in voting games with simultaneous voting.

3.4 Comparative statics

In this subsection, we present a general comparative static result. Throughout, we assume that parameter values are generic and all MVEs are unique (e.g., the sufficient conditions for uniqueness in Theorem 4 are satisfied).

We say that environments E^1 and $\tilde{E}^1$ *coincide on* $S' \subset S$, if for each $i \in N$ and for any state $x \in S'$, we have $u_{E^1,i}(x) = u_{\tilde{E}^1,i}(x)$, $W_{E^1,x} = W_{\tilde{E}^1,x}$, $F_{E^1}|_{S'} = F_{\tilde{E}^1}|_{S'}$ (in the sense that for $x, y \in S'$, $y \in F_{E^1}(x) \Leftrightarrow y \in F_{\tilde{E}^1}(x)$). The next result shows that there is a simple way of characterizing the equilibrium transition mapping of one environment at the steady state of the other. For this result, we will assume that MVE is unique (e.g., the assumptions of Theorem 4 are satisfied for all subsets $S' \subset S$).[24]

Theorem 6 ***(General Comparative Statics)*** *Suppose that environments E^1 and $\tilde{E}^1$ coincide on $S' = [1, s] \subset S$ and that there is a unique MVE in both environments. For MVE ϕ_{E^1} in E^1, suppose that $\phi_{E^1}(x) = x$ for some $x \in S'$. Then for MVE $\tilde{\phi}_{\tilde{E}^1}$ in $\tilde{E}_1$ we have $\tilde{\phi}_{E^1}(x) \geq x$.*

The theorem says that if x is a steady state in environment E^1 and environments E^1 and $\tilde{E}^1$ coincide on a subset of states $[1, s]$ that includes x, then the MVE in $\tilde{E}^1$ will either stay at x or induce a transition to a greater state than x. Of course, the two environments can be swapped: if $y \in S'$ is such that $\tilde{\phi}_{\tilde{E}^1}(y) = y$, then $\phi_{E^1}(y) \geq y$. Moreover, since the ordering of states can be reversed, a similar result applies when $S' = [s, m]$ rather than $[1, s]$.

The intuition for Theorem 6 is instructive. The fact that $\phi_{E^1}(x) = x$ implies that in environment E^1, there is no winning coalition wishing to move from x to $y < x$. But when restricted to S', economic payoffs and the distribution of political power are the same in environment $\tilde{E}^1$ as in E^1, so in environment $\tilde{E}^1$ there will also be no winning coalition supporting the move to $y < x$. This implies $\tilde{\phi}_{\tilde{E}^1}(x) \geq x$. Note, however, that $\tilde{\phi}_{\tilde{E}^1}(x) > x$ is possible even though $\phi_{E^1}(x) = x$, since the differences in economic payoffs or distribution of political power in states outside S' may make a move to higher states more attractive for some winning coalition in $\tilde{E}^1$. Interestingly, since the difference between two environments outside S' is left totally unrestricted,

[24] The theorem immediately extends to the case where we consider two stochastic environments $\mathcal{E} = \{E^1, \dots, E^h\}$ and $\tilde{\mathcal{E}} = \left\{\tilde{E}^1, E^2, \dots, E^h\right\}$ (i.e., with only the initial environments being different), and assume that $\pi^{\mathcal{E}}\left(E^1, E^k\right) = \pi^{\tilde{\mathcal{E}}}\left(\tilde{E}^1, E^k\right)$ for any $k > 1$ and $\pi^{\mathcal{E}}(E, E') = \pi^{\tilde{\mathcal{E}}}(E, E')$ for any $E, E' \in \{E^2, \dots, E^h\}$.

A similar result can also be established without uniqueness. For example, one can show that if for some $x \in S'$, for each MVE ϕ_{E^1} in E^1, $\phi_{E^1}(x) \geq x$, with at least one MVE ϕ such that $\phi_{E^1}(x) = x$, then all MVE $\tilde{\phi}_{\tilde{E}^1}$ in E^1 satisfy $\tilde{\phi}_{\tilde{E}^1}(x) \geq x$. Because both the statements of these results and the proofs are more involved, we focus here on situations in which MVE are unique.

this last possibility can happen even if in environment $\tilde{E}^1$ payoffs outside S' are *lower* for all players (this could be, for example, because even though all players' payoffs decline outside S', this change also removes some "slippery slope" previously discouraging a winning coalition from moving to some state $z > x$).

Theorem 6 compares MVE in two distinct environments E^1 and $\tilde{E}^1$ (or two distinct stochastic environments, $\mathcal{E}$ and $\tilde{\mathcal{E}}$, as noted in footnote 24). In this sense, we can think of it as a comparative static with respect to an unanticipated shock (taking us from one environment to the other). The next corollary states a similar result when there is a stochastic transition from one environment to another.

Corollary 1 *Suppose that $\mathcal{E} = \{E^1, E^2\}$, E^1 and E^2 coincide on $S' = [1, s] \subset S$, and the MVE is unique in both environments. Suppose also that for MVE $\phi = (\phi_{E^1}, \phi_{E^2})$ in $\mathcal{E}$ and some $x \in S'$, $\phi_{E^1}(x) = x$, and this state x is reached before a switch from environment E^1 to E^2 occurs at time t. Then ϕ_{E^2} is such that that along the equilibrium path in environment E^2, we have $s_\tau \geq x$ for all $\tau \geq t$.*

This corollary states that if steady state x is reached before a shock changes the environment — in a way that only higher states are affected as a result of this change in environment — then the equilibrium after the change can only move society further towards the direction where the shock happened (or stay where it was); in particular, the equilibrium will never involve moving back to a lower state than x. A straightforward implication is that the only way society can stay in the set of states $[1, x-1]$ is not to leave the set before the shock arrives.

An interesting application of this corollary is when we consider x as a "minimal sufficiently democratic state"; states to the right of x as further refinements of democracy; and environment E^2 as representing (the strengthening of) a threat to democracy. Then the corollary implies that this threat to democracy may disrupt the emergence of this minimal democracy if it arrives early. But if it arrives late, after this minimal democratic state — which thus can be considered as a "democratic threshold" — has already been reached, it would not create a reversal. Interestingly, and perhaps paradoxically, Corollary 1 implies that such a threat, if it arrives late, may act as an impetus for additional transitions in a further democratic direction, even though it would have prevented the emergence of this minimum democratic state had it arrived early.

Example B4 in Appendix B below demonstrates that the requirement that E^1 and E^2 coincide for some states cannot be dispensed with, in part because when this assumption is relaxed, slippery slope considerations can lead to counter-intuitive dynamics.

Futher comparative statics results are also provided in Appendix B. First, we show in Theorem B1 that when the discount factor is sufficiently low and two environments coincide on a subset of states, the equilibrium path is monotone everywhere (i.e., it does not change direction even as shocks arrive), and as a result, equilibrium paths with and without shocks can be ranked.

In Theorem B2, we show that if the sets of winning coalitions in some states to the right ($x > s$) change such that the sets of QMVs expand further towards the right (for example, because some players on the right become additional veto players), then the transition mapping is unaffected for states on the left that are not directly affected by the change (i.e., $x < s$). Applied to the dynamics of democratization, this theorem implies that an absolute monarch's decision of whether to move to a constitutional monarchy is not affected by the power that the poor will be able to secure in this new regime provided that the monarch himself still remains a veto player.

3.5 Monotone vs non-monotone MVE

We have so far focused on monotone MVE. In many interesting cases this is without loss of generatlity, as the following theorem establishes.

Theorem 7 ***(Monotonicity of MVE)*** *All MVE are generically monotone if*

1. *in all environments, the sets of QMVs in two different states have either zero or exactly one player in common: for all* $E \in \mathcal{E}, x, y \in S : x \neq y \Rightarrow |M_{E,x} \cap M_{E,y}| \leq 1$*,* ***or***

2. *in all environments, only one-step transitions are possible.*

The first part of the theorem weakens the first condition in Theorems 2 and 4 that the set of QMVs in each state is a singleton, while the second part only requires that there are one-step transitions (relative to the stronger conditions in these previous theorems). As a result, the conditions in Theorem 7 are strictly weaker than those in Theorem 2 and 4.

Example B5 in Appendix B shows that both conditions in Theorem 7 cannot be simultaneously dispensed with.

Our last result in this section shows that even if non-monotone MVE exist, they will still induce paths that are monotone except for possible changes in direction due to shocks. In particular, we say that mapping $\phi = \{\phi_E\}_{E \in \mathcal{E}}$ induces paths that are *monotone between shocks* if for any $E \in \mathcal{E}$ and $x \in S$, $\phi_E(x) \geq x$ implies $\phi_E^2(x) \geq \phi_E(x)$; in other words, the MVE generates paths that are monotone so long the environment does not change due to an exogenous shock. The next theorem shows that all equilibrium paths are monotone between shocks.

Theorem 8 ***(Monotone Paths)*** *Any MVE ϕ (not necessarily monotone) generically induces paths that are monotone between shocks.*

3.6 Infinitely many environments

Our analysis so far has been conducted under the assumption of a finite number of environments, which greatly simplified the analysis and enabled sharp results. Here, we show that a monotone MVE exists even with infinitely many environments (shocks). In particular, we assume that we have countably many environments $\mathcal{E} = \{E^1, E^2, \ldots\}$ with transition probabilities $\pi(E, E')$ and such that Assumption 1 holds (and the Assumptions 2-5 for each E^i). The proof of this theorem, like those of all remaining results in the paper, is provided in Appendix B.

Theorem 9 ***(Existence with Infinitely Many Environments)*** *Suppose that utilities are bounded in all environments (i.e., there exists $M > 0$ such that for every $E \in \mathcal{E}$, $s \in S$ and $i \in N$, $|u_{E,i}(s)| < M$). Then there exists a monotone MVE.*

4 Applications

In this section, we discuss two applications of our general framework. The first one, on radical politics, is the most detailed. We then discuss a model of experimentation over institutions.

4.1 Radical politics

In this subsection, we apply our general framework to the study of radical politics, already briefly introduced in Example 1 in the Introduction. We first describe the initial environment, E^1. There is a fixed set of n players $N = \{-l, \ldots, r\}$ (so $n = l + r + 1$), which we interpret as groups of individuals with the same preferences (e.g., ethnicities, economic interests or ideological groupings) that have already solved their within-group collective action problem.

The weight of each group $i \in N$ is denoted by γ_i and represents, for example, the number of individuals within the group and thus the group's political power. Throughout this subsection, we assume "genericity" of $\{\gamma_i\}$, in the sense that there are no two disjoint combinations of groups with exactly the same weight.[25] Group 0 is chosen such that it contains the median

[25] See Acemoglu, Egorov and Sonin (2008) for an extended discussion of this assumption.

voter. Individuals in group i have preferences (net of repression costs) given by

$$w_i(p) = -(p - b_i)^2,$$

where p is the policy choice of society and b_i is the political bliss point of group i. We assume that $\{b_i\}$ is increasing in i, which ensures that preferences satisfy increasing differences (Assumption 2). For example, those with high index can be interpreted as the "rich" or "right-wing" groups that prefer the pro-rich (pro-right-wing) policy.

As in Example 1, the set of states is $S = \{-l - r, \ldots, l + r\}$, and so the total number of states is $m = 2l + 2r + 1 = 2n - 1$. States correspond to different combinations of political rights. Political rights of certain groups can be reduced by repression (which is potentially costly as described below). The set of groups that are *not* repressed in state s is denoted by H_s, where $H_s = \{-l, \ldots, r + s\}$ for $s \leq 0$ and $H_s = \{-l + s, \ldots, r\}$ for $s > 0$.[26] Only the groups that are not repressed participate in politics. This implies that in state 0, which corresponds to "democracy" (with no repression of any group), group 0 contains the median voter. In states below 0, some groups with right-wing preferences are repressed, and in the leftmost state $s = -l - r$, only the group $-l$ participates in decision-making (hence, all other groups are repressed). Similarly, in states above 0 some of the left-wing groups are repressed (in rightmost state $s = l + r$ only group r has power). This structure ensures that Assumption 4 is satisfied, and we also assume that all transitions across states are feasible, so that Assumption 5 also holds.

Policy p and transitions across states are decided by a simple majority of those individuals who have political rights (i.e., belong to groups that are not repressed). This implies that policy will always be chosen as the political bliss point of the QMV (given political rights), b_{M_s}. Our assumptions so far (in particular, the genericity of $\{\gamma_i\}$) ensure that M_s contains a single group. The cost of repressing agents in group j is denoted by C_j and is assumed to be incurred by all players. So, stage payoffs are given by

$$\begin{aligned} u_i(s) &= w_i(p) - \sum\nolimits_{j \notin H_s} \gamma_j C_j, \\ &= -(b_{M_s} - b_i)^2 - \sum\nolimits_{j \notin H_s} \gamma_j C_j. \end{aligned}$$

In what follows, we refer to the leftmost group $-l$ as radicals. We assume that the radical

[26] We could allow for the repression of any combination of groups, thus having to consider $2^n - 1$ rather than $2n-1$ states, but choose not to do so to save on notation. Partial repression of some groups could also be allowed, with similar results.

group $-l$ is smaller than the next group: $\gamma_{-l} < \gamma_{-l+1}$, which implies that radicals can implement their preferred policy *only if* they repress all of the groups in society.[27]

We model power shifts by introducing h "radical" environments, $R^{-l-r}, \ldots, R^{-l-r+h-1}$, each with probability λ_j for $j = 1, \ldots, m$ at each date starting from E^1. Environment R^j is the same as E^1, except that in environment R^j, if the current state is one of $-l-r, \ldots, j$, the radical group, $-l$, acquires the ability to force a transition to any other state (in the process incurring the costs of repression). In particular, the radicals can choose to "grab power" by repressing all other groups and transitioning to state $s = -l - r$.[28] Therefore, in state s, the probability of the radicals having an opportunity to grab power is $\mu_s = \sum_{j=s}^{l+r} \lambda_j$, which is naturally (weakly) decreasing in s.

We also assume that in each period in any of the environments R^j, there is a probability ν of returning to the initial environment, E^1. This is equivalent to a transition to the "final" environment E^f identical to E^1 in terms of payoffs and winning coalitions (but there will be no further possibility of radicals coming to power after that). Clearly, $\nu = 0$ corresponds to a permanent shock, and as ν increases, the expected length of the period during which radicals can dictate transitions declines. Note, however, that if radicals grab power permanently the first time they get the opportunity and impose a transition to state $s = -l - r$ (in which they are the dictator), then they will remain in power even after there is a transition to environment E^f.

The next proposition uses Theorems 1 and 2 to establish the existence of a unique MVE, and then characterizes it in a baseline environment where there is no possibility of a radical takeover of power. The environment without radicals can be represented by E^f (since from E^f there is no further transition and thus no possibility of a radical takeover of power), and we use this convention to avoid introducing further notation.

Proposition 1 ***(Equilibria without radicals)*** *Without the possibility of radicals grabbing power (i.e., in environment E^f), there exists a unique MVE represented by $\phi_{E^f} : S \to S$. In this equilibrium:*

1. *Democracy is stable: $\phi_{E^f}(0) = 0$.*

[27] Though in this subsection we focus on left-wing radicals, our theory can be directly applied to the study of right-wing radicals and can also be readily extended to study environments in which both types of radical are present.

[28] In the context of the Bolshevik Revolution, this corresponds to assuming that in some possible environments (i.e., with some probability), Bolsheviks would be able to grab control with Kerensky in power but not necessarily with some government further to the right.

2. *For any costs of repression $\{C_j\}_{j \in N}$, there is never more repression than in the initial state: i.e., if $s < 0$ then $\phi_{E^f}(s) \in [s, 0]$, and if $s > 0$, then $\phi_{E^f}(s) \in [0, s]$.*

3. *Consider repression costs parametrized by k: $C_j = kC_j^*$, where $\left\{C_j^*\right\}$ are positive constants. There exists $k^* > 0$ such that: if $k > k^*$, then $\phi_{E^f}(s) = 0$ for all s, and if $k < k^*$, then $\phi_{E^f}(s) \neq 0$ for some s.*

Without radicals, democracy is stable because the median voter knows that she will be the one setting policy in the future (and can do so without incurring any cost of repression). This does not mean, however, that there is no repression starting in any state. Rather, other states may also be stable, meaning that agents can pay the cost of repression and stay away from $s = 0$. For instance, starting from a situation in which there is repression of the left, the QMV in that state may not find it beneficial to reduce repression because this will typically lead to policies further to the left (relative to the political bliss point of the QMV). But this type of repression is also limited by the cost of repression. If these costs are sufficiently high, then repression becomes unattractive starting from any state, and democracy becomes the only stable state.

The next proposition shows how political dynamics change when there is a risk of a radical takeover of power. This and the following proposition both utilize Theorems 3 and 4 to establish the existence of a unique MVE in the presence of shocks (that potentially shift power to radicals), and then use the same backward induction approach outlined in Section 3.2 (for establishing Theorem 3) to characterize behavior before the arrival of shocks as a function of the continuation play after the arrival of shocks.

Proposition 2 ***(Radicals)*** *There exists a unique MVE. Suppose that when the society is at state s, there is a transition to environment R^z (where $z \geq s$) so that radicals can grab power. Then, when they have the opportunity, the radicals move to state $s = -l - r$ (repressing all other groups) under a wider set of parameters when: (a) they are more radical (meaning their ideal point b_{-l} is lower, i.e., further away from 0); (b) they are "weaker" (i.e., z is smaller) in the sense that there is a smaller set of states in which they are able to control power.*

This proposition is intuitive. When they have more radical preferences, radicals value more the prospect of imposing their political bliss point, and are thus willing to incur the costs of repressing all other groups to do so. Radicals are also "more likely" to repress all other groups when they are "weaker" because when z is lower, there is a greater range of states where they cannot control future transitions, encouraging an immediate transition to $s = -l - r$.

To state our next proposition, we return to the (counterfactual) expected continuation utility of a group from permanently staying in a state $s \in S$ until a shock changes the environment, and following the MVE play thereafter. This continuation utility was was defined in Section 3.2 (in particular, footnote 20), and it is given (up to a scalar factor $1 - \beta(1 - \mu_1)$) by:[29]

$$\tilde{U}_i(s) = u_i(s) + \beta \sum_{z=-l-r}^{-l-r+h-1} \lambda_z V_{R^z,i}(s).$$

Proposition 3 ***(Repression by moderates anticipating radicals)*** *The transition mapping before radicals come to power, ϕ_{E^1}, satisfies the following properties.*

1. *If $s \leq 0$, then $\phi_{E^1}(s) \geq s$.*

2. *If $\tilde{U}_0(0) < \tilde{U}_0(s)$ for some $s > 0$, then there is a state $x \geq 0$ such that $\phi_{E^1}(x) > x$. In other words, there exists some state in which there is an increase in the repression of the left in order to decrease the probability of a radical takeover of power.*

3. *If for all states $y > x \geq 0$, $\tilde{U}_{M_x}(y) < \tilde{U}_{M_x}(x)$, then for all $s \geq 0$, $\phi_{E^1}(s) \leq s$. In other words, repression of the left never increases when the cost of repression increase (e.g., letting $C_j = kC_j^*$, repression weakly declines when k increases).*

The first part of the proposition indicates that there is no reason for repression of the right to increase starting from states below $s = 0$; rather, in these states the tendency is to reduce repression. However, the second part shows that if the median voter (in democracy) prefers a more repressive state *when she could counterfactually ensure no further repression unless radicals come to power* (which she cannot do because she is not in control in that state), then there is at least one state x from which there will be an increase in repression against the left (which does not necessarily have to be $s = 0$).[30] An implication of this result is that, off the threat of radical disappears, there will be a decline in repression starting in state $x > 0$. The third part of the proposition provides a sufficient condition for the opposite result.

The next proposition is a direct consequence of our general comparative static results given in Theorem 6, and shows how these results can be applied to reach substantive conclusions in specific settings.

[29] Observe also that $\left\{\tilde{U}_i(s)\right\}$ are defined only in terms of strategies played in environments R^z and E^f, and do not depend on strategies played in E^1. Hence, they can be computed directly as functions of the underlying parameters.

[30] Notably, even if there are "slippery slope" considerations (as defined in footnote 3) making some types of repressions undesirable, these will not be sufficient to prevent *all* repression

Proposition 4 ***(Comparative statics of repression)*** *Suppose that there is a state $s \geq 0$ (i.e., democracy or some state favoring the right), which is stable in E^1 for some set of probabilities $\{\mu_j\}$. Consider a change from $\{\mu_j\}$ to $\left\{\mu'_j\right\}$ such that $\mu'_j = \mu_j$ for $j \geq s$. Then there will be (weakly) less repression of the left after the change, i.e., $\phi'_{E^1}(s) \leq \phi_{E^1}(s) = s$.*

The intuition is the same as Theorem 6: if the probabilities of a radical takeover of power change, but only in states that already had repression against the left, and we are in a stable state without repression against the right, then this can only reduce repression. If there is now a lower likelihood of a radical grab of power, then this leads to less repression. But, paradoxically, even if there is a higher likelihood of such a grab, there may be less repression as the "slippery slope" considerations become less powerful.

Our final result deals with strategic complementarity in repressions. To state this result, consider a change in the costs of repression so that it becomes cheaper for radicals to repress right-wing groups. In particular, the stage payoff function of radicals changes to

$$u_{-l}(s) = -\left(b_{M_s} - b_{-l}\right)^2 - \rho \sum\nolimits_{j \notin H_s} \gamma_j C_j$$

for $s < 0$ and $\rho \in [0, 1]$. Clearly, $\rho = 1$ corresponds to our baseline environment, and a decrease in ρ implies that radicals can repress right-wing groups with less cost to themselves. Then:

Proposition 5 ***(Strategic Complementarity)*** *Suppose that $\lambda_z = 0$ for all $z > 0$ (meaning that radicals can only seize power if they are not currently repressed). Consider a change in the radicals' repression costs to $\rho' < \rho$ and denote the MVE before and after the change by, respectively, ϕ and ϕ'. Then if $\phi_{E^1}(s) > s$ for some $s \geq 0$, then $\phi'_{E^1}(s) > s$.*

Put differently, the proposition implies that if $\phi_{E^1}(0) > 0$, then $\phi'_{E^1}(0) > 0$, so that repression of the radicals is more likely when they themselves have lower costs of repressing other groups. At the root of this result is a strategic complementarity in repression: anticipating greater repression by radicals in future radical environments, the current political system now becomes more willing to repress the radicals. One interesting implication of this result is that differences in repression of opposite ends of the political spectrum across societies may result from small differences in (institutional or social) costs of repression rather than a "culture of repression" in some countries. Thus, the repression of first left- and then right-wing groups in early 20th-century Russia, contrasted with a lack of such systematic repression in Britain, may not just be a reflection of a Russian culture of repression, but a game-theoretic consequence of the anticipation of different patterns of repression in different political states in Russia.

4.2 Institutional experimentation

Our second application is one of collective experimentation over institutions. In many institutional reforms, which are marred with uncertainty, a key concern of incumbent decision-makers is the possibility that they may lose political control and may not be able to reverse certain aspects of prior reforms even if these turn out to be highly detrimental. These issues are illustrated, for example, by trade-offs post-socialist countries faced during their transitions. A key uncertainty this process concerned the optimal sequencing of institutional reforms, especially about property rights protection and legal system, and privatization (e.g., Roland, 2000). An attractive strategy under such uncertainty might be experimentation, for example, starting with the privatization of some large state-owned enterprises. But this early privatization may then cause both the establishment of politically powerful strong vested interests and also backlash from voters depending on its efficiency and distributional consequences.[31]

Formally, there are again several players (representing groups) indexed by $i = 1, ..., n$. The stage payoff of player i in state s when policy p is implemented is given by

$$w_i(s,p) = B_s - (b_i - p)^2,$$

where b_i is its bliss point and B_s denotes utility from state s which is shared by all players (e.g., quality of government or public goods provision) and will be modeled below. We assume that b_i is increasing in i which ensures that Assumption 2 holds.

We assume that there are n states, and that in state $i < n$, player i is the unique QMV (decision-maker) and sets the policy and decides on transitions to a different state. The value of parameter B in these states is also known, and assumed, for simplicity, to be weakly increasing in i: $B_1 \leq B_2 \leq \cdots \leq B_{n-1}$.

In state n, policy is chosen by player n, but the value of B_n and the identity of the decision-maker in state n are not known *ex ante*. In particular, B_n takes the value B^h with probability γ and the value $B^l < B^h$ with probability $1-\gamma$. Moreover, we simplify the discussion by assuming that in state n, player n always decides the policy, but controls transitions only with probability μ (i.e., player n is the unique QMV, or $M_n = n$), and with probability $1 - \mu$, it is player $n - 1$ who retains control over transitions (i.e., $M_n = n - 1$). This structure ensures that Assumption

[31] Indeed, Hellman (1998) observes that big winners from the early stage of reforms later became major obstacles to the next stage of reform. In Russia, banks created at the beginning of the reform process were later strongly opposed to government attempts to bring down inflation (Shleifer and Treisman, 2000).

4 holds. Again for simplicity, we also assume that B_n and M_n are independent, and that the society always learns about B_n and M_n at the same time.[32]

Learning takes place in two ways. First, if society moves to state n, the true values of B_n and M_n will be revealed. Second, in each period, there is probability $\lambda \in [0,1]$ that these values will be revealed even when society is not in state n. This could be, for example, because there is passive learning from another country in the midst of a similar experiment, or current political dynamics will provide insights about what will happen in state n. The initial environment is denoted by E^0 and society starts in one of the states $1,\ldots,n-1$. The realizations of B_n and M_n define four additional environments $E^{h,n-1}$, $E^{l,n-1}$, $E^{h,n}$, $E^{l,n}$. A MVE is therefore a collection of five mappings ϕ_0, $\{\phi_{B_n,M_n}\}$ that satisfy Definition 3.

Several comments are in order. First, this model is related to Fernandez and Rodrik (1991) and particularly to Strulovici's (2010) important paper on strategic experimentation by voting, but with a crucial difference. In both of these models, learning is about individual idiosyncratic preferences, whereas in our paper learning is about characteristics of different states that affect all individuals. In this sense, the experimentation is over institutions, rather than over individual preferences. Second, the assumptions are meant to capture the uncertainty over both the payoff implications of moving to new states (that have not been tried yet) and the uncertainty over who controls political power in these states. For instance, in addition to the post-socialist transition example discussed above, we can think of state n as corresponding to a reform deregulating a particular industry. The benefits of deregulation will be learned after it has been tried, but other evidence or research may reveal its value even without active experimentation. There is also some possibility that industry insiders maybe able to amass significant power and prevent a reversal of this deregulation even if it is revealed to be a failure. The rest of the players are ranked in terms of their dislike of this deregulation, and the assumption that when industry insiders capture the power over the form of regulation, transitions are controlled by the neighboring group is for simplicity.

The next proposition follows directly from Theorems 3 and 4 by verifying that our baseline assumptions are satisfied.

Proposition 6 *In the environment described above, there exists a unique MVE given by the monotone mappings* ϕ_0, $\{\phi_{B_n,M_n}\}$.

[32] Both assumptions can be relaxed relatively straightforwardly. For example, we could assume that $n-1$ is initially in control, but every period agent n may succeed in consolidating power.

The key question in this model is whether there will be experimentation with state n. Experimentation is represented by $\phi_0(n-1) = n$, i.e., by whether there will be a move to state n while there is uncertainty about its payoff and power implications. We assume in what follows that

$$B^l < B_{n-1} + (b_n - b_{n-1})^2 < B^h.$$

This ensures that group $n-1$ would prefer to move to state n if it knew that $B_n = B^h$, but not when $B_n = B^l$. Then from increasing differences (Assumption 2), we also have that group n strictly prefers state n when $B_n = B^h$ (but may or may not do so if $B_n = B^l$).

Proposition 7 *Let* $Y \equiv \frac{\gamma}{1-\gamma}\frac{B^h-(b_n-b_{n-1})^2-B_{n-1}}{B_{n-1}+(b_n-b_{n-1})^2-B^l}$.

(i) Suppose that $B_{n-1} - (b_n - b_{n-1})^2 > B^l$. *Then* $\phi_{l,\cdot}(n) = n-1$ *and* $\phi_{h,\cdot}(n) = n$, *and there is experimentation if and only if* $Y > 1 - \beta(1-\lambda)$. *This condition does not depend on* μ*; experimentation will take place for a wider set of parameter values when* β *is higher or* λ *is lower.*

(ii) Conversely, suppose that $B_{n-1} - (b_n - b_{n-1})^2 < B^l$. *Then* $\phi_{l,n-1}(n) = n-1$ *and* $\phi_{\cdot,\cdot}(n) = n$ *otherwise. Society experiments if and only if* $Y > \frac{(1-\beta+\beta\lambda)(1-\beta+\beta\mu)}{1-\beta}$, *which holds for a wider set of parameter values when* λ *is lower or* μ *is lower. Moreover, if* $\lambda + \mu \geq 1$, *then an increase in* β *makes the set of parameter values for which experimentation takes place smaller, and if* $\lambda + \mu < 1$, *the effect of* β *is nonmononotone: it is inverse U-shaped, reaching a local maximum in the interior and local minima at* $\beta = 0$ *and* 1.

The decision by group $n-1$ to experiment therefore depends on Y, which is, very intuitively, the ratio of potential gain from being in state n with $B_n = B^h$, as compared to the baseline ("safe option") of B_{n-1}, to (the absolute value of) the potential loss if $B_n = B^l$, weighted, naturally, with the probabilities of the two outcomes, γ and $1-\gamma$, respectively ($Y > 1$ if and only if $\mathbb{E}B_n - (b_n - b_{n-1})^2 > B_{n-1}$). Society experiments if the ratio Y exceeds a certain threshold. Unsurprisingly, experimentation is "more likely" if λ is low; intuitively, if the society is very likely to learn B_n without trying it, it makes more sense to wait until it happens. For fixed payoffs, a Shigher γ also makes experimentation more likely, as increases the odds of a high value of B_n. Furthermore, if the interests of groups $n-1$ and n regarding experimentation are aligned (so group n prefers state $n-1$ if $B_n = B^l$), then a high discount factor makes experimentation more likely. Indeed, in this case, if $B_n = B^l$, then the low payoff will be experienced for at most one period, and if β is high, the relative impact of this period to the lifetime payoff grows smaller. In addition, the society experiments for any β if $Y > 1$, i.e, if even the average payoff $\mathbb{E}B_n - (b_n - b_{n-1})^2$ exceeds B_{n-1}.

The results are different in the second case, where group n prefers state n regardless of the realization of B_n and will stay in this state if it can. In this case, experimentation is risky and need not happen even if $Y > 1$ (provided that $\lambda \neq 0$): in this case, instead of taking a chance, group $n-1$ may find it prudent to wait and find out the value of B_n. These considerations are more pronounced if the likelihood of group n seizing power is higher, so experimentation is less likely for high μ. The comparative statics with respect to β is ambiguous, because of two effects. On the one hand, similar to the previous case, a higher discount factor decreases the significance of one period of experimentation, and this makes experimentation more likely. On the other hand, a higher discount factor also makes waiting to learn B_n without taking risks more attractive. It turns out that for low λ and μ the first effect dominates; for high λ and μ (or high β) the second one does.

The next result shows that the response of experimentation to changes in riskiness of the experiment is potentially non-monotone.

Proposition 8 *Suppose* $\mathbb{E}B_n - (b_n - b_{n-1})^2 > B_{n-1}$*;* $\lambda, \mu > 0$*, and* β *is sufficiently close to 1. For a fixed* $\mathbb{E}B_n$ *and* γ*, vary the spread* $\Delta \equiv B^h - B^l$*. Then the decision to experiment is non-monotone in* Δ*: there exist two thresholds* $\Delta_1 < \Delta_2$ *such that there is experimentation if* $\Delta < \Delta_1$ *or* $\Delta > \Delta_2$*, but no experimentation if* $\Delta \in (\Delta_1, \Delta_2)$*.*

If the spread between B^h and B^l is small, then the downside risk from experimentation for group $n-1$ is small, even if this experimentation leads to society being stuck forever in state n; hence experimentation takes place for small Δ. As this spread increases, the downside risk to group $n-1$ becomes substantial because, when it controls political power, group n will prefer to stay in state n permanently; in this case, group $n-1$ prefers to wait rather than experiment. However, if this spread becomes sufficiently large, the interests of groups $n-1$ and n become aligned; in this case, the effective risk of having to stay in state n forever disappears, and experimentation again takes place. Overall, therefore, experimentation is less likely to take place when the downside risk is moderate, but more likely when this risk is low or high — because this risk also affects the nature of the conflict of interest between groups.

5 Conclusion

This paper has provided a general framework for the analysis of dynamic political economy problems, including democratization, extension of political rights or repression of different groups. The distinguishing feature of our approach is that it enables an analysis of nonstationary, stochastic environments (which allow for anticipated and unanticipated shocks changing the distribution of political power and economic payoffs) under fairly rich heterogeneity and general political or economic conflict across groups.

We assume that the payoffs are defined either directly on states or can be derived from states, which represent economic and political institutions. For example, different distribution of property rights or adoption of policies favoring one vs. another group correspond to different states. Importantly, states also differ in their distribution of political power: as states change, different groups become politically pivotal (and in equilibrium different coalitions may form). Our notion of equilibrium is Markov Voting Equilibrium, which requires that economic and political changes — transitions across states — should take place if there exists a subset of players with the power to implement such changes and who will obtain higher expected discounted utility by doing so.

We assume that both states and players are "ordered" (e.g., states go from more right-wing to more left-wing, or less to more democratic, and players are ordered according to their ideology or income level). Our most substantive assumptions are that, given these orders, stage payoffs satisfy a "single crossing" (increasing differences) type assumption, and the distribution of political power also shifts in the same direction as economic preferences (e.g., individuals with preferences further to the right gain relatively more from moving towards states further to the right, and their political power does not decrease if there is a transition towards such a state).

Under these assumptions, we prove the existence of a pure-strategy equilibrium, provide conditions for its uniqueness, and show that a steady state always exists (though it generally depends on the order and exact timing of shocks). We also provide some comparative static results that apply at this level of generality. For example, if there is a change from one environment to another (with different economic payoffs and distribution of political power) but the two environments coincide up to a certain state s' and before the change the steady state of equilibrium was at some state $x \leq s'$, then the new steady state after the change in environment can be no smaller than x.

We then use this framework to study the dynamics of repression in the presence of radical groups that can stochastically grab power depending on the distribution of political rights in society. We characterize the conditions under which the presence of radicals leads to greater repression (of less radical groups) and identify a novel strategic complementarity in repression. We also provide an application to the problem of collective experimentation over different institutions.

Our framework can be extended and applied in several different directions, which constitute interesting area for future research. The first is to incorporate greater individual-level heterogeneity, which can change over time (e.g., a type of "social mobility"), a topic we are actively pursuing. More challenging is the study of problems in which heterogeneity cannot be reduced to a single dimension, which opens the door for more complex strategic interactions and dynamics — and a broader set of applications. Some of the important applications of the framework we have proposed, which constitute interesting area for future research, go beyond political economy and include problems in organizational economics (in particular focusing on the internal politics of the firm) and international relations (relationships between countries and dynamics of secessions and civil wars).

Appendix A

Lemmas

We start with a number of lemmas, which play a central role in establishing important properties of MVE and form the foundation of all of our main proofs.

Lemma 1 *Suppose that vector* $\{w_i(\cdot)\}$ *satisfies increasing differences on* $N \times \{x, y\}$ *for some* $x, y \in S$. *Let*

$$P = \{i \in N : w_i(y) > w_i(x)\},$$

and take any $s \in S$. *Then* $P \in W_s$ *if and only if* $M_s \subset P$. *A similar statement is true for relations* $\geq$, $<$, $\leq$.

Proof. "If": Suppose $M_s \subset P$, so for each $i \in M_s$, $w_i(y) > w_i(x)$. Consider two cases. If $y > x$, then increasing differences implies that $w_j(y) > w_j(x)$ for all $j \geq \min M_s$. On the other hand, $[\min M_s, n]$ is a winning coalition (if not, $i = M_s - 1$ would be a QMV by definition, but such $i \notin M_s$). If $y < x$, then, similarly, $w_j(y) > w_j(x)$ for all $j \leq \max M_s$, which is a winning coalition for similar reasons. In either case, P contains a subset (either $[\min M_s, n]$ or $[1, \max M_s]$) which is a winning coalition, and thus $P \in W_s$.

"Only if": Suppose $P \in W_s$. Consider the case $y > x$. Let $i = \min P$; then increasing differences implies that for all $j \geq i$, $w_j(y) > w_j(x)$. This means that $P = [i, n]$, and is thus a connected coalition. Since P is winning, we must have $i \leq j \leq n$ for any $j \in M_s$ by definition of M_s, and therefore $M_s \subset P$. The case where $y < x$ is similar, so $M_s \subset P$.

The proofs for relations $\geq$, $<$, $\leq$ are similar and are omitted. ■

For each $s \in S$, let us introduce the binary relation $>_s$ on the set of n-dimensional vectors to designate that there exists a winning coalition in s strictly preferring one payoff vector to another. Formally:

$$w^1 >_s w^2 \Leftrightarrow \left\{i \in N : w_i^1 > w_i^2\right\} \in W_s.$$

The relation $\geq_s$ is defined similarly. Lemma 1 now implies that if a vector $\{w_i(x)\}$ satisfies increasing differences, then for any $s \in S$, the relations $>_s$ and $\geq_s$ are transitive on $\{w_i(x)\}_{x \in S}$. Notice that for this result, we need only two assumptions: Assumption 3 on winning coalitions in state s ensures existence of the (nonempty) set of QMVs M_s, and we need vector $\{w_i(x)\}_{x \in S}$ to satisfy increasing differences.

Lemma 2 *Suppose Assumption 2 holds. Then, for a mapping* $\phi : S \to S$*, the vector* $\left\{V_i^{\phi}(s)\right\}_{i\in N}^{s\in S}$*, given by* (4)*, satisfies increasing differences if*

1. ϕ *is monotone; or*

2. *for all* $x \in S$*,* $|\phi(x) - x| \leq 1$*.*

Proof. Part 1. Take $y > x$ and any $i \in N$. We have:

$$\begin{aligned} V_i^{\phi}(y) - V_i^{\phi}(x) &= u_i(y) + \sum\nolimits_{k=1}^{\infty} \beta^k u_i\left(\phi^k(y)\right) - u_i(x) - \sum\nolimits_{k=1}^{\infty} \beta^k u_i\left(\phi^k(x)\right) \\ &= (u_i(y) - u_i(x)) + \sum\nolimits_{k=1}^{\infty} \beta^k \left(u_i\left(\phi^k(y)\right) - u_i\left(\phi^k(x)\right)\right). \end{aligned}$$

The first term is (weakly) increasing in i if $\{u_i(s)\}_{i\in N}^{s\in S}$ satisfies increasing differences, and the second is (weakly) increasing in i as $\phi^k(y) \geq \phi^k(x)$ for $k \geq 1$ due to monotonicity of ϕ. Consequently, (4) is (weakly) increasing in i.

Part 2. If ϕ is monotone, then Part 1 applies. Otherwise, for some $x < y$ we have $\phi(x) > \phi(y)$, and this means that $y = x + 1$; there may be one or more such pairs. Notice that for such x and y, we have $\phi(x) = y$ and $\phi(y) = x$. Consider

$$\begin{aligned} V_i^{\phi}(y) - V_i^{\phi}(x) &= \left(u_i(y) + \sum\nolimits_{k=1}^{\infty} \beta^{2k-1} u_i(x) + \sum\nolimits_{k=1}^{\infty} \beta^{2k} u_i(y)\right) \\ &\quad - \left(u_i(x) + \sum\nolimits_{k=1}^{\infty} \beta^{2k-1} u_i(y) + \sum\nolimits_{k=1}^{\infty} \beta^{2k} u_i(x)\right) \\ &= \frac{1}{1+\beta}(u_i(y) - u_i(x)); \end{aligned}$$

this is (weakly) increasing in i.

Let us now modify stage payoffs and define

$$\tilde{u}_i(x) = \begin{cases} u_i(x) & \text{if } \phi(x) = x \text{ or } \phi^2(x) \neq x; \\ (1-\beta) V_i(x) & \text{if } \phi(x) \neq x = \phi^2(x). \end{cases}$$

Consider mapping $\tilde{\phi}$ given by

$$\tilde{\phi}(s) = \begin{cases} \phi(x) & \text{if } \phi(x) = x \text{ or } \phi^2(x) \neq x; \\ x & \text{if } \phi(x) \neq x = \phi^2(x). \end{cases}$$

This $\tilde{\phi}$ is monotone and $\{\tilde{u}_i(x)\}_{i\in N}^{x\in S}$ satisfies increasing differences. By Part 1, the continuation values $\left\{\tilde{V}_i^{\tilde{\phi}}(x)\right\}_{i\in N}^{x\in S}$ computed for $\tilde{\phi}$ and $\{\tilde{u}_i(x)\}_{i\in N}^{x\in S}$ using (4) satisfy increasing differences as well. But by construction, $\tilde{V}_i^{\tilde{\phi}}(x) = V_i^{\phi}(x)$ for each i and s, and thus $\left\{V_i^{\phi}(x)\right\}_{i\in N}^{x\in S}$ satisfies increasing differences. ■

Lemma 3 ***(Monotone Deviation Principle)*** *Suppose that $\phi : S \to S$ is feasible (part 1 of Definition 3) and monotone but the core property (part 2 of Definition 3) is violated. Then there exist $x, y \in S$ such that $y \in F(x)$,*

$$V^{\phi}(y) >_x V^{\phi}(\phi(x)). \tag{A1}$$

and the mapping $\phi' : S \to S$ given by

$$\phi'(s) = \begin{cases} \phi(s) & \text{if } s \neq x \\ y & \text{if } s = x \end{cases} \tag{A2}$$

is monotone.

Proof. Existence of $x, y \in S$ such that $y \in F(x)$ and (A1) holds follows from failure of part 2 of Definition 3. We show that for some pair of such x, y, (A2) is monotone.

Suppose, to obtain a contradiction, that for each $x, y \in S$ such that $y \in F(x)$ and (A1) holds, ϕ' given by (A2) is not monotone. Take $x, y \in S$ such that $|y - \phi(x)|$ is minimal among all pairs $x, y \in S$ such that $y \in F(x)$ and (A1) holds (informally, we consider the shortest deviation). By our assertion, ϕ' is not monotone. Since ϕ is monotone and ϕ and ϕ' differ by the value at x only, there are two possibilities: either for some $z < x$, $y = \phi'(x) < \phi(z) \leq \phi(x)$ or for some $z > x$, $\phi(x) \leq \phi(z) < \phi'(x) = y$. Assume the former (the latter case may be considered similarly). Let s be defined by

$$s = \min(z \in S : \phi(z) > y);$$

in the case under consideration, the set of such z is nonempty (e.g., x is its member, and z found earlier is one as well), and hence state s is well-defined. We have $s < x$; since ϕ is monotone, $\phi(s) \leq \phi(x)$.

Notice that a deviation in state s from $\phi(s)$ to y is monotone: indeed, there is no state $\tilde{z}$ such that $\tilde{z} < s$ and $y < \phi(\tilde{z}) \leq \phi(s)$ by construction of s, and there is no state $\tilde{z} > s$ such that $\phi(s) \leq \phi(\tilde{z}) < y$ as this would contradict $\phi(s) > y$. Moreover, it is feasible, so $y \in F(s)$: this is automatically true if $y = s$; if $y > s$, this follows from $s < y < \phi(s)$; and if $y < s$, this follows from $y = \phi'(x)$ and $y < s \leq x$. By assertion, this deviation cannot be profitable, i.e., $V^{\phi}(y) \not>_s V^{\phi}(\phi(s))$. By Lemma 2, since $y < \phi(s)$, $V^{\phi}_{\max M_s}(y) \leq V^{\phi}_{\max M_s}(\phi(s))$. Since $s < x$, Assumption 4 implies (for $i = \max M_x$) $V^{\phi}_i(y) \leq V^{\phi}_i(\phi(s))$.

On the other hand, (A1) implies $V^{\phi}_i(y) > V^{\phi}_i(\phi(x))$. We therefore have

$$V^{\phi}_i(\phi(s)) \geq V^{\phi}_i(y) > V^{\phi}_i(\phi(x)) \tag{A3}$$

and thus, by Lemma 2, since $\phi(s) < \phi(x)$ (we know $\phi(s) \leq \phi(x)$, but $\phi(s) = \phi(x)$ would contradict (A3)),

$$V^{\phi}(\phi(s)) >_x V^{\phi}(\phi(x)).$$

Notice, however, that $y < \phi(s) < \phi(x)$ implies that $|\phi(s) - \phi(x)| < |y - \phi(x)|$. This contradicts the choice of y such that $|y - \phi(x)|$ is minimal among pairs $x, y \in S$ such that $y \in F(x)$ and (A1) is satisfied. This contradiction proves that our initial assertion was wrong, and this proves the lemma. ■

Lemma 4 ***(No Double Deviation)*** *Let $a \in [1, m-1]$, and let $\phi_1 : [1, a] \to [1, a]$ and $\phi_2 : [a+1, m] \to [a+1, m]$ be two monotone mappings which are MVE on their respective domains. Let $\phi : S \to S$ be defined by*

$$\phi(s) = \begin{cases} \phi_1(s) & \text{if } s \leq a \\ \phi_2(s) & \text{if } s > a \end{cases} \tag{A4}$$

Then exactly one of the following is true:

1. *ϕ is a MVE on S;*

2. *there is $z \in [a+1, \phi(a+1)]$ such that $z \in F(a)$ and $V^{\phi}(z) >_a V^{\phi}(\phi(a))$;*

3. *there is $z \in [\phi(a), a]$ such that $z \in F(a+1)$ and $V^{\phi}(z) >_{a+1} V^{\phi}(\phi(a+1))$.*

Proof. We show first that if [1] is the case, then [2] and [3] are not satisfied. We then show that if [1] does not hold, then either [2] or [3] are satisfied, and complete the proof by showing that [2] and [3] are mutually exclusive.

First, suppose, to obtain a contradiction, that both [1] and [2] hold. Then [2] implies that for some $z \in [a+1, \phi(a+1)]$ such that $z \in F(a)$, $V^{\phi}(z) >_a V^{\phi}(\phi(a))$, but this contradicts that ϕ is MVE, so [1] cannot hold. We can similarly prove that if [1] holds, then [3] is not satisfied.

Second, suppose that [1] does not hold. Notice that for any $x \in S$, $\phi(x) \in F(x)$ and $V^{\phi}(\phi(x)) \geq_x V^{\phi}(x)$, because these properties hold for ϕ_1 if $x \in [1, a]$ and for ϕ_2 if $x \in [a+1, m]$. Consequently, if ϕ is not MVE, then it is because the (core) condition in Definition 3 is violated. Lemma 3 then implies existence of a monotone deviation, i.e., $x, y \in S$ such that $y \in F(x)$ and $V^{\phi}(y) >_x V^{\phi}(\phi(x))$. Since ϕ_1 and ϕ_2 are MVE on their respective domains, we must have that either $x \in [1, a]$ and $y \in [a+1, m]$ or $x \in [a+1, m]$ and $y \in [1, m]$. Assume the former; since the deviation is monotone, we must have $x = a$ and $a + 1 \leq y \leq \phi(a+1)$. Hence, we have

$V^{\phi}(y) >_a V^{\phi}(\phi(a))$, and this shows that [2] holds. If we assumed the latter, we would similarly get that [3] holds. Hence, if [1] does not hold, then either [2] or [3] does.

Third, suppose that both [2] and [3] hold. Let

$$\begin{aligned} x &\in \arg \max_{z \in [\phi(a), \phi(a+1)] \cap F(a)} V^{\phi}_{\min M_a}(z), \\ y &\in \arg \max_{z \in [\phi(a), \phi(a+1)] \cap F(a+1)} V^{\phi}_{\max M_{a+1}}(z); \end{aligned}$$

then $x \geq a+1 > a \geq y$. By construction, $V^{\phi}_{\min M_a}(x) > V^{\phi}_{\min M_a}(y)$ and $V^{\phi}_{\max M_{a+1}}(y) > V^{\phi}_{\max M_{a+1}}(x)$ (the inequalities are strict because they are strict in [2] and [3]). But this violates the increasing differences that $\left\{V^{\phi}_i(s)\right\}^{s \in S}_{i \in N}$ satisfies as ϕ is monotone (indeed, $\min M_a \leq \max M_{a+1}$ by Assumption 4). This contradiction proves that [2] and [3] are mutually exclusive, which completes the proof. ■

Lemma 5 ***(Extension of Equilibrium)*** *Let $\tilde{S} = [1, m-1]$. Suppose that $\phi : \tilde{S} \to \tilde{S}$ is a monotone MVE and that $F(m) \neq \{m\}$. Let*

$$a = \max\left(\arg \max_{b \in [\phi(m-1), m-1] \cap F(m)} V^{\phi}_{\max M_m}(b)\right). \tag{A5}$$

If

$$V^{\phi}(a) >_m u(m) / (1-\beta), \tag{A6}$$

then mapping $\phi' : S \to S$ defined by

$$\phi'(s) = \begin{cases} \phi(s) & \text{if } s < m \\ a & \text{if } s = m \end{cases}$$

is a monotone MVE. A similar statement, mutatis mutandis, applies for $\tilde{S} = [2, m]$.

Proof. Mapping ϕ' satisfies property 1 of Definition 3 by construction. Let us show that it satisfies property 2. Suppose, to obtain a contradiction, that this is not the case. By Lemma 3, there are states $x, y \in S$ such that

$$V^{\phi'}(y) >_x V^{\phi'}(\phi'(x)), \tag{A7}$$

and this deviation is monotone. Suppose first that $x < m$, then $y \leq \phi(m) = a \leq m-1$. For any $z \leq m-1$, $(\phi')^k(z) = \phi^k(z)$ for all $k \geq 0$, and thus $V^{\phi'}(z) = V^{\phi}(z)$; therefore, $V^{\phi}(y) >_x V^{\phi}(\phi(x))$. However, this would contradict that ϕ is a MVE on $\tilde{S}$. Consequently, $x = m$. If $y < m$, then (A7) implies, given $a = \phi'(m)$,

$$V^{\phi}(y) >_m V^{\phi}(a). \tag{A8}$$

Since the deviation is monotone, $y \in [\phi(m-1), m-1]$, but then (A8) contradicts the choice of a in (A5). This implies that $x = y = m$, so (A7) may be rewritten as

$$V^{\phi'}(m) >_m V^{\phi}(a). \tag{A9}$$

But since

$$V^{\phi'}(m) = u(m) + \beta V^{\phi}(a), \tag{A10}$$

(A9) implies

$$u(m) >_m (1-\beta) V^{\phi}(a).$$

This, however, contradicts (A6), which proves that ϕ' satisfies property 2 of Definition 3.

To prove that ϕ' is MVE, we need to establish that it satisfies property 3 of Definition 3, i.e.,

$$V^{\phi'}(\phi'(x)) \geq_x V^{\phi'}(x) \tag{A11}$$

for each $x \in S$. If $x \in \tilde{S}$ (i.e., $x < m$), then $(\phi')^k(x) = \phi^k(x)$ for any $k \geq 0$, so (A11) is equivalent to $V^{\phi}(\phi(x)) \geq_x V^{\phi}(x)$, which is true for $x < m$, because ϕ is MVE on $\tilde{S}$. It remains to prove that (A11) is satisfied for $x = m$. In this case, (A11) may be rewritten as

$$V^{\phi}(a) \geq_m V^{\phi'}(m). \tag{A12}$$

Taking (A10) into account, (A12) is equivalent to $(1-\beta) V^{\phi}(a) \geq_m u(m)$,which is true, provided that (A6) is satisfied. We have thus proved that ϕ' is MVE on S, which completes the proof. ■

Proofs of Theorems 1-8

Proof of Theorem 1. We prove this result by induction by the number of states. For any set X, let Φ^X be the set of monotone MVE, so we have to prove that $\Phi^X \neq \varnothing$.

Base: If $m = 1$, then $\phi : S \to S$ given by $\phi(1) = 1$ is monotone MVE for trivial reasons, so $\Phi^S \neq \varnothing$ is $|S| = 1$.

Induction Step: Suppose that if $|S| < m$, then $\Phi^S \neq \varnothing$. Let us prove this if $|S| = m$. Consider the set $A = [1, m-1]$, and for each $a \in A$, consider two monotone MVE $\phi_1^a : [1,a] \to [1,a]$ and $\phi_2^a : [a+1, m] \to [a+1, m]$. Without loss of generality, we may assume that

$$\begin{aligned} \phi_1^a &\in \arg \max_{\phi \in \Phi^{[1,a]}, z \in [\phi(a), a] \cap F(a+1)} V^{\phi}_{\max M_{a+1}}(z), \\ \phi_2^a &\in \arg \max_{\phi \in \Phi^{[a+1,m]}, z \in [a+1, \phi(a+1)] \cap F(a)} V^{\phi}_{\min M_a}(z) \end{aligned}$$

(whenever $[\phi(a), a] \cap F(a+1) = \varnothing$ or $[a+1, \phi(a+1)] \cap F(a)$ are empty, we pick any ϕ_1^a or ϕ_2^a, respectively). For each $a \in A$, define $\phi^a : S \to S$ by

$$\phi^a(s) = \begin{cases} \phi_1^a(s) & \text{if } s \leq a \\ \phi_2^a(s) & \text{if } s > a \end{cases}.$$

Let us define function $f : A \to \{1, 2, 3\}$ as follows. By Lemma 4, for every split $S = [1, a] \cup [a+1, m]$ given by $a \in A$ and for MVE ϕ_1^a and ϕ_2^a, exactly one of three properties hold; let $f(a)$ be the number of the property. Then, clearly, if for some $a \in A$, $f(a) = 1$, then ϕ^a is a monotone MVE by construction of function f.

Now let us consider the case where for every $a \in A$, $f(a) \in \{2, 3\}$. We have the following possibilities.

First, suppose that $f(1) = 2$. This means that (since $\phi_1^a(1) = 1$ for $a = 1$)

$$\arg \max_{z \in [1, \phi(2)] \cap F(1)} V_{\min M_1}^{\phi^1}(z) \subset \left[2, \phi^1(2)\right]. \tag{A13}$$

Let

$$b \in \arg \max_{z \in [2, \phi(2)] \cap F(1)} V_{\min M_1}^{\phi^1}(z) \tag{A14}$$

and define $\phi' : S \to S$ by

$$\phi'(s) = \begin{cases} b & \text{if } s = 1 \\ \phi^1(s) & \text{if } s > 1 \end{cases}; \tag{A15}$$

let us prove that ϕ' is a MVE. Notice that (A13) and (A14) imply

$$V_{\min M_1}^{\phi^1}(b) > V_{\min M_1}^{\phi^1}(1).$$

By Lemma 2, since $b > 1$,

$$V^{\phi^1}(b) >_1 V^{\phi^1}(1). \tag{A16}$$

Notice, however, that

$$V^{\phi^1}(1) = u(1) / (1 - \beta),$$

and also $V^{\phi^1}(b) = V^{\phi_2^1}(b)$; therefore, (A16) may be rewritten as

$$V^{\phi_2^1}(b) >_1 u(1) / (1 - \beta).$$

By Lemma 5, $\phi' : S \to S$ defined by (A15), is a MVE.

Second, suppose that $f(m-1) = 3$. In this case, using the first part of Lemma 5, we can prove that there is a MVE similarly to the previous case.

Finally, suppose that $f(1)=3$ and $f(m-1)=2$ (this already implies $m\geq 3$), then there is $a\in[2,m-1]$ such that $f(a-1)=3$ and $f(a)=2$. Define, for $s\in S\setminus\{a\}$ and $i\in N$,

$$V_i^*(s)=\begin{cases} V_i^{\phi_1^{a-1}}(s) & \text{if } s<a \\ V_i^{\phi_2^a}(s) & \text{if } s>a \end{cases}.$$

Let us first prove that there exists $b\in\left(\left[\phi_1^{a-1}(a-1),a-1\right]\cup\left[a+1,\phi_2^a(a+1)\right]\right)\cap F(a)$ such that

$$V^*(b)>_a u(a)/(1-\beta), \tag{A17}$$

and let B be the set of such b (so $B\subset\left(\left[\phi_1^{a-1}(a-1),a-1\right]\cup\left[a+1,\phi_2^a(a+1)\right]\right)\cap F(a)$). Indeed, since $f(a-1)=3$,

$$\arg\max_{z\in\left[\phi^{a-1}(a-1),\phi^{a-1}(a)\right]\cap F(a)} V_{\max M_a}^{\phi^{a-1}}(z)\subset\left[\phi^{a-1}(a-1),a-1\right]. \tag{A18}$$

Let

$$b\in\arg\max_{z\in\left[\phi^{a-1}(a-1),a-1\right]\cap F(a)}\left(V_{\max M_a}^{\phi^{a-1}}(z)\right), \tag{A19}$$

then (A18) and (A19) imply

$$V_{\max M_a}^{\phi^{a-1}}(b)>V_{\max M_a}^{\phi^{a-1}}(a). \tag{A20}$$

By Lemma 2, since $b<a$,

$$V^{\phi^{a-1}}(b)>_a V^{\phi^{a-1}}(a). \tag{A21}$$

We have, however,

$$V^{\phi^{a-1}}(a)=V^{\phi_2^{a-1}}(a)=u(a)+\beta V^{\phi_2^{a-1}}\left(\phi_2^{a-1}(a)\right)\geq_a u(a)+\beta V^{\phi_2^{a-1}}(a)=u(a)+\beta V^{\phi^{a-1}}(a)$$

($V^{\phi^{a-1}}(a)=V^{\phi_2^{a-1}}(a)$ by definition of ϕ^{a-1}, and the inequality holds because ϕ_2^{a-1} is MVE on $[a,m]$). Consequently, (A20) and (A21) imply (A17). (Notice that using $f(a)=2$, we could similarly prove that there is $b\in[a+1,\phi^a(a+1)]$ such that (A17) holds.)

Let us now take some QMV in state a, $j\in M_a$, and state $d\in B$ such that

$$d=\arg\max_{b\in B}V_j^*(b), \tag{A22}$$

and define monotone mapping $\phi:S\to S$ as

$$\phi(s)=\begin{cases} \phi_1^{a-1}(s) & \text{if } s<a \\ d & \text{if } s=a \\ \phi_2^a(s) & \text{if } s>a \end{cases}$$

(note that $V^\phi(s)=V^*(s)$ for $x\neq a$). Let us prove that ϕ is a MVE on S.

By construction of d in (A22), we have that $b \in \left[\phi_1^{a-1}(a-1), \phi_2^a(a+1)\right] \cap F(a)$ implies

$$V^{\phi}(b) \not>_a V^{\phi}(d).$$

This is automatically true for $b \in B$, whereas if $b \notin F(a) \backslash B$ and $b \neq a$, the opposite would imply $V^{\phi}(b) >_a u(a)/(1-\beta)$, which would contradict $b \notin B$; finally, if $b = a$, $V^{\phi}(a) >_a V^{\phi}(d)$ is impossible, as this would imply $u(a) >_a (1-\beta) V^{\phi}(d)$ contradicting (A17), given the definition of d (A22). Now, Lemma 5 implies that $\phi' = \phi|_{[1,a]}$ is a MVE on $[1, a]$.

Suppose, to obtain a contradiction, that ϕ is not MVE. Since ϕ is made from MVE ϕ' on $[1, a]$ and MVE ϕ_2^a on $[a+1, m]$, properties 1 and 3 of Definition 3 are satisfied, and by Lemma 4 there are only two possible monotone deviations that may prevent ϕ from being MVE. First, suppose that for some $y \in [a+1, \phi_2^a(a+1)] \cap F(a)$,

$$V^{\phi}(y) >_a V^{\phi}(d). \tag{A23}$$

However, this would contradict (A22) (and if $y \notin B$, then (A23) is impossible as $d \in B$). The second possibility is that for some $y \in [d, a] \cap F(a+1)$, we have

$$V^{\phi}(y) >_{a+1} V^{\phi}(\phi_2^a(a+1)).$$

This means that $V^{\phi}_{\max M_{a+1}}(y) > V^{\phi}_{\max M_{a+1}}(\phi_2^a(a+1))$. At the same time, for any $x \in [a+1, \phi_2^a(a+1)] \cap F(a)$, we have $V^{\phi}_{\max M_{a+1}}(x) \leq V^{\phi}_{\max M_{a+1}}(\phi_2^a(a+1))$ (otherwise Lemma 2 would imply a profitable deviation to x). This implies that for any such x, $V^{\phi}_{\max M_{a+1}}(y) > V^{\phi}_{\max M_{a+1}}(x)$. Now, recall that

$$\phi_1^a \in \arg \max_{\phi \in \Phi^{[1,a]}, z \in [\phi(a), a] \cap F(a)} V^{\phi}_{\max M_{a+1}}(z).$$

This means that there is $z \in [\phi_1^a(a), a] \cap F(a)$ such that

$$V^{\phi_1^a}_{\max M_{a+1}}(z) \geq V^{\phi}_{\max M_{a+1}}(y),$$

and thus for any $x \in [a+1, \phi_2^a(a+1)] \cap F(a)$,

$$V^{\phi_1^a}_{\max M_{a+1}}(z) > V^{\phi}_{\max M_{a+1}}(x).$$

But $\phi_1^a = \phi^a$ on the left-hand side, and $\phi = \phi^a$ on the right-hand side. We therefore have that the following maximum is achieved on $[\phi^a(a), a]$:

$$\arg \max_{z \in [\phi^a(a), \phi^a(a+1)] \cap F(a)} V^{\phi^a}_{\max M_{a+1}}(z) \subset [\phi^a(a), a],$$

i.e., that [3] in Lemma 4 holds. But this contradicts that $f(a) = 2$. This contradiction completes the induction step, which proves existence of a monotone MVE for any S.

Finally, suppose that ϕ is a monotone MVE; take any s_0. If $\phi(s_0) \geq s_0$, then monotonicity implies $\phi^2(s_0) \geq \phi(s_0)$ etc, and thus the sequence $\{\phi^k(s_0)\}$ is weakly increasing in k. It must therefore have a limit. A similar reasoning applies if $\phi(s_0) < s_0$, which completes the proof. ■

Proof of Theorem 2. We need to establish that the equilibrium is generically unique. For the purpose of this proof and other proofs in the paper, we call the set of parameters *generic* if β and $\{\pi(E, E')\}_{E,E' \in \mathcal{E}}$ satisfy the following: For any agent i and any set of mappings $\{\phi_E : S \to S\}_{E \in \mathcal{E}}$, the continuation values that solve (2) are such that for any environment $E \in \mathcal{E}$ and any two *different* states $x, y \in S$, $V^{\phi}_{E,i}(x) \neq V^{\phi}_{E,i}(y)$. In other words, this says that an agent is never indifferent between two states, regardless of continuation paths that will follow. Note that even though the statement involves continuation values, it is in fact an assumption on primitives, because the solution to (2) is uniquely determined by the primitives on the model. Indeed, one can rewrite (2) as $(\mathbf{I} + \Omega) V^{\phi}_{E,i}(s) = u_{E,i}(s)$, where $\mathbf{I}$ is the $mh \times mh$ identity matrix, and Ω is a matrix the elements of which depend on β and $\{\pi(E, E')\}_{E,E' \in \mathcal{E}}$. Ω defines a contraction mapping (in the sup norm), and thus $\mathbf{I} + \Omega$ is invertible, and $V^{\phi}_{E,i}(s) = (\mathbf{I} + \Omega)^{-1} u_{E,i}(s)$. This gives us no more than $n \times m^{mh} \times h \times \frac{m(m-1)}{2}$ linear conditions on utilities $\{u_{E,i}(s)\}$, which proves that the set of parameter values for which the condition above fails indeed has Lebesgue measure zero both in the set of feasible payoffs $\{u_{E,i}(s)\}^{E \in \mathcal{E}}_{s \in S}$ for fixed β and $\{\pi(E, E')\}_{E,E' \in \mathcal{E}}$ and in the set of all parameters $\left(\beta, \{\pi(E, E')\}_{E,E' \in \mathcal{E}}, \{u_{E,i}(s)\}^{E \in \mathcal{E}}_{s \in S}\right)$.

From now on, suppose that parameters satisfy the above condition. Under either of the assumptions of this theorem, any MVE is monotone; this follows from Theorem 7 which is proved below.

Suppose, to obtain a contradiction, that there are two MVE ϕ_1 and ϕ_2; then they are monotone by the argument above. Without loss of generality, assume that m is the minimal number of states for which this is possible, i.e., if $|S| < m$, then MVE is unique. Obviously, $m \geq 2$. Consider the set $Z = \{x \in S \mid \phi_1(x) \neq \phi_2(x)\}$, and denote $a = \min Z$, $b = \max Z$. Without loss of generality, assume that ϕ_1 and ϕ_2 are enumerated such that $\phi_1(a) < \phi_2(a)$.

Let us first show that if $\phi_1(x) = x$ or $\phi_2(x) = x$, then $x = 1$ or $x = m$. Indeed, suppose first that $\phi_1(x) = x$ and consider $\phi_2(x)$. If $\phi_2(x) < x$, then $\phi_1|_{[1,x]} \neq \phi_2|_{[1,x]}$ are two MVE for the set of states $[1, x]$, which contradicts the choice of m. If $\phi_2(x) > x$, we get a similar contradiction for $[x, m]$, and if $\phi_2(x) = x$, we get a contradiction by considering $[1, x]$ if $a < x$ and $[x, m]$ if

$a > x$. The case where $\phi_2 = x$ may be considered similarly. At this point, the proofs for the two parts diverge.

Part 1. Let us first prove that the following is true (auxiliary result): $a < m$; $b > 1$; if $x \in [\max\{2, a\}, b]$, then $\phi_1(x) < x \le \phi_2(x)$; if $x \in [a, \min\{b, m-1\}]$, then $\phi_1(x) \le x < \phi_2(x)$.

Assume first, to obtain a contradiction, that $a = m$. Then $Z = \{m\}$, so $\phi_1|_{[1,m-1]} = \phi_2|_{[1,m-1]}$; in this case, $\phi_1(m) \ne \phi_2(m)$ is impossible for generic parameter values (see the definition above). We would get a similar contradiction if $b = 1$, which proves that $a < m$ and $b > 1$, thus proving the first part of the auxiliary result.

Let us now show that for $x \in [a, b] \setminus \{1, m\}$, we have that either $\phi_1(x) < x < \phi_2(x)$ or $\phi_2(x) < x < \phi_1(x)$. Indeed, neither $\phi_1(x) = x$ nor $\phi_2(x) = x$ is possible. If $\phi_1(x) < x$ and $\phi_2(x) < x$, then $\phi_1|_{[1,x]}$ and $\phi_2|_{[1,x]}$ are two different MVE on $[1, x]$, which is impossible; we get a similar contradiction if $\phi_1(x) > x$ and $\phi_2(x) > x$. This also implies that if $a < x < b$, then $x \in Z$.

We now prove that for any $x \in Z$, $\phi_1(x) < \phi_2(x)$. Indeed, suppose that $\phi_2(x) > \phi_1(x)$ (equality is impossible as $x \in Z$); then $x > a \ge 1$. If $x < m$, then, as we proved, we must have $\phi_2(x) < x < \phi_1(x)$, and if $x = m$, then $\phi_2(x) < \phi_1(x) \le m = x$. In either case, $\phi_2(x) < x$, and since $\phi_2(a) > \phi_1(a) \ge 1$, then by monotonicity of ϕ_2 there must be $y : 1 \le a < y < x \le m$ such that $\phi_2(y) = y$, but we proved that this is impossible. Hence, $\phi_1(x) < \phi_2(x)$ for any $x \in Z$, and using the earlier result, we have $\phi_1(x) < x < \phi_2(x)$ for any $x \in Z \setminus \{1, m\}$.

To complete the proof of the auxiliary result, it suffices to show that $\phi_1(1) = 1$ and $\phi_2(m) = m$. Suppose, to obtain a contradiction, that $\phi_1(1) > 1$. We then have $\phi_2(1) > 1$, then $\phi_1(2) \ge 2$ and $\phi_2(2) \ge 2$ and thus $\phi_1|_{[2,m]}$ and $\phi_2|_{[2,m]}$ are MVE on $[2, m]$, and since $b \ne 1$, they must be different, which would again contradict the choice of m. We would get a similar contradiction if $\phi_2(m) = m$. This completes the proof of the auxiliary result.

To complete the proof of the theorem, notice that the auxiliary result implies, in particular, that $Z = [a, b] \cap S$, so Z has no "gaps". We define function $g : Z \to \{1, 2\}$ as follows. If $V_{M_x}^{\phi_1}(\phi_1(x)) > V_{M_x}^{\phi_2}(\phi_2(x))$, then $g(x) = 1$, and if $V_{M_x}^{\phi_1}(\phi_1(x)) < V_{M_x}^{\phi_2}(\phi_2(x))$, then $g(x) = 2$ (the case $V_{M_x}^{\phi_1}(\phi_1(x)) = V_{M_x}^{\phi_2}(\phi_2(x))$ is ruled out by the genericity assumption).[33] Intuitively g picks the equilibrium (left or right) that agent M_x prefers.

Let us prove that $g(a) = 2$ and $g(b) = 1$. Indeed, suppose that $g(a) = 1$; since $a < m$,

[33] In particular, the auxiliary result implies that for all iterations $k \ge 1$, $\phi_1^k(x) < \phi_2^k(x)$. Then $V_{M_x}^{\phi_1}(\phi_1(x)) = V_{M_x}^{\phi_2}(\phi_2(x))$ would imply that $V_{M_x}^{\phi}(\phi_1(x)) = V_{M_x}^{\phi}(\phi_2(x))$ for ϕ with the following properties: $\phi(y) = \phi_1(y)$ if $y < x$, $\phi(y) = \phi_2(y)$ if $y > x$, and $\phi(x) = x$. But this is ruled out.

we must have $\phi_1(a) \leq a < \phi_2(a)$ (with equality if $a = 1$ and strict inequality otherwise). Consider two cases. If $a > 1$, then for $x < a$, $\phi_1(x) = \phi_2(x)$, and since $\phi_1(a) < a$, then $V_{M_a}^{\phi_1}(\phi_1(a)) = V_{M_a}^{\phi_2}(\phi_1(a))$. But $g(a) = 1$ would imply that $V_{M_a}^{\phi_1}(\phi_1(a)) > V_{M_a}^{\phi_2}(\phi_2(a))$, and thus $V_{M_a}^{\phi_2}(\phi_1(a)) > V_{M_a}^{\phi_2}(\phi_2(a))$, which contradicts that ϕ_2 is MVE. If $a = 1$, then $g(a) = 1$ would imply that $V_{M_1}^{\phi_1}(1) > V_{M_1}^{\phi_2}(\phi_2(1))$. But $\phi_1(1) = 1$, which means $\frac{u_{M_1}(1)}{1-\beta} > V_{M_1}^{\phi_2}(\phi_2(1))$, thus $u_{M_1}(1) + \beta V_{M_1}^{\phi_2}(\phi_2(1)) > V_{M_1}^{\phi_2}(\phi_2(1))$. The left-hand side equals $V_{M_1}^{\phi_2}(1)$, and thus we have $V_{M_1}^{\phi_2}(1) > V_{M_1}^{\phi_2}(\phi_2(1))$. This contradicts that ϕ_2 is an MVE, thus proving that $g(a) = 2$. We can similarly prove that $g(b) = 1$.

Clearly, there must be two states $s, s+1 \in Z$ such that $g(s) = 2$ and $g(s+1) = 1$. For such s, let us construct mapping ϕ as follows:

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if } x \leq s \\ \phi_2(x) & \text{if } x > s \end{cases};$$

then $\phi(s) \leq s < \phi_2(s)$ (the first inequality is strict unless $s = 1$) and $\phi(s+1) \geq s+1 > \phi_1(s+1)$ (the first inequality is strict unless $s+1 = m$), which implies, in particular, that ϕ is monotone. Now, $g(s) = 2$ implies that $V_{M_s}^{\phi_2}(\phi_2(s)) > V_{M_s}^{\phi_1}(\phi_1(s))$. But $V_{M_s}^{\phi_2}(\phi_2(s)) = V_{M_s}^{\phi}(\phi_2(s))$ and $V_{M_s}^{\phi_1}(\phi_1(s)) = V_{M_s}^{\phi}(\phi_1(s))$, and thus $V_{M_s}^{\phi}(\phi_2(s)) > V_{M_s}^{\phi}(\phi(s))$ (note also that $s+1 \leq \phi_2(s) \leq \phi_2(s+1)$). Similarly, $g(s+1) = 1$ implies $V_{M_{s+1}}^{\phi}(\phi_1(s+1)) > V_{M_{s+1}}^{\phi}(\phi(s+1))$. But this contradicts Lemma 4 for mapping ϕ (since $\phi_2(s) > s$ and $\phi_1(s+1) < s+1$). This contradiction completes the proof.

Part 2. If for some x, $\phi_1(x) < x < \phi_2(x)$ or vice versa, then for all $i \in M_x$, there must be both a state $x_1 < x$ and a state $x_2 > x$ such that $u_i(x_1) > u_i(x)$ and $u_i(x_2) > u_i(x)$, which contradicts the assumption in this case. Since for $1 < x < m$, $\phi(x) \neq x$, we get that $\phi_1(x) = \phi_2(x)$ for such x. Let us prove that $\phi_1(1) = \phi_2(1)$. If this is not the case, then $\phi_1(1) = 1$ and $\phi_2(1) = 2$ (or vice versa). If $m = 2$, then monotonicity implies $\phi_2(2) = 2$, and if $m > 2$, then, as proved earlier, we must have $\phi_2(x) = x+1$ for $1 < x < m$ and $\phi_2(m) = m$. In both cases, we have $\phi_1(x) = \phi_2(x) > 1$ for $1 < x \leq m$. Hence, $V_i^{\phi_1}(2) = V_i^{\phi_2}(2)$ for all $i \in N$. Since ϕ_1 is MVE, we must have $u_i(1)/(1-\beta) \geq V_i^{\phi_1}(2)$ for $i \in M_1$, and since ϕ_2 is MVE, we must have $V_i^{\phi_2}(2) \geq u_i(1)/(1-\beta)$. This is only possible if $V_i^{\phi_1}(2) = u_i(1)/(1-\beta)$, which is equivalent to $V_i^{\phi_1}(2) = V_i^{\phi_1}(1)$. However, if parameter values are generic according to the definition above, this cannot be true, and this proves that $\phi_1(1) = \phi_2(1)$. We can likewise prove that $\phi_1(m) = \phi_2(m)$, thus establishing uniqueness. ■

Proof of Theorem 3. The existence is proved in the text. Since, on equilibrium path, there is only a finite number of shocks, from some period t on the environment will be the same, say E^x. Since ϕ_{E^x} is monotone, the sequence $\{s_t\}$ has a limit by Theorem 1. The fact that this limit may depend on the sequence of shock realizations is shown by Example B2. ■

Proof of Theorem 4. Part 1. Without loss of generality, suppose that h is the minimal number for which two monotone MVE $\phi = \{\phi_E\}_{E\in\mathcal{E}}$ and $\phi' = \left\{\phi'_E\right\}_{E\in\mathcal{E}}$ exist. For generic parameter values, if we take $\tilde{\mathcal{E}} = \left\{E^2, \ldots, E^h\right\}$ with the same environments $E^2, \ldots, E^h$ and the same transition probabilities, we will have a unique monotone MVE $\tilde{\phi} = \{\phi_E\}_{E\in\mathcal{E}'} = \left\{\phi'_E\right\}_{E\in\mathcal{E}'}$ by assumption. Now, with the help of transformation used in Subection 3.2 in the proof of Theorem 3 we get that ϕ_{E^1} and ϕ'_{E^1} must be MVE in a certain (stationary) environment $\tilde{E}$. However, by Theorem 2 such MVE is unique, which leads to a contradiction.

Part 2. The proof is similar to that of Part 1. The only step is that we need to verify that we can apply Part 2 of Theorem 2 to the (stationary) environment $\tilde{E}$. In general, this will not be the case. However, it is easy to notice (by examining the proof of Part 2 of Theorem 2) that instead of single-peakedness, we could require a weaker condition: that for each $s \in S$ there is $i \in M_s$ such that there do not exist $x < s$ and $y > s$ such that $u_i(x) \geq u_i(s)$ and $u_i(y) \geq u_i(s)$.

We can now prove that if $\{u_i(s)\}_{i\in N}^{s\in S}$ satisfy this property and ϕ is MVE, then $\left\{V_i^{\phi}(s)\right\}_{i\in N}^{s\in S}$ also does. Indeed, suppose, to obtain a contradiction, that for some $s \in S$, for all $i \in M_s$ there are $x_i < s$ and $y_i > s$ such that $V_i^{\phi}(x_i) \geq V_i^{\phi}(s)$ and $V_i^{\phi}(y_i) \geq V_i^{\phi}(s)$; without loss of generality, we may assume that x_i and y_i minimize $|x_i - s|$ and $|y_i - s|$ among such x_i and y_i.

Consider the case $\phi(s) > s$. This implies that for all $i \in M_s$, there is $a > s$ such that $u_i(a) > u_i(s)$, and therefore for all $i \in M_s$ and all $a < s$, $u_i(z) < u_i(s)$. Moreover, for all $i \in M_s$, $u_i(z) < V_i^{\phi}(s)/(1-\beta)$. Take $j = \max M_s$, and let $z = x_j$. We cannot have $\phi(z) \leq z$, because then $V_j^{\phi}(\phi(z)) \geq V_j^{\phi}(s)$ would be impossible. Thus, $\phi(z) > z$, and in this case we must have $\phi(z) > s$, To see this, notice that $V_j^{\phi}(z) = u_j(z) + \beta V_j^{\phi}(\phi(z))$. If $\phi(z) < s$, then $V_j^{\phi}(z) \geq V_j^{\phi}(s)$ and $u_j(z) < V_i^{\phi}(s)/(1-\beta)$, implying $V_j^{\phi}(\phi(z)) > V_j^{\phi}(s)$ and thus contradicting the choice of $z = x_j$. If $\phi(z) = s$, then $V_j^{\phi}(z) = u_j(z) + \beta V_j^{\phi}(\phi(z))$ contradicts $V_j^{\phi}(z) \geq V_j^{\phi}(s)$ and $u_j(z) < V_i^{\phi}(s)/(1-\beta)$. Consequently, $\phi(z) > s$. Monotonicity of ϕ implies $s < \phi(z) \leq \phi(s)$. Now, $V_j^{\phi}(z) \geq V_j^{\phi}(s)$ and $u_j(z) < u_j(s)$ implies $V_j^{\phi}(\phi(z)) > V_j^{\phi}(\phi(s))$ (and in particular, $\phi(z) < \phi(s)$). Since $j = \max M_s$, we have $V^{\phi}(\phi(z)) >_s V^{\phi}(\phi(s))$. Since $s < \phi(z) < \phi(s)$, $\phi(z) \in F_s$, and therefore a deviation in s from $\phi(s)$ to $\phi(z)$ is feasible and profitable. This contradicts that ϕ is a MVE. We would get a similar contradiction if we assumed that $\phi(s) < s$.

Finally, assume $\phi(s) = s$. Then take any $i \in M_s$, and suppose, without loss of generality, that for any $a < s$, $u_i(a) < u_i(s)$. Then, since for all such a, $\phi^k(s) \leq s$ for all $k \geq 1$, we must have $V_i^{\phi}(a) < V_i^{\phi}(s)$, which contradicts the assertion. This proves the auxiliary result.

We have thus proved that under the assumptions of the theorem, the environment $\tilde{E}$ constructed in the proof of 3 satisfies the requirements Part 2 of Theorem 2. The rest of the proof follows immediately. ■

Proof of Theorem 5. Part 1. It suffices to prove this result for the stationary case. For each $s \in S$ take any protocol such that if $\phi(s) \neq s$, then $\theta_s(|F_s| - 1) = \phi(s)$ (i.e., the desired transition is the last one to be considered). We claim that there is a strategy profile σ such that if for state s, $\phi(s) = s$, then no alternative is accepted, and if $\phi(s) \neq s$, then no alternative is accepted until the last stage, and in this last stage, the alternative $\phi(s)$, is accepted.

Indeed, under such a profile, the continuation strategies are given by (4). To show that such an outcome is possible in equilibrium, consider first periods where $\phi(s) \neq s$. Consider the subgame reached if no alternatives were accepted before the last one. Since by property 3 of Definition 3, $V^{\phi}(\phi(s)) \geq_s V^{\phi}(s)$, it is a best response for players to accept $\phi(s)$. Let us now show, by backward induction, that if stage k, $1 \leq k \leq |F_s| - 1$ is reached without any alternatives accepted, then there is an equilibrium where $\phi(s)$ is accepted in the last stage. The base was just proved. The induction step follows from the following: if at stage k, alternative $y = \theta_s(k)$ is under consideration, then accepting it yields a vector of payoffs $V^{\phi}(y)$, and rejecting it yields, by induction, $V^{\phi}(\phi(s))$. Since by property 2 of Definition 3, $V^{\phi}(y) \not>_s V^{\phi}(\phi(s))$, it is a best response to reject the alternative y. Consequently, $\phi(s)$ will be accepted by induction. This proves the induction step, and therefore $\phi(s)$ is the outcome in a period which started with s. Now consider a period where $\phi(s) = s$. By backward induction, we can prove that there is an equilibrium where no proposal is accepted. Indeed, the last proposal $\theta_s(|F_s| - 1)$ may be rejected, because $V^{\phi}(\theta_s(|F_s| - 1)) \not>_s V^{\phi}(s)$ by property 2 of Definition 3. Going backward, if for some stage k, s is the outcome once $\theta_s(k)$ was rejected, sufficiently many players may reject $\theta_s(k)$, because $V^{\phi}(\theta_s(k)) \not>_s V^{\phi}(s)$. This proves that in periods where $\phi(s) = s$, it is possible to have an equilibrium where no proposal is accepted. Combining the equilibrium strategies for different initial s in the beginning of the period, we get a MPE which induces transition mappings $\phi(s)$.

Part 2. If the transition mapping is monotone, then continuation utilities $\left\{V_{E,i}^{\phi}(s)\right\}_{i \in N}^{s \in S} = \left\{V_{E,i}^{\sigma}(s)\right\}_{i \in N}^{s \in S}$ satisfy increasing differences for any $E \in \mathcal{E}$ by Lemma 2. Again, the proof that ϕ is MVE reduces to the stationary case. For each state s, we consider the set $J_s \subset \{1, \ldots, |F_s| - 1\}$ of stages k where the alternative under consideration, $\theta_s(k)$, is accepted if this stage is reached. Naturally, $\phi(s) = s$ if and only if $J_s = \varnothing$, and if $J_s \neq \varnothing$, then $\phi(s) = \theta_s(\min J_s)$. Moreover, one can easily prove by induction that for any $j, k \in J_s$ such that $j \leq k$, $V^{\phi}(\theta_s(j)) \geq_s V^{\phi}(\theta_s(k))$ (this follows from transitivity of $\geq_s$ established in Lemma 1), and thus for any $j \in J_s$, $V^{\phi}(\theta_s(j)) \geq_s V^{\phi}(s)$.

Take any $s \in S$. Property 1 of Definition 3 holds trivially, because only states in F_s are considered as alternatives and may be accepted. Let us show that Property 2 holds. First, consider the case $\phi(s) = s$. Suppose, to obtain a contradiction, that for some $y \in F_s$, $V^{\phi}(y) >_s V^{\phi}(s)$. Suppose that this y is considered at stage k. But then, if stage k is reached, a winning coalition of players must accept y, because rejecting it leads to s. Then $k \in J_s$, contradicting $J_s = \varnothing$ for such s. Second, consider the case $\phi(s) \neq s$. Again, suppose that for some $y \in F_s$, $V^{\phi}(y) >_s V^{\phi}(\phi(s))$; notice that $y \neq s$, because $V^{\phi}(\phi(s)) = V^{\phi}(\theta_s(\min J_s)) \geq_s V^{\phi}(s)$. Let k be the stage where y is considered. If $k < \min J_s$, so y is considered before $\phi(s)$, then a winning coalition must accept y, which implies $k \in J_s$, contradicting $k < \min J_s$. If, on the other hand, $k > \min J_s$, then notice that $k \notin J_s$ (otherwise, $V^{\phi}(y) >_s V^{\phi}(\phi(s))$ is impossible). If $k > \max J_s$, then we have $V^{\phi}(y) >_s V^{\phi}(\phi(s)) = V^{\phi}(\theta_s(\min J_s)) \geq_s V^{\phi}(s)$, which means that this proposal must be accepted, so $k \in J_s$, a contradiction. If $k < \max J_s$, then we can take $l = \min\{J_s \cap [k+1, |F_s| - 1]\}$. Since $V^{\phi}(y) >_s V^{\phi}(\phi(s)) = V^{\phi}(\theta_s(\min J_s)) \geq_s V^{\phi}(\theta_s(l))$, it must again be that y is accepted, so $k \in J_s$, again a contradiction. In all cases, the assertion that such y exists leads to a contradiction, which proves that Property 2 holds.

Finally, we show that Property 3 of Definition 3 holds. This is trivial if $\phi(s) = s$. Otherwise, we already proved that for all $j \in J_s$, $V^{\phi}(\theta_s(j)) \geq_s V^{\phi}(s)$; in particular, this is true for $j = \min J_s$. Consequently, $V^{\phi}(\phi(s)) \geq_s V^{\phi}(s)$. This completes the proof that ϕ is a MVE. ■

Proof of Theorem 6. Suppose, to obtain a contradiction, that $\tilde{\phi}_{\tilde{E}^1}(x) < x$. Then $\phi_{E^1}|_{S'}$ and $\tilde{\phi}_{\tilde{E}^1}|_{S'}$ are mappings from S' to S' such that both are MVE on the restricted environment $E|_{S'}$, which is identical to $\tilde{E}|_{S'}$. Moreover, these MVE are different, as $\phi_{E^1}(x) = x > \tilde{\phi}_{\tilde{E}^1}(x)$. However, this violates uniqueness, completing the proof. ■

Proof of Corollary 1. Consider an alternative set of environments $\mathcal{E}' = \{E^0, E^2\}$, where E^0 coincides with E^2 on S, but the transition probabilities are the same as in $\mathcal{E}$. Clearly, ϕ' defined by $\phi'_{E^0} = \phi'_{E^2} = \phi_{E^2}$ is a MVE in $\mathcal{E}'$. Let us now consider environments $\tilde{E}^0$ and $\tilde{E}^1$

obtained from $\mathcal{E}'$ and $\mathcal{E}$, respectively, using the procedure from Section 3.2. Suppose, to obtain a contradiction, that $\phi_{E^2}(x) < x$, then environments $\tilde{E}^0$ and $\tilde{E}^1$ coincide on $[1, x]$ by construction. Theorem 6 then implies that, since $\phi_{E^1}(x) = x$, then $\phi'_{E^0}(x) \geq x$ (since ϕ'_{E^0} and ϕ_{E^1} are the unique MVE in $\tilde{E}^0$ and $\tilde{E}^1$, respectively). But by definition of ϕ', $x \leq \phi'_{E^0}(x) = \phi_{E^2}(x) < x$, a contradiction. This contradiction completes the proof. ■

Proof of Theorem 7. Take generic parameter values (see the proof of Theorem 2).

Part 1. It suffices to prove this result in stationary environments. By Theorem 8, there are no cycles, and thus for any $x \in S$, the sequence $x, \phi(x), \phi^2(x), \ldots$ has a limit. Suppose, to obtain a contradiction, that MVE ϕ is non-monotone, which means there are states $x, y \in S$ such that $x < y$ and $\phi(x) > \phi(y)$. Without loss of generality we can assume that x and y are such that the set $Z = \{x, \phi(x), \phi^2(x), \ldots; y, \phi(y), \phi^2(y), \ldots\}$ has fewest different states. In that case, mapping ϕ is monotone on the set $Z \setminus \{x, y\}$, which implies that $\{V_i^s\}_{i \in N}^{s \in Z \setminus \{x,y\}}$ satisfies increasing differences. By property 2 of Definition 3 applied to state x, we get

$$V_{\max M_x}(\phi(x)) \geq V_{\max M_x}(\phi(y)), \tag{A24}$$

and if we apply it to state y,

$$V_{\min M_y}(\phi(y)) \geq V_{\min M_y}(\phi(x)). \tag{A25}$$

Since $\max M_x \leq \min M_y$ by assumption, (A24) implies

$$V_{\min M_y}(\phi(x)) \geq V_{\min M_y}(\phi(y)).$$

For generic parameter values, this inequality is strict, and thus contradicts (A25).

Part 2. Again, consider stationary environments only. If ϕ is non-monotone, then for some $x, y \in S$ we have $x < y$ and $\phi(x) > \phi(y)$, which in this case implies $\phi(x) = y = x + 1$ and $\phi(y) = x$. However, if parameters are generic, this contradicts Theorem 8. This contradiction completes the proof. ■

Proof of Theorem 8. It suffices to prove that within any stationary environment E, a path that starts with any state s is monotone. We first rule out cycles, where for some x, $\phi(x) \neq x$, but $\phi^k(x) = x$ for some $k > 1$. Without loss of generality, let k be the minimal one for which this is true, and x be the highest element in the cycle. In this case, the we have, for any $i \in N$,

$$\begin{aligned} V_i(x) - V_i(\phi(x)) &= u_i(x) + \beta V_i(\phi(x)) - V_i(\phi(x)) = u_i(x) - (1-\beta) V_i(\phi(x)) \\ &= \sum\nolimits_{j=1}^{k-1} \frac{(1-\beta)\beta^{j-1}}{1-\beta^k} \left(u_i(x) - u_i\left(\phi^j(x)\right)\right), \end{aligned}$$

which is increasing in i, since each term is increasing in i as $x > \phi^j(x)$ for $j = 1, \ldots, k-1$. This means that $\{V_i(s)\}_{i \in N}^{s \in \{\phi(x), x\}}$ satisfies the increasing differences. Because of that, Property 3 of Definition 3, when applied to state x, implies that $V_i(\phi(x)) \geq V_i(x)$ for all $i \in M_x$. However, if we take $y = \phi^{k-1}(x)$ (so $\phi(y) = x$), then Property 2 of Definition 3 would imply that $V_i(x) \geq V_i(\phi(x))$ for at least one $i \in M_y$. Increasing differences implies that $V_i(x) \geq V_i(\phi(x))$ for at least one $i \in M_x$, and therefore for such i, $V_i(x) = V_i(\phi(x))$. This cannot hold for generic parameter values; this contradictions proves that cycles are ruled out.

Now, to prove that any path is monotone, assume the opposite, and take x that generates the shortest non-monotone path (i.e., such that the sequence $x, \phi(x), \phi^2(x), \ldots$ has the fewest different states). In that case, either $\phi(x) > x$, but $\phi^2(x) < \phi(x)$ or vice versa; without loss of generality consider the former case. Denote $y = \phi(x)$; then the sequence $y, \phi(y), \phi^2(y), \ldots$ is monotone by construction of x. Consequently, $\{V_i(s)\}_{i \in N}^{s \in \{y, \phi(y), \phi^2(y), \ldots\}}$ satisfies increasing differences. By Property 3 of Definition 3 applied to state y, for all $i \in M_y$, $V_i(\phi(y)) \geq V_i(y)$; for generic parameter values, this inequality is strict. Since $\phi(y) < y$, this is true for $i \in [1, \max M_y]$; now, $x < y$ implies $\max M_x \leq \max M_y$, and therefore, for all $i \in M_x$, $V_i(\phi(y)) > V_i(y)$. However, this contradicts Property 2 of Definition 3, applied to state x. This contradiction completes the proof. ■

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Appendix B — For Online Publication

B1 Additional Results

Additional comparative statics

In the next two Theorems, parameters are assumed to be generic (see the proof of Theorem 2).

Theorem B1 ***(Monotonicity everywhere)*** *Suppose that $\mathcal{E} = \left\{E^1, E^2\right\}$, $0 < \pi\left(E^1, E^2\right) < 1$, $\pi\left(E^2, E^1\right) = 0$, and E^1 and E^2 coincide on $S' = [1, s] \subset S$. Then there exists $\beta_0 > 0$ such that if $\beta < \beta_0$, then in the unique MVE ϕ, if the initial state is $s_0 \in S'$ such that $\phi_{E^1}(s_0) \geq s_0$, then the entire path $s_0, s_1, s_2, \ldots$ (induced both under environment E^1 and after the switch to E^2) is monotone. Moreover, if the shock arrives at time t, then for all $\tau \geq t$, $s_\tau \geq \tilde{s}_\tau$, where $\tilde{s}_\tau$ is the hypothetical path if the shock never arrives.*

Proof. Let us first prove this result for the case where each QMV is a singleton. Both before and after the shock, the mapping that would map any state x to a state which maximizes the stage payoff $u_{M_x}(y)$ would be a monotone MVE for $\beta < \beta_0$. By uniqueness, ϕ_{E^1} and ϕ_{E^2} would be these mappings under E^1 and E^2, respectively. Now it is clear that if the shock arrives at period t, and the state at the time of shock is $x = s_{t-1}$, then $\phi_{E^2}(x)$ must be either the same as $\phi_{E^1}(x)$ or must satisfy $\phi_{E^2}(x) > s$. In either case, we get a monotone sequence after the shock. Moreover, the sequence is the same if $s_\tau \leq s$, and if $s_\tau > s$, then we have $s_\tau > s \geq \tilde{s}_\tau$ automatically.

The general case may be proved by observing that a mapping that maps each state x to an alternative which maximizes by $u_{\min M_x}(y)$ among the states such that $u_i(y) \geq u_i(x)$ for all $i \in M_x$ is a monotone MVE. Such mapping is generically unique, and by the assumption of uniqueness it coincides with the mapping ϕ_{E^1} if the environment is E^1 and it coincides with ϕ_{E^2} if the environment is E^2. The remainder of the proof is analogous. ■

Theorem B2 ***(Additional veto players)*** *Suppose that stationary environments E^1 and E^2 have the same payoffs, $u_{E^1,i}(x) = u_{E^2,i}(x)$, that the same transitions are feasible ($F_{E^1} = F_{E^2}$) and that $M_{E^1,x} = M_{E^2,x}$ for $x \in [1, s]$ and $\min M_{E^1,x} = \min M_{E^2,x}$ for $x \in [s+1, m]$. Suppose also that the MVE ϕ_1 in E^1 and MVE ϕ_2 in E^2 are unique on any subset of $[1, s]$. Then $\phi_1(x) = \phi_2(x)$ for any $x \in [1, s]$.*

Proof. It is sufficient, by transitivity, to prove this Theorem for the case where $\max M_{E^1,x} \neq \max M_{E^2,x}$ for only one state $x \in [s+1, m]$. Moreover, without loss of generality, we can assume that $\max M_{E^1,x} < \max M_{E^2,x}$. Notice that if $\phi_1(x) \geq x$, then ϕ_1 is MVE in environment E^2, and by uniqueness must coincide with ϕ_2.

Consider the remaining case $\phi_1(x) < x$; it implies $\phi_1(x-1) \leq x-1$. Consequently, $\phi_1|_{[1,x-1]}$ is MVE under either environment restricted on $[1, x-1]$ (they coincide on this interval). Suppose, to obtain a contradiction, that $\phi_1|_{[1,s]} \neq \phi_2|_{[1,s]}$; since $x > s$, we have $\phi_1|_{[1,x-1]} \neq \phi_2|_{[1,x-1]}$. We must then have $\phi_2(x-1) > x-1$ (otherwise there would be two MVE $\phi_1|_{[1,x-1]}$ and $\phi_2|_{[1,x-1]}$ on $[1, x-1]$, and therefore $\phi_2(x) \geq x$. Consequently, $\phi_2|_{[x,m]}$ is MVE on $[x, m]$ under environment E^2 restricted on $[x, m]$. Let us prove that $\phi_2|_{[x,m]}$ is MVE on $[x, m]$ under environment E^1 restricted on $[x, m]$ as well. Indeed, if it were not the case, then there must be a monotone deviation, as fewer QMVs (in state x) imply that only Property 2 of Definition 3 may be violated. Since under E^1, state x has fewer QMVs than under E^2, it is only possible if $\phi_2(x) > x$, in which case $\phi_2(x+1) \geq x+1$. Then $\phi_2|_{[x+1,m]}$ would be MVE on $[x+1, m]$, and by Lemma 5 we could get MVE $\tilde{\phi}_2$ on $[x, m]$ under environment E^1. This MVE $\tilde{\phi}_2$ would be MVE on $[x, m]$ under environment E^2. But then under environment E^2 we have two MVE, $\tilde{\phi}_2$ and $\phi_2|_{[x,m]}$ on $[x, m]$, which is impossible.

We have thus shown that $\phi_1|_{[1,x-1]}$ is MVE on $[1, x-1]$ under both E^1 and E^2, and the same is true for $\phi_2|_{[x,m]}$ on $[x, m]$. Take mapping ϕ given by

$$\phi(y) = \begin{cases} \phi_1(y) & \text{if } y < x \\ \phi_2(y) & \text{if } y > x \end{cases}.$$

Since $\phi_1|_{[1,x-1]} \neq \phi_2|_{[1,x-1]}$ and $\phi_1|_{[x,m]} \neq \phi_2|_{[x,m]}$ ($\phi_1(x-1) \leq x-1$, $\phi_2(x-1) > x-1$, $\phi_1(x) < x$, $\phi_2(x) \geq x$), ϕ is not MVE in E^1 nor it is in E^2. By Lemma 4, in both E^1 and E^2 only one type of monotone deviation (at $x-1$ to some $z \in [x, \phi_2(x)]$ or at x to some $z \in [\phi_1(x-1), x]$) is possible. But the payoffs under the first deviation are the same under both E^1 and E^2; hence, in both environments it is the same type of deviation.

Suppose that it is the former deviation, at $x-1$ to some $z \in [x, \phi_2(x)]$. Consider the following restriction on feasible transitions:

$$\tilde{F}(a) = \begin{cases} F(a) & \text{if } a \geq x; \\ F(a) \cap [1, x-1] & \text{if } a < x; \end{cases}$$

denote the resulting environments by E^1 and E^2. This makes the deviation impossible, and thus ϕ is MVE in E^1 (in E^2 as well). However, ϕ_1 is also MVE in E^1, as it is not affected by the change is feasibility of transitions, and this contradicts uniqueness. Finally, suppose that the deviation is at x to some $z \in [\phi_1(x-1), x]$. Then consider the following restriction on feasible transitions:

$$\bar{F}(a) = \begin{cases} F(a) & \text{if } a < x; \\ F(a) \cap [x, m] & \text{if } a \geq x; \end{cases}$$

denote the resulting environments by $\bar{E}^1$ and $\bar{E}^2$. This makes the deviation impossible, and thus ϕ is MVE in E^2. However, ϕ_2 is also MVE in E^1, as it is not affected by the change in feasibility. Again, this contradicts uniqueness, which completes the proof. ■

Extension: Continuous spaces

In this subsection, we show how our results can be extended to economies with a continuum of states and/or a continuum of players.

Suppose that the set of states is $S = [s_l, s_h]$, and the set of players is given by $N = [i_l, i_h]$. (The construction and reasoning below are easily extendable to the case where the are a finite number of players but a continuum of states, or vice versa.) We assume that each player has a utility function $u_i(s) : S \to \mathbb{R}$, which is continuous as a function of $(i, s) \in N \times S$ and satisfies increasing differences: for all $i > j$, $x > y$,

$$u_i(x) - u_i(y) \geq u_j(x) - u_j(y).$$

The mapping F, which describes feasible transitions, is assumed to be upper-hemicontinuous on S and to satisfy Assumption 5. Finally, for each state s there is a set of winning coalitions W_s, which are assumed to satisfy Assumption 3. As before, for each state s, we have a non-empty set of QMVs M_s (which may nevertheless be a singleton). We make the following version of Monotone QMV assumption: functions $\inf M_s$ and $\sup M_s$ are continuous and increasing functions of s.

For simplicity, let us focus on the case without shocks and on monotone transition functions $\phi : S \to S$ (this function may be discontinuous). MVE is defined as in Definition 3. The following result establishing the existence of MVE.

Theorem B3 ***(Existence in Continuous Spaces)*** *With a continuum of states and/or players, there exists a MVE ϕ. Moreover, take any sequence of sets of states $S_1 \subset S_2 \subset \cdots$ and any sequence of players $N_1 \subset N_2 \subset \cdots$ such that $\bigcup_{j=1}^{\infty} S_j$ is dense in S and $\bigcup_{j=1}^{\infty} N_j$ is dense in N. Consider any sequence of monotone functions $\{\phi_j : S_j \to S_j\}_{j=1}^{\infty}$ which are MVE (not necessarily unique) in the environment*

$$E^j = \left(N, S, \beta, \{u_i(s)\}_{i \in N_j}^{s \in S_j}, \{W_s\}_{s \in S_j}, \{F_j(s)\}_{s \in S_j}\right).$$

Existence of such MVE is guaranteed by Theorem 1, as all assumptions are satisfied. Then there is a subsequence $\{j_k\}_{k=1}^{\infty}$ *such that* $\left\{\phi_{j_k}\right\}_{k=1}^{\infty}$ *converges pointwise on* $\bigcup_{j=1}^{\infty} S_j$*, to some MVE* $\phi : S \to S$*.*

Proof. Take an increasing sequence of sets of points, $S_1 \subset S_2 \subset S_3 \subset \cdots$, so that $\bigcup\limits_{i=1}^{\infty} S_i$ is dense. For each S_i, take MVE ϕ_i. We know that ϕ_i is a monotone function on S_i; let us complement it to a monotone (not necessarily continuous) function on S which we denote by $\tilde{\phi}_i$ for each i. Since $\tilde{\phi}_i$ are monotone functions from a bounded set to a bounded set, there is a subsequence $\tilde{\phi}_{i_k}$ which converges to some $\tilde{\phi}$ pointwisely. (Indeed, we can pick a subsequence which converges on S_1, then a subsequence converging on S_2 etc; then use a diagonal process. After it ends, the set of points where convergence was not achieved is at most countable, so we can repeat the diagonal procedure.) To show that $\tilde{\phi}$ is a MVE, suppose not, then there are two points x and y such that y is preferred to $\tilde{\phi}(x)$ by all members of M_x. Here, we need to apply a continuity argument and say that it means that the same is true for some points in some S_i. But this would yield a contradiction. ■

Proofs of Theorem 9 and Propositions 1-8

Proof of Theorem 9. For each $i \geq 1$, consider a truncated set of environments $\mathcal{E}^i = \left\{E^1, E^2, \ldots, E^i\right\}$ and transition mappings such that for all $1 \leq j,\, k \leq i$,

$$\pi^{\mathcal{E}^i}\left(E^j, E^k\right) = \begin{cases} 0 & \text{if } j > k \\ \pi\left(E^j, E^k\right) & \text{if } j < k \\ \pi\left(E^j, E^k\right) + \sum_{l=i+1}^{\infty} \pi\left(E^j, E^l\right) & \text{if } j = k \end{cases}$$

(in other words, we replace transitions to high-numbered environments with staying at the same environment; in particular, E^i is a static environment). In this case, for $\mathcal{E}^i$, Theorem 3 is applicable, and there is a MVE $\phi^{\mathcal{E}^i} = \left\{\phi_{E^j}^{\mathcal{E}^i}\right\}_{1 \leq j \leq i}$.

Take the first environment E^1 and consider the sequence $\left\{\phi_{E^1}^{\mathcal{E}^i}\right\}_{i=1}^{\infty}$. Since the number of mappings $\phi : S \to S$ is finite, there will be a mapping ϕ_{E^1} which equals $\phi_{E^1}^{\mathcal{E}^i}$ for an infinite number of i; let the set of such i be called Z_1. Now, take the second environment E^2 and consider the sequence $\left\{\phi_{E^2}^{\mathcal{E}^i}\right\}_{i \in Z_1 \setminus \{1\}}$ (we subtract the first element because $\phi_{E^2}^{\mathcal{E}^1}$ is not defined). Again, there is a mapping ϕ_{E^2} which equals $\phi_{E^2}^{\mathcal{E}^i}$ for an infinite number of $i \in Z_1$; let the set of such i be called $Z_2 \subset Z_1$. Proceeding in a similar way, for every $j > 2$ we will construct ϕ_{E^j} that equals $\phi_{E^j}^{\mathcal{E}^i}$ for an infinite number of $i \in Z_{j-1}$. We claim that the set of mappings $\phi = \{\phi_{E^j}\}_{j=1}^{\infty}$ constructed in this way is MVE.

Suppose that it is not; then Definition 3 is violated for some j. Property 1 (feasibility) holds true by construction. Suppose Property (core) is violated. Then for some $x, y \in S$, we have $y \in F_{E^j}(x)$ and $Y \equiv \left\{i \in N : V_{E^j,i}^{\phi}(y) > V_{E^j,i}^{\phi}(\phi_{E^j}(x))\right\} \in W_{E^j,x}$. Let $\varepsilon = \min_{i \in Y}\left(V_{E^j,i}^{\phi}(y) - V_{E^j,i}^{\phi}(\phi_{E^j}(x))\right) > 0$. Now take $k \in \mathbb{N}$ so large that $4M\frac{\beta^{k-j}}{1-\beta} < \varepsilon$.

Now, consider a truncated environment $\mathcal{E}^l$, where $l \in Z^k$. By construction, in $\mathcal{E}^l$, there is MVE $\phi^{\mathcal{E}^l} = \left\{\phi_{E^r}^{\mathcal{E}^l}\right\}_{1 \leq r \leq l}$ which coincides with ϕ for $r \leq k$: $\phi_{E^r}^{\mathcal{E}^l} = \phi_{E_r}$ for such r. By the choice of k, we must have that for all $i \in Y$, the continuation payoffs of player i under $\phi^{\mathcal{E}^l}$ satisfy the same inequality: $V_{E^j,i}^{\phi^{\mathcal{E}^l}}(y) > V_{E^j,i}^{\phi^{\mathcal{E}^l}}(\phi_{E^j}(x))$. But this violates the definition of MVE for $\phi^{\mathcal{E}^l}$ in $\mathcal{E}^l$, which is a contradiction.

Finally, if Property 3 (status quo) were violated, we would get a similar contradiction. This completes the proof. ■

Proof of Proposition 1. Part 1. We start by proving that there exists a unique monotone MVE. To show this, we need to establish that all requirements for existence and generic uniqueness are satisfied.

(Increasing differences) Consider player i and take two states x, y with $x > y$. The policy in state x is b_{M_x} and in state y, it is b_{M_y}. Since $M_x \geq M_y$ and b is increasing in the identity of the player, we have $b_{M_x} \geq b_{M_y}$. Take the difference

$$\begin{aligned} u_i(x) - u_i(y) &= -(b_{M_x} - b_i)^2 - \sum\nolimits_{j \notin H_x} \gamma_j C_j - \left(-\left(b_{M_y} - b_i\right)^2 - \sum\nolimits_{j \notin H_y} \gamma_j C_j\right) \\ &= \left(b_{M_x} - b_{M_y}\right)\left(2b_i - b_{M_x} - b_{M_y}\right) - \sum\nolimits_{j \notin H_x} \gamma_j C_j + \sum\nolimits_{j \notin H_y} \gamma_j C_j. \end{aligned}$$

This only depends on i through b_i, which is increasing in b_i. Hence, increasing differences is satisfied.

(Monotone QMV) The QMV in state s is M_s. If $s \geq 0$, then an increase in s implies that players on the right get more power, and $s \leq 0$, then a decrease in s implies that players on the left get more power.

(Feasibility) All transitions are feasible, and thus the assumption holds trivially.

(QMV are singletons) This holds generically, when no two disjoint sets of players have the same power.

This establishes that there is a unique monotone MVE. To show that $\phi(0) = 0$, suppose not. Without loss of generality, $\phi(0) > 0$. Then if $s_1 = 0$, monotonicity implies that $s_t > 0$ for all

$t > 1$. But $M_0 = 0$, thus $b_{M_0} = b_0$ and $u_{M_0}(0) = 0$, while $u_{M_0}(s) < 0$ for $s \neq 0$. This shows that if $\phi(0) > 0$, there is a profitable deviation to 0. This contradiction completes the proof.

Part 2. Consider the case $s < 0$ (the case $s > 0$ is considered similarly). Since $\phi(0) = 0$, monotonicity implies that $\phi(s) \leq 0$. To show that $\phi(s) \geq s$, suppose, to obtain a contradiction, that $\phi(s) < s$. Then, starting from the initial state $s_1 = s$, the equilibrium path will involve $s_t < s$ for all $t > 1$. Notice, however, that for the QMV M_s, $u_{M_s}(s) = -\sum_{j \notin H_s} \gamma_j C_j$, and for $x < s$, $u_{M_s}(x) = -(b_{M_x} - b_{M_s}) - \sum_{j \notin H_x} \gamma_j C_j < u_{M_s}(s)$, as H_x is a strict superset of H_s. Again, there is a profitable deviation, which completes the proof.

Part 3. Consider the mapping ϕ such that $\phi(s) = 0$ for all s. Under this mapping, continuation utilities are given by

$$V_i^{\phi}(s) = -(b_{M_s} - b_i)^2 - k \sum_{j \notin H_s} \gamma_j C_j^* - \frac{\beta}{1-\beta}(b_0 - b_i)^2 .$$

Now, the two conditions required to hold for ϕ to be an MVE simplify to:

$$\begin{aligned} \text{for any } s, x &: \quad V_{M_s}^{\phi}(0) \geq V_{M_s}^{\phi}(x); \\ \text{for any } s &: \quad V_{M_s}^{\phi}(0) \geq V_{M_s}^{\phi}(s); \end{aligned}$$

clearly, the second line of inequalities is a subset of the first. This simplifies to

$$\text{for any } s, x\text{: } k \sum_{j \notin H_x} \gamma_j C_j^* \geq (b_{M_s})^2 - (b_{M_x} - b_{M_s})^2 .$$

Clearly, as k increases, the number of equations that are true weakly increases. Furthermore, for k high enough, the left-hand side becomes arbitrarily large for all x except for $x = 0$ where it remains zero, but for $x = 0$, $b_{M_x} = 0$ and thus the right-hand side is zero as well. Finally, if k is small enough, the left-hand side is arbitrarily close to 0 for all s and x, and thus the inequality will be violated, e.g., for $s = x = 1$. This proves that there is a unique positive k^* with the required property. ■

Proof of Proposition 2. **Part 1.** The equilibrium exists and is unique because the required properties hold in each of the environments, and thus Theorems 3 and 4 are applicable.

Let ϕ_{E^f} be the mapping after radicals have left. Since the environment E^f allows for no further stochastic shocks, ϕ_{E^f} coincides with ϕ from Proposition 1 (i.e., if radicals are impossible). Now take any radical environment R^z (so states $x \leq z$ are controlled by radicals). Notice that $\phi_{R^z}(s)$ is the same for all $s \leq z$ (otherwise, setting $\phi_{R^z}(s) = \phi_{R^z}(z)$ for all $s < z$ would yield another MVE, thus violating uniqueness). Consider two situations: $z < 0$ and $z \geq 0$.

Suppose first that $z < 0$. Then $\phi_{R^z}(0) = 0$ (similar to the proof of Part 1 of Proposition 1), and thus by monotonicity $\phi_{R^z}(s) \in [-l-r, 0]$. For any x such that $z < x < 0$, $\phi_{R^z}(x) \geq x$ (again, similar to that proof). Notice that as b_{-l} varies, the mapping $\phi_{R^z}|_{[z+1,l+r]}$ does not change. Indeed, equilibrium paths starting from $x \geq z+1$ remain within that range, and thus continuation utilities of M_x for any $x \geq z+1$ do not depend on b_{-l}; moreover, a deviation from $x \geq z+1$ to some $y \leq z$ cannot be profitable for obvious reasons. The state $\phi_{R^z}(z)$ is such that it maximizes the continuation utility of the radical $-l$ among the following alternaties: moving to some state $y \leq z$, staying there until transition to environment E^f and moving according to ϕ_{E^f}, and moving to some state $y > z$, moving according to ϕ_{R^z} until the transition to E^f and according to ϕ_{E^f} after the transition. Notice that as b_{-l} decreases, the continuation utilities of the radical $-l$ under all these options, except of moving to state $y = -l-r$, strictly decrease, while the payoff of that option remains unchanged (and equal to $-\frac{1}{1-\beta}k\sum_{j>-l}\gamma_j C_j^*$). Hence, a decrease in b_{-l} makes this transition more likely starting from state z, and thus for all $s \leq z$.

Now suppose that $z \geq 0$. Trivially, we must have $\phi_{R^z}(z) \leq 0$. In this case, $\phi_{R^z}|_{[z+1,l+r]}$ may depend on b_{-l}, moving to $y \in [z+1, l+r]$ is suboptimal for the radical anyway. So in this case, the equilibrium $\phi_{R^z}(z)$ maximizes the radical's continuation utility among the options of moving to some $y \leq 0$, staying there until transition to E^f, and then moving according to ϕ_{E^f}. Again, only for $y = -l-r$ the continuation payoff remains unchanged as b_{-l} decreases, and for all other options it decreases. Hence, in this case, too, a lower b_{-l} makes $\phi_{R^z}(z) = -l-r$ more likely. Moreover, since the equilibrium path starting from any $y \leq 0$ will only feature states $s \leq 0$, and for all possible $y \leq 0$, the path for lower y is first-order stochastically dominated by the path for higher y, an increase in k makes $\phi_{R^z}(z) = -l-r$ less likely.

It remains to prove that an increase in z decreases the chance of transition to $-l-r$ for any given $s \leq z$. This is equivalent to saying that a higher z decreases the chance that $\phi_{R^z}(-l-r) = -l-r$. Suppose that z increases by one. If $z \geq 0$ (thus increasing to $z+1 \geq 1$), then $\phi_{R^z}(-l-r)$ does not change as moving to $y \geq 1$ was dominated anyway. If $z < 0$ (thus increasing to $z+1 \leq 0$), then this increase does not change $\phi_{R^z}|_{[z+2,l+r]}$, and thus the only change is the option to stay in $z+1$ as long as the shock leading to E^f does not arrive. This makes staying in $-l-r$ weakly less attractive for the radical, and for some parameter values may make him switch.

Part 2. Suppose, to obtain a contradiction, that for some $s \leq 0$, $\phi_{E^1}(s) < s$. Without loss of generality we may assume that this is the lowest such s, meaning $\phi_{E^1}(s)$ is ϕ_{E^1}-stable. Consider a deviation at s from $\phi_{E^1}(s)$ to s. This deviation has the following effect on continuation utility. First, in the period of deviation, the QMV M_s gets a higher state payoff. Second, the continuation utilities if a transition to R^z for some z takes place immediately after that may differ (if there is no shock, then both paths will converge at $\phi_{E^1}(s)$ thus yielding the same continuation utilities). Now consider two cases: if $z \geq s$, then the radicals are in power in both s and $\phi_{E^1}(s)$. As showed in the proof of Part 1, the radicals will transit to the same state, thus resulting in the same path and continuation utilities. If, however, $z < s$, then the transition in R^z will be chosen by M_s if he stayed in s, hence, this transition will maximize his continuation payoff under R^z, and this need not be true if he moved to $\phi_{E^1}(s)$ (regardless of whether or not radicals rule in this state). In all cases, the continuation utility after the current period is weakly higher if he stayed in s than if he moved to $\phi_{E^1}(s) < s$, and taking into account the first effect, we have a strictly profitable deviation. This contradicts the definition of MVE, which completes the proof. ■

Proof of Proposition 3. Part 1. Suppose, to obtain a contradiction, that $\phi_{E^1}(s) \leq x$ for all $x \geq 0$. By Part 2 of Proposition 2, $\phi_{E^1}(s) \geq s$ for $s \leq 0$, which now implies $\phi_{E^1}(0) = 0$.

Part 2. As in Theorem 3, we may treat the environment E^1 as static, with $W_i(s)$ as quasi-utilities and $\tilde{\beta} = \beta(1-\mu)$ as the discount factor. Assume, to obtain a contradiction, that for all $x \geq 0$, $\phi_{E^1}(s) \leq s$. The payoff from staying in 0 for player $M_0 = 0$ is $V_0(0) = \frac{\tilde{U}_0(0)}{1-\tilde{\beta}}$. By definition of MVE, $V_{M_s}(\phi_{E^1}(s)) \geq V_{M_s}(s)$, and since continuation utilities satisfy increasing differences, $\phi_{E^1}(s) \leq s$, and $M_0 \leq M_s$, it must be that $V_0(\phi_{E^1}(s)) \geq V_0(s)$. Since $V_0(s) = \tilde{U}_0(s) + \tilde{\beta} V_0(\phi_{E^1}(s))$, we have $V_0(\phi_{E^1}(s)) \geq \frac{\tilde{U}_0(s)}{1-\tilde{\beta}}$. Consequently, it must be that $V_0(\phi_{E^1}(s)) > V_0(0)$. This is impossible if $\phi_{E^1}(s) = 0$, and it suggests a profitable deviation at 0 from 0 to s otherwise. This contradiction proves that such x exists.

Part 3. Suppose, to obtain a contradiction, that for some $s > 0$, $\phi_{E^1}(s) > s$. Without loss of generality, assume that $\phi_{E^1}(s)$ is itself ϕ_{E^1}-stable. By definition of MVE, $V_{M_s}(\phi_{E^1}(s)) \geq V_{M_s}(s)$. This is equivalent to $\frac{\tilde{U}_{M_s}(\phi_{E^1}(s))}{1-\tilde{\beta}} \geq \tilde{U}_{M_s}(s) + \frac{\tilde{\beta}\tilde{U}_{M_s}(\phi_{E^1}(s))}{1-\tilde{\beta}}$, thus implying $\tilde{U}_{M_s}(\phi_{E^1}(s)) \geq \tilde{U}_{M_s}(s)$. Setting $y = \phi_{E^1}(s)$ and $x = s$, we have $y > x \geq 0$ and $\tilde{U}_{M_x}(y) \geq \tilde{U}_{M_x}(x)$, a contradiction. This completes the proof. ■

Proof of Proposition 4. This is an immediate corollary of Theorem 6. ■

Proof of Proposition 5. All our baseline assumptions hold for trivial reasons, however, we need to verify that the increasing differences (Assumption 2) hold when one of the agents is group $-l$. Take another group $x > -l$; we have

$$u_s(x) - u_s(-l) = \begin{cases} (b_x - b_{-l})\left(2b_{M_s} - b_{M_x} - b_{M_{-l}}\right) - (1-\rho)\sum_{j \notin H_s} \gamma_j C_j & \text{if } s < 0 \\ (b_x - b_{-l})\left(2b_{M_s} - b_{M_x} - b_{M_{-l}}\right) & \text{if } s \geq 0 \end{cases}.$$

But $b_{M_s}$is increasing in s, and $\sum_{j \notin H_s} \gamma_j C_j$ is decreasing while remaining positive. This implies that $u_s(x) - u_s(-l)$ is increasing in s, so all assumptions hold.

Take some ρ and ρ' such that $\rho > \rho'$. Suppose, to obtain a contradiction, that $\phi_{E^1}(0) > 0$, but $\phi'_{E^1}(0) = 0$. Since radicals cannot come to power at state 1, we must have $\phi_{E^1}(1) \in \{0,1\}$, and $\phi'_{E^1}(1) \in \{0,1\}$. We therefore have $\phi_{E^1}(0) = \phi_{E^1}(1) = 1$.

It is easy to check that for any radical environment R^z and for any $x \leq z$, $\phi'_{R^z}(x) \leq \phi_{R^z}(x) \leq 0$, and therefore, if in period t, the environment is R^z and the state is $s_t = s'_t \leq z$, then for all $\tau \geq t$ and for all realizations of shocks, we have $s'_\tau \leq s_\tau \leq 0$. From this, we have that $V_{R^z,0}(0) = V'_{R^z,0}(0)$ and $V_{R^z,0}(1) = V'_{R^z,0}(1)$ whenever $z < 0$ (indeed, the equilibrium paths in these cases in R^z and E^f are the same and do not involve states $x < 0$).

Notice also that the mapping $\phi_{R^z}|_{[0,r]} = \phi_{E^f}|_{[0,r]}$ for $z < 0$. Denote $\lambda^* = \mu_{-l-r} - \mu_0$, so λ^* is the probability of a shock to a radical environments other than R^0.

Let us prove that $\phi_{E^1}(0) = 1$ implies $\phi_{R^0}(0) = 1$. Indeed, from $\phi_{E^1}(0) = 1$, we have $\tilde{u}_{E^1,0}(1) \geq \tilde{u}_{E^1,0}(0)$. By definition,

$$\begin{aligned} \tilde{u}_{E^1,0}(1) &= u_0(1) + \beta\left(\lambda^* V_{E^f,0}(1) + \lambda_0 V_{R^0,0}(1)\right), \\ \tilde{u}_{E^1,0}(0) &= u_0(0) + \beta\left(\lambda^* V_{E^f,0}(0) + \lambda_0 V_{R^0,0}(0)\right). \end{aligned}$$

But $u_0(1) < u_0(0)$ and, clearly, $V_{E^f,0}(1) < 0 = V_{E^f,0}(0)$. This means $V_{R^0,0}(1) > V_{R^0,0}(0)$, implying that $\phi_{R^0}(0) = 1$ (which in turn implies $\phi_{R^0}(1) = 1$).

Now, notice that we have similar formulas for $\tilde{u}_{E^1,0}(1)$ and $\tilde{u}_{E^1,0}(0)$, and moreover, $V_{E^f,0}(1) = V'_{E^f,0}(1)$ and $V_{E^f,0}(0) = V'_{E^f,0}(0)$. Therefore,

$$\begin{aligned} \tilde{u}_{E^1,0}(1) - \tilde{u}'_{E^1,0}(1) &= \beta\lambda_0\left(V_{R^0,0}(1) - V'_{R^0,0}(1)\right), \\ \tilde{u}_{E^1,0}(0) - \tilde{u}'_{E^1,0}(0) &= \beta\lambda_0\left(V_{R^0,0}(0) - V'_{R^0,0}(0)\right). \end{aligned}$$

But $\phi_{R^0}(0) = \phi_{R^0}(1) = 1$ implies $V_{R^0,0}(1) = V'_{R^0,0}(1)$. On the other hand, $V_{R^0,0}(0) \geq V'_{R^0,0}(0)$. Together, this all implies that

$$\left(\tilde{u}_{E^1,0}(1) - \tilde{u}'_{E^1,0}(1)\right) - \left(\tilde{u}_{E^1,0}(0) - \tilde{u}'_{E^1,0}(0)\right) \leq 0.$$

Since $\tilde{u}_{E^1,0}(1) \geq \tilde{u}_{E^1,0}(0)$, it must be that $\tilde{u}'_{E^1,0}(1) \geq \tilde{u}'_{E^1,0}(0)$. This means $\tilde{u}'_{E^1,M_1}(1) \geq \tilde{u}'_{E^1,M_1}(0)$, implying $\phi'_{E^1}(1) = 1$. But then $\tilde{u}'_{E^1,0}(1) \geq \tilde{u}'_{E^1,0}(0)$ is incompatible with $\phi'_{E^1}(0) = 0$. This contradicts our initial assertion, which completes the proof. ■

Proof of Proposition 6. First note that the notion of MVE in this environment needs to be refined slightly since society cannot start a period in state n. Let us first define payoffs. For $E \in \{E^{h,n-1}, E^{l,n-1}, E^{h,n}, E^{l,n}\}$, $i \in N$ and $s \in S$, $u_{E,i}(s) = B_s - (b_i - b_s)^2$ (with $B_n = B^h$ for $E^{h,n-1}$ and $E^{h,n}$ and $B_n = B^l$ for $E^{l,n-1}$ and $E^{l,n}$). For the initial environment E^1, $i \in N$ and $s \in S \setminus \{n\}$, $u_{E^1,i}(s) = u_i(s) = B_s - (b_i - b_s)^2$. We do not define $u_{E^1,i}(n)$. Given these definitions, an MVE is again a collection of mappings $\phi = (\phi_{E^1}, \phi_{h,n-1}, \phi_{l,n-1}, \phi_{h,n}, \phi_{l,n})$, where ϕ_{E^1} is mapping $S \setminus \{n\} \to S$ and $\phi_{\cdot,\cdot}$ are mappings $S \to S$ such that Definition 3 is satisfied for continuation utilities found from (2), with the only caveat that in environment E^1, if $\phi_{E^1}(s) = n$, then a shock (or, more precisely, one of four possible shocks) happens with probability 1 rather than λ. Notice that the right-hand sides in (2) are well-defined even in E^1 and s such that $\phi_{E^1}(s) = n$, precisely because $V_{E^1}(n)$ would enter with coefficient 0. In what follows, we define environment $\tilde{E}^1$ that satisfies all the assumptions and show that its MVE corresponds to MVE in E^1; this will ensure existence and (generic) uniqueness. The environments $E^{h,n-1}$, $E^{l,n-1}$, $E^{h,n}$, $E^{l,n}$ are stationary, and Theorems 1 and 2 apply (it is trivial to verify that Assumptions 2–5 are satisfied). Therefore, MVE in these environments exist and are (generically) unique. Denote these MVE by $\phi_{h,n-1}$, $\phi_{l,n-1}$, $\phi_{h,n}$, $\phi_{l,n}$, respectively; let the continuation values of player i in state s in these environments under these MVE be $V_i^{h,n-1}(s)$, $V_i^{l,n-1}(s)$, $V_i^{h,n}(s)$, $V_i^{l,n}(s)$, respectively. Let

$$\bar{V}_i(s) = \gamma\mu V_i^{h,n}(s) + \gamma(1-\mu) V_i^{h,n-1}(s) + (1-\gamma)\mu V_i^{h,n}(s) + (1-\gamma)(1-\mu) V_i^{h,n-1}(s);$$

in other words, $\bar{V}_i(s)$ is the expected continuation utility if it is known that the shock (learning) will happen in the current period, but the exact realizations of B_n and M_n are unknown yet.

Consider an alternative environment $\tilde{E}^1$, which is obtained from E^1 (which means, in particular, that transition probabilities to environments $E^{h,n-1}$, $E^{l,n-1}$, $E^{h,n}$, $E^{l,n}$ are preserved) by making the following definitions for state n. Assume $F_{\tilde{E}^1}(n) = \{n\}$ (i.e., no transitions are feasible), $M_{\tilde{E}^1,n} = \{n\}$ (we assume group n to make decisions on transitions, although this is inconsequential since transitions are ruled out), and, most importantly, $u_{\tilde{E}^1,i}(n) = (1-\beta)\bar{V}_i(n)$ (so, each agent is assumed to get a per-period share of his expected payoff from moving to state n and learning its true payoffs). The environment $\tilde{E}^1$ constructed in this way satisfies all assumptions of Theorems 3 and 4, and therefore it has a (generically) unique MVE, which we denote by ϕ_0. (The only part that needs to be verified is Assumption 2 (increasing differences) at the pair of states $n-1, n$, which holds because for each i, $V_i^{h,n-1}(n)$, $V_i^{l,n-1}(n)$, $V_i^{h,n}(n)$, $V_i^{l,n}(n)$ are weighted averages of $u_i(n-1)$ and $u_i(n)$, and thus so is $\bar{V}_i(n)$.) We argue that ϕ_0 is a MVE in the original environment E^1 (after dropping $\phi_0(n)$, which is irrelevant because of the automatic learning).

Consider the continuation utilities under mapping ϕ_0 in E^1 and $\tilde{E}^1$; denote them $V_i(s)$ and $\tilde{V}_i(s)$, respectively. For $s = n$, we have

$$\tilde{V}_i(n) = \frac{u_{\tilde{E}^1,i}(n)}{1-\beta} = \bar{V}_i(n);$$

indeed, even though agent i gets per-period utility equal to $u_{\tilde{E}^1,i}(n)$ only until a shock takes place, the average continuation utility after the shock is $\bar{V}_i(n)$ by definition of $\bar{V}_i(n)$, and thus this equality holds. The continuation utility $V_i(n)$ is not well-defined, because the society cannot start a period in state n in environment E^1. Now, supoose $s < n$ and consider two cases. If $\phi_0(s) \neq n$, then

$$\begin{aligned}
\tilde{V}_i(s) &= u_i(s) + \beta\gamma\bar{V}_i(s) + \beta(1-\gamma)\tilde{V}_i(\phi_0(s)); \\
V_i(s) &= u_i(s) + \beta\gamma\bar{V}_i(s) + \beta(1-\gamma)V_i(\phi_0(s)).
\end{aligned}$$

In the case $\phi_0(s) = n$, we have

$$\begin{aligned}
\tilde{V}_i(s) &= u_i(s) + \beta\bar{V}_i(n); \\
V_i(s) &= u_i(s) + \beta\bar{V}_i(n).
\end{aligned}$$

Consequently, the vector of continuation utilities $\{V_i(s)\}_{i\in N}^{s\in S\setminus\{n\}}$ is equal to the corresponding vector in $\tilde{E}^1$, $\{V_i(s)\}_{i\in N}^{s\in S\setminus\{n\}}$. This implies that mapping ϕ_0 satisfies all the parts of Definition 3 in E^1(as adapted above) and is therefore a MVE.

A similar argument would prove that any MVE ϕ' in E^1 would correspond to a MVE in $\tilde{E}^1$ if we defined $\phi'(n) = n$. Now, uniqueness of MVE in $\tilde{E}^1$ proves uniqueness of MVE in E^1, which completes the proof. ■

Proof of Proposition 7. Part 1. In this case, $\phi_{h,\cdot}(n) = n$ and $\phi_{l,\cdot}(n) = n-1$ regardless of whether group $n-1$ or n is in power in state n. In E^0 (and thus $\tilde{E}^0$) defined in the proof of Proposition 6), the only two possibilities are $\phi_0(n-1) = n$ and $\phi_0(n-1) = n$; at the same time, $\phi_0(n) = n$. Let us suppose that $\phi_0(n-1) = n$ and compute the continuation utilities of group $n-1$ under these mappings. We have:

$$\begin{aligned}
\tilde{V}_{n-1}(n) &= \mathbb{E}B_n - (b_n - b_{n-1})^2 + \beta\left(\gamma\frac{1}{1-\beta}\left(B^h - (b_n - b_{n-1})^2\right) + (1-\gamma)\frac{1}{1-\beta}B_{n-1}\right), \\
\tilde{V}_{n-1}(n-1) &= B_{n-1} + \beta\lambda\left(\gamma\frac{1}{1-\beta}\left(B^h - (b_n - b_{n-1})^2\right) + (1-\gamma)\frac{1}{1-\beta}B_{n-1}\right) + \beta(1-\lambda)V_{n-1}(n).
\end{aligned}$$

Mapping ϕ_0 is an equilibrium if and only if $\tilde{V}_{n-1}(n) \geq \tilde{V}_{n-1}(n-1)$, which is equivalent to

$$\begin{aligned}
&(1-\beta(1-\lambda))\left(\mathbb{E}B_n - (b_n - b_{n-1})^2\right) \\
&\quad + (1-\beta(1-\lambda)-\lambda)\beta\left(\gamma\frac{1}{1-\beta}\left(B^h - (b_n - b_{n-1})^2\right) + (1-\gamma)\frac{1}{1-\beta}B_{n-1}\right) \geq B_{n-1}.
\end{aligned}$$

Simplifying and substituting $\mathbb{E}B_n = \gamma B^h + (1-\gamma)B^l$, we get

$$\gamma\left(B^h - (b_n - b_{n-1})^2 - B_{n-1}\right) \geq (1-\beta(1-\lambda))(1-\gamma)\left(B_{n-1} + (b_n - b_{n-1})^2 - B^l\right);$$

under our assumptions, both sides are positive, and thus this is equivalent to $Y \geq 1-\beta(1-\lambda)$. This means that if this inequality is satisfied, then $\phi_0(n-1) = n$ in the equilibrium; otherwise, we get a contradiction, in which case it must be that $\phi_0(n-1) = n-1$. Since $1-\beta(1-\lambda)$ does not depend on μ, decreases in β and increases in λ, the result follows.

Part 2. In this case, we have $\phi_{l,n-1}(n) = n-1$, whereas $\phi_{h,\cdot}(n) = n$ and $\phi_{l,n}(n) = n$. Again, suppose that $\phi_0(n-1) = n$; then we have

$$\begin{aligned}
\tilde{V}_{n-1}(n) &= \mathbb{E}B_n - (b_n - b_{n-1})^2 \\
&\quad + \beta\left(\gamma\frac{1}{1-\beta}\left(B^h - (b_n - b_{n-1})^2\right) + (1-\gamma)\mu\frac{1}{1-\beta}\left(B^l - (b_n - b_{n-1})^2\right) + (1-\gamma)(1-\mu)\frac{1}{1-\beta}\right. \\
\tilde{V}_{n-1}(n-1) &= B_{n-1} \\
&\quad + \beta\lambda\left(\gamma\frac{1}{1-\beta}\left(B^h - (b_n - b_{n-1})^2\right) + (1-\gamma)\frac{1}{1-\beta}B_{n-1}\right) + \beta(1-\lambda)V_{n-1}(n).
\end{aligned}$$

The equilibrium condition $\tilde{V}_{n-1}(n) \geq \tilde{V}_{n-1}(n-1)$ is equivalent to

$$\begin{aligned}
&(1-\beta(1-\lambda))\left(\mathbb{E}B_n - (b_n - b_{n-1})^2 + \beta(1-\gamma)\mu\frac{1}{1-\beta}\left(B^l - (b_n - b_{n-1})^2 - B_{n-1}\right)\right) \\
&\quad + (1-\beta(1-\lambda)-\lambda)\beta\left(\gamma\frac{1}{1-\beta}\left(B^h - (b_n - b_{n-1})^2\right) + (1-\gamma)\frac{1}{1-\beta}B_{n-1}\right) \geq B_{n-1}.
\end{aligned}$$

Simplifying as in the proof of Part 1, we obtain that this is equivalent to

$$\gamma\left(B^h - (b_n - b_{n-1})^2 - B_{n-1}\right) \geq \frac{(1-\beta+\beta\lambda)(1-\beta+\beta\mu)}{1-\beta}(1-\gamma)\left(B_{n-1} + (b_n - b_{n-1})^2 - B^l\right);$$

this is, in its turn, equivalent to $Y \geq \frac{(1-\beta+\beta\lambda)(1-\beta+\beta\mu)}{1-\beta}$. Thus, this is precisely the condition under which the society experiments. The right-hand side is increasing in λ and μ. Differentiating it with respect to β and simplifying yields

$$\frac{d}{d\beta}\frac{(1-\beta+\beta\lambda)(1-\beta+\beta\mu)}{1-\beta} = \lambda\mu - (1-\lambda)(1-\mu)(1-\beta)^2 .$$

Now, if $\lambda+\mu > 1$, then $\frac{\lambda\mu}{(1-\lambda)(1-\mu)} > 1$, and the derivative is positive for all β. If $\lambda+\mu < 1$, then $\frac{\lambda\mu}{(1-\lambda)(1-\mu)} < 1$, and the derivative is decreasing in β, changing its sign exactly once at $\beta = 1-\sqrt{\frac{\lambda\mu}{(1-\lambda)(1-\mu)}}$. This completes the proof. ■

Proof of Proposition 8. If $\lambda > 0$ and $\mu > 0$, then $\frac{(1-\beta+\beta\lambda)(1-\beta+\beta\mu)}{1-\beta} \to \infty$ as $\beta \to 1$; consider β to be large enough so that if $B^l = B_{n-1} - (b_n - b_{n-1})^2$, the condition $Y \geq \frac{(1-\beta+\beta\lambda)(1-\beta+\beta\mu)}{1-\beta}$ is violated. The condition $\mathbb{E}B_n - (b_n - b_{n-1})^2 > B_{n-1}$ implies $Y > 1$ and, moreover, that Y is monotonically decreasing in Δ and that for B^l less than but close to $B_{n-1} + (b_n - b_{n-1})^2$, Y becomes arbitrarily large. Thus, if we denote the value of Δ under which $Y = \frac{(1-\beta+\beta\lambda)(1-\beta+\beta\mu)}{1-\beta}$ by Δ_1, we will have that for $\Delta < \Delta_1$, the society experiments.

Now, let Δ_2 be the value of Δ that solves $B^l = B_{n-1} - (b_n - b_{n-1})^2$; out choice of β ensures that $\Delta_2 > \Delta_1$. For $\Delta \in (\Delta_1, \Delta_2)$, $Y < \frac{(1-\beta+\beta\lambda)(1-\beta+\beta\mu)}{1-\beta}$, and the society does not experiment. However, for $\Delta > \Delta_2$, $B^l < B_{n-1} - (b_n - b_{n-1})^2$, and group n will move to state $n-1$ if $B_n = B^l$. In this case, the society experiments whenever $Y > 1-\beta(1-\lambda)$; since $Y > 1$, this condition holds. Therefore for $\Delta > \Delta_2$, the society experiments. This completes the proof. ■

B2 Examples

Example B1 ***(Example with single-peaked preferences and two MVE)*** There are three states A, B, C, and two players 1 and 2. The decision-making rule is unanimity in state A and dictatorship of player 2 in states B and C. Payoffs are given by

	A	B	C
1	2	25	20
2	1	20	25

Then ϕ_1 given by $\phi_1(A,B,C) = (B,C,C)$ and ϕ_2 given by $\phi_2(A,B,C) = (C,C,C)$ are both MVE when the discount factor is any $\beta \in [0,1)$.

Example B2 ***(Example where the limit state depends on the timing of shocks)*** There are two environments, E^1 and E^2, with the probability of transition $\pi(E^1, E^2) = 0.1$. There are two states A, B, and two players 1 and 2. In both environments, the decision-making rule is dictatorship of player 1 in state A and dictatorship of player 2 in state B. All transitions are feasible, and the discound factor is $\beta = 0.9$. Payoffs are given by

$$\begin{array}{ccc} E^1 & A & B \\ 1 & 5 & 20 \\ 2 & 20 & 30 \end{array}, \quad \begin{array}{ccc} E^2 & A & B \\ 1 & 30 & 20 \\ 2 & 20 & 30 \end{array}.$$

Then the mapping ϕ is given by $\phi_{E^1}(A, B) = (B, B)$; $\phi_{E^2}(A, B) = (A, B)$. Suppose that $s_0 = 1$. Then, if the shock arrives in period $t = 1$, the limit state is A, and if the shock arrives later, the limit state is B.

Example B3 ***(Continuation utilities need not satisfy single-peakedness***) There are four states and three players, player 1 is the dictator in state A, player 2 is the dictator in state B, and player 3 is the dictator in states C and D. The payoffs are given by the following matrix:

$$\begin{array}{ccccc} & A & B & C & D \\ 1 & 20 & 30 & 90 & 30 \\ 2 & 5 & 20 & 85 & 90 \\ 3 & 5 & 25 & 92 & 99 \end{array}.$$

All payoffs are single-peaked. Suppose $\beta = 0.5$; then the unique equilibrium has $\phi(A) = C$, $\phi(B) = \phi(C) = \phi(D) = D$. Let us compute the continuation payoffs of player 1. We have: $V_1(A) = 40$, $V_1(B) = 30$, $V_1(C) = 50$, $V_1(D) = 30$; the continuation utility of player 1 is thus not single-peaked.

Example B4 ***(Importance of E and $\tilde{E}$ coinciding on some space for comparative statics)*** Suppose that in environment E^1, there are two players and three states, all transitions are feasible, and $\beta = 0.99$. Player 1 is the dictator in states A and B, and player 2 is the dictator in state C. The payoffs are given by the matrix:

$$\begin{array}{cccc} E & A & B & C \\ 1 & 80 & 75 & 10 \\ 2 & 20 & 60 & 15 \end{array}.$$

The environment $\tilde{E}$ has the same feasible transitions, winning coalitions and the discount factor, but the payoffs are

$$\begin{array}{cccc} \tilde{E} & A & B & C \\ 1 & 80 & 85 & 30 \\ 2 & 20 & 70 & 35 \end{array}.$$

Let ϕ_E and $\phi_{\tilde{E}}$ be the MVE in these environments.

In environment $\tilde{E}$, both players have stronger preferences for higher than lower states, as compared to environment E (the differences in utilities between B and A, C and B are increased by 10). However, it is not true that $\phi_{\tilde{E}}(x) \geq \phi_E(x)$ for any x (and even for x satisfying $\phi_E(x) = x$, as in Theorem 6). Indeed, $\phi_E(A, B, C) = (A, A, C)$, and $\phi_{\tilde{E}}(A, B, C) = (B, B, B)$. Thus, $\phi_E(C) = C > B = \phi_{\tilde{E}}(C)$.

However, the following result is (generically) true: if (1) for any states $x < y$ and any player i, $u_{\tilde{E},i}(y) - u_{\tilde{E},i}(x) \geq u_{E,i}(y) - u_{E,i}(x)$; (2) for any state x, $M_{\tilde{E}}(x) \geq M_E(x)$ (in the sense $\min M_{\tilde{E}}(x) \geq \min M_E(x)$ and $\max M_{\tilde{E}}(x) \geq \max M_E(x)$); and (3) for any states $x < y$, $y \in F_E(x)$ implies $y \in F_{\tilde{E}}(x)$ and $x \in F_{\tilde{E}}(y)$ implies $x \in F_E(y)$, then there exists $\beta_0 > 0$ such that for $\beta < \beta_0$, $\phi_E(x) = x$ implies $\phi_{\tilde{E}}(x) \geq x$.

Example B5 ***(Example of non-monotone MVE)*** There are three states A, B, C, and two players 1 and 2. The decision-making rule is unanimity in all states, and all transitions are possible. Payoffs are given by

id	*A*	*B*	*C*
1	30	50	40
2	10	40	50

Suppose β is relatively close to 1, e.g., $\beta = 0.9$. This case does not satisfy either set of conditions of Theorem 7. It is straightforward to verify that there is a non-monotone MVE $\phi(A) = \phi(C) = C$, $\phi(B) = B$. (There is also a monotone equilibrium with $\phi(A) = \phi(B) = B$, $\phi(C) = C$.)

Example B6 ***(No MVE with infinite number of shocks)*** Below is an example with finite number of states and players and finite number of environments such that all assumptions, except for the assumption that the number of shocks is finite, are satisfied, but there is no Markov Voting Equilibrium in pure strategies.

There are three environments E^1, E^2, E^3, three states $A = 1, B = 2, C = 3$, and three players $1, 2, 3$. The history of environments follows a simple Markov chain; in fact, in each period the environment is drawn separately. More precisely,

$$\begin{aligned}
\pi\left(E^1\right) \quad &: \quad = \pi\left(E^1, E^1\right) = \pi\left(E^2, E^1\right) = \pi\left(E^3, E^1\right) = \frac{1}{2}; \\
\pi\left(E^2\right) \quad &: \quad = \pi\left(E^1, E^2\right) = \pi\left(E^2, E^2\right) = \pi\left(E^3, E^2\right) = \frac{2}{5}; \\
\pi\left(E^3\right) \quad &: \quad = \pi\left(E^1, E^2\right) = \pi\left(E^2, E^3\right) = \pi\left(E^3, E^3\right) = \frac{1}{10}.
\end{aligned}$$

The discount factor is $\frac{1}{2}$.

The following matrices describe stage payoffs, winning coalitions, and feasible transitions.

Environment E^1	*State* A	*State* B	*State* C
Winning coalition	*Dictatorship of Player* 1		
Feasible transitions	*to* A, B	*to* B	*to* C
Player 1	60	150	-800
Player 2	30	130	60
Player 3	-100	60	50

Environment E^2	*State* A	*State* B	*State* C
Winning coalition	*Dictatorship of Player* 2		
Feasible transitions	*to* A	*to* A, B	*to* C
Player 1	100	80	-800
Player 2	80	70	60
Player 3	-100	60	50

Environment E^3	*State* A	*State* B	*State* C
Winning coalition	*Dictatorship of Player* 3		
Feasible transitions	*to* A	*to* B, C	*to* C
Player 1	100	80	-800
Player 2	80	70	60
Player 3	-100	60	50

It is straightforward to see that Sincreasing differences holds; moreover, payoffs are single-peaked, and in each environment and each state, the set of QMVs is a singleton.

The intuition behind the example is the following. The payoff matrices in environment E^2 and E^3 coincide, so "essentially", there are two equally likely environments E^1 and "$E^2 \cup E^3$". Both player 1 and 2 prefer state B when the environment is E^2 and state A when the environment is E^1; given the payoff matrix and the discount factor, player 1 would prefer to move from A to B when in E^1, and knowing this, player 2 would be willing to move to A when in E^2. However, there is a chance that the environment becomes E^3 rather than E^2, in which case a "maniac" player 3 will become able to move from state B (but not from A!) to state C; the reason for him to do so is that although he likes state B (in all environments), he strongly dislikes A, and thus if players 1 and 2 are expected to move between these states, player 3 would rather lock the society in state C, which is only slighly worse for him than B.

State C, however, is really hated by player 1, who would not risk the slightest chance of getting there. So, if player 3 is expeced to move to C when given such chance, player 1 would not move from A to B when the environment is E^1, because player 3 is only able to move to C from B. Now player 2, anticipating that if he decides to move from B to A when the environment is E^2, the society will end up in state A forever; this is something player 2 would like to avoid, because state A is very bad for him when the environment is E^1. In short, if player 3 is expected to move to C when given this chance, then the logic of the previous paragraph breaks down, and neither player 1 nor player 2 will be willing to move when they are in power. But in this case, player 3 is better off staying in state B even when given a chance to move to C, as he trades off staying in B forever versus staying in C forever. These considerations should prove that there is no MVE.

More formally, note that there are only eight candidate mappings to consider (some transitions are made infeasible precisely to simplify the argument; alternatively, we could allow any transitions and make player 1 the dictator in state A when the environment is E^3). We consider these eight mappings separately, and point out the deviation. Obviously, the only values of the transition mappings to be specified are $\phi_{E^1}(A)$, $\phi_{E^2}(B)$, and $\phi_{E^3}(B)$.

1. $\phi_{E^1}(A) = A$, $\phi_{E^2}(B) = A$, $\phi_{E^3}(B) = B$. Then $\phi'_{E^3}(B) = C$ is a profitable deviation.
2. $\phi_{E^1}(A) = B$, $\phi_{E^2}(B) = A$, $\phi_{E^3}(B) = B$. Then $\phi'_{E^3}(B) = C$ is a profitable deviation.
3. $\phi_{E^1}(A) = A$, $\phi_{E^2}(B) = B$, $\phi_{E^3}(B) = B$. Then $\phi'_{E^1}(A) = B$ is a profitable deviation.
4. $\phi_{E^1}(A) = B$, $\phi_{E^2}(B) = B$, $\phi_{E^3}(B) = B$. Then $\phi'_{E^2}(B) = A$ is a profitable deviation.
5. $\phi_{E^1}(A) = A$, $\phi_{E^2}(B) = A$, $\phi_{E^3}(B) = C$. Then $\phi'_{E^2}(B) = B$ is a profitable deviation.
6. $\phi_{E^1}(A) = B$, $\phi_{E^2}(B) = A$, $\phi_{E^3}(B) = C$. Then $\phi'_{E^1}(A) = A$ is a profitable deviation.
7. $\phi_{E^1}(A) = A$, $\phi_{E^2}(B) = B$, $\phi_{E^3}(B) = C$. Then $\phi'_{E^3}(B) = B$ is a profitable deviation.
8. $\phi_{E^1}(A) = B$, $\phi_{E^2}(B) = B$, $\phi_{E^3}(B) = C$. Then $\phi'_{E^3}(B) = B$ is a profitable deviation.

This proves that there is no MVE in pure strategies (i.e., in the sense of Definition 3).