Not-For-Publication Appendix for "A Political Theory of Populism": Proofs

The following Lemma clarifies the role and consequences of Assumption 1.

Lemma 1 *Suppose that Assumption 1 holds. Then:*

1. $f(x) < \frac{2}{b\left(\frac{W}{\alpha b^2}+1\right)} < \frac{2}{b}$ *and* $f(x) < \frac{1}{\sqrt{2\pi}b} < \frac{2}{5b}$.
2. $|f'(x)| < \frac{5/4}{(W/\alpha)+b^2}$

Proof of Lemma 1. Part 1. We have

$$\frac{\sigma}{b} = \frac{4}{5}\frac{\sigma}{b} + \frac{1}{5}\frac{\sigma}{b} > \frac{4}{5}\frac{W}{4\alpha b^2} + \frac{1}{5} = \frac{1}{5}\left(\frac{W}{\alpha b^2}+1\right).$$

Hence,

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}} < \frac{1/b}{\sqrt{2\pi}\frac{\sigma}{b}} < \frac{1}{b}\frac{5}{\sqrt{2\pi}\left(\frac{W}{\alpha b^2}+1\right)} < \frac{2}{b\left(\frac{W}{\alpha b^2}+1\right)}.$$

Also, we have

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}} < \frac{1}{\sqrt{2\pi}b}.$$

Part 2. For a normal distribution, $\max|f'(x)|$ is obtained at $x = \pm\sigma$ and equals $\frac{1}{\sqrt{2\pi e}\sigma^2}$. Assumption 1 implies

$$\sigma^2 = \frac{4}{5}\frac{\sigma}{b}\frac{\sigma}{b}b^2 + \frac{1}{5}b^2 > \frac{4}{5}\frac{W}{4\alpha b^2}\times b^2 + \frac{1}{5}b^2 = \frac{1}{5}\left(\frac{W}{\alpha}+b^2\right).$$

Consequently,

$$\frac{1}{\sqrt{2\pi e}\sigma^2} < \frac{5}{\sqrt{2\pi e}}\frac{1}{(W/\alpha)+b^2} < \frac{5}{4}\frac{1}{(W/\alpha)+b^2}.$$

■

Proof of Proposition 1. Proved in the text. ■

Proof of Proposition 2. Part 1. We first prove existence. Consider equation (17). For $p = 0$, the left-hand side is negative, and for p high it is positive, hence there is a positive solution $p > 0$. By taking $h = -p$ and then c from (A2), we find that there is a solution (h, c) to the system (15)–(16). Then h and c constitute best responses in problems (11) and (13) as the maximands are concave. Indeed, differentiating (12) and (14) yields, respectively (using Lemma 1):

$$\begin{aligned}
& -2\alpha+\left(W+(1-\mu)\,\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2\right)f'\left(\frac{h+c}{2}-x\right) \\
< \; & -2\alpha+\alpha\left(\frac{W}{\alpha}+b^2\right)\frac{5/4}{(W/\alpha)+b^2}<-\frac{3}{4}\alpha<0; \text{ and} \\
& -2\,(\alpha+\beta)+\left(W+(1-\mu)\,\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2+(\chi+\mu-\mu\chi)\left(\frac{\beta^2b^2}{\alpha+\beta}-K\right)\right)f'\left(\frac{h+c}{2}-x\right) \\
< \; & -2\,(\alpha+\beta)+\alpha\left(\frac{W}{\alpha}+b^2\right)\frac{5/4}{(W/\alpha)+b^2}+\frac{\beta^2b^2}{\alpha+\beta}\frac{5/4}{(W/\alpha)+b^2} \\
< \; & -2\,(\alpha+\beta)+\frac{5}{4}\alpha+\frac{5}{4}\beta<0.
\end{aligned}$$

It remains to prove that corrupt politicians and the lobby are better off with bribing, i.e., the gain from bribing is high enough. To do this, take the h and c that solve (15)–(16) and let $\tilde{c}$ be the policy that the corrupt type would choose on his own in period 1 (yet anticipating a bribe in period 2). This $\tilde{c}$ would solve the problem, taking $\pi(x)$ as given by (10):

$$\max_{x\in\mathbb{R}}\left\{-\alpha x^2+\left(W+\chi\left(\frac{\beta^2b^2}{\alpha+\beta}-K\right)\right)\pi(x)-(1-\mu)\left(\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2\right)(1-\pi(x))\right\}, \quad \text{(A1)}$$

and since there is an extra term in this expression relative to (11) (coming from the corrupt politician's surplus from a bribe), he would choose $\tilde{c}<h$. Denote the joint expected utility of the lobby and the corrupt politician if policy x is chosen by $W(x)$; we need to prove that $W(c)-W(\tilde{c})>K$.

Let $n\equiv\tilde{c}+\frac{\beta b}{\alpha+\beta}$. Then $W(c)\geq W(n)$ as c maximizes $W(x)$. Now, using the fact that $\pi(\tilde{c})-\pi(n)\leq(n-\tilde{c})\sup_{x\in[\tilde{c},n]}f(x)\leq\frac{\beta b}{\alpha+\beta}f(0)$, we get

$$\begin{aligned}
W(n)-W(\tilde{c})-K &= -\alpha n^2-\beta(n-b)^2+\alpha\tilde{c}^2+\beta(\tilde{c}-b)^2-(H+R)(\pi(\tilde{c})-\pi(n))-K \\
&= \frac{\beta b}{\alpha+\beta}\left(2\beta b-(\alpha+\beta)\left(2\tilde{c}+\frac{\beta b}{\alpha+\beta}\right)\right)-(H+R)(\pi(\tilde{c})-\pi(n))-K \\
&= -2\beta b\tilde{c}+\left(\frac{\beta^2b^2}{\alpha+\beta}-K\right)(1-(\chi+\mu-\mu\chi)(\pi(\tilde{c})-\pi(n)))-H(\pi(\tilde{c})-\pi(n)) \\
&> -2\beta bh-H\frac{\beta b}{\alpha+\beta}f(0) \\
&= \frac{\beta b}{\alpha}Hf\left(\frac{c-h}{2}\right)-H\frac{\beta b}{\alpha+\beta}f(0).
\end{aligned}$$

Denote $\Delta \equiv c - h$. It suffices to prove that $\frac{f(\Delta/2)}{f(0)} \geq \frac{\alpha}{\alpha+\beta}$, which is equivalent to $\frac{\Delta^2}{\sigma^2} \leq 8\ln\left(1+\frac{\beta}{\alpha}\right)$. Notice that Δ satisfies

$$\Delta \equiv c - h = \frac{\beta b}{(\alpha+\beta)} - \frac{h}{(\alpha+\beta)}\left(\frac{\beta H - \alpha R}{H}\right). \tag{A2}$$

Using (A2),

$$\Delta \equiv c - h = \frac{\beta b}{(\alpha+\beta)} - \frac{h}{(\alpha+\beta)}\left(\frac{\beta H - \alpha R}{H}\right) < \frac{\beta}{\alpha+\beta}(b-h) < \frac{2\beta b}{\alpha+\beta},$$

where we used that $p = -h < b$. Then, from (12),

$$-h = \frac{1}{2\alpha} H \times f\left(\frac{\Delta}{2}\right) < \frac{1}{2\alpha}\left(W + \alpha b^2\right)\frac{2}{b\left(\left(W/\alpha b^2\right)+1\right)} < b.$$

Now, using $\sigma > b$ we get $\frac{\Delta}{\sigma} < 2\frac{\beta}{\alpha+\beta}$, so $\frac{\Delta^2}{\sigma^2} < 4\frac{(\beta/\alpha)^2}{1+(\beta/\alpha)^2}$. The result follows as $4\frac{x^2}{1+x^2} < 8\ln(1+x)$ for all positive x, which completes the existence part.

We next prove uniqueness. Doing so within the class of equilibria with bribing in the first period where $h < c$ is trivial: this follows from that in (12), c is increasing in h, and in (14), c is decreasing in h. Therefore, we need to rule out other possibilities.

If there is bribing and $h = c$, then citizens are indifferent between the incumbent and the challenger. In this case, by assumption, they vote for the incumbent, the incumbent is sure to win, and therefore can choose any policy, but this would imply that honest ones choose 0 and corrupt ones choose $\frac{\beta b}{\alpha+\beta}$, which contradicts $h = c$. Finally, consider the case $h > c$. Then citizens reelect the incumbent if and only if $s > \frac{h+c}{2}$, and the probability of the reelection is

$$\pi(x) = \Pr\left(x + z \geq \frac{h+c}{2}\right) = 1 - F\left(\frac{h+c}{2} - x\right) = F\left(x - \frac{h+c}{2}\right).$$

Honest politicians would solve (11) and dishonest politicians (with the lobby) would solve (13); since honest ones choose h over c and corruptible do the opposite, we have

$$\begin{aligned} -\alpha h^2 + HF\left(\frac{h-c}{2}\right) &\geq -\alpha r^2 + H\left(1 - F\left(\frac{h-c}{2}\right)\right); \\ -\alpha r^2 - \beta(c-b)^2 + (H+R)\left(1 - F\left(\frac{h-c}{2}\right)\right) &\geq -\alpha h^2 - \beta(h-b)^2 + (H+R)F\left(\frac{h-c}{2}\right). \end{aligned}$$

Adding these inequalities and simplifying, we obtain

$$(h-b)^2-(c-b)^2 \geq R\left(2F\left(\frac{h-c}{2}\right)-1\right).$$

The right-hand side is positive due to $h>c$, which, together with $h>c$, implies $h>b$. The policy choice h must satisfy the following first-order condition:

$$-2\alpha h+Hf\left(\frac{h-c}{2}\right)=0.$$

If $h>b$, then

$$f\left(\frac{h-c}{2}\right)>\frac{2\alpha b}{H}>\frac{2b}{(W/\alpha)+b^2}=\frac{2}{b\left((W/\alpha b^2)+1\right)},$$

but this contradicts Lemma 1. Therefore, there is no other equilibria with bribing.

To rule out equilibria without bribing in the first period, consider the following cases. Suppose $h<c$, then again the incumbent is reelected iff $s\leq\frac{h+c}{2}$. But we argued above that without bribing in the first period, the corrupt politician wants to be reelected more (he expects to be bribed in the second), and thus he must choose $c<h$, which contradicts $h<c$. A similar contradiction follows when we start with the hypothesis that $c<h$. Finally, if $h=c$, then the incumbent is reelected anyway, in which case it is optimal for the corrupt politician and the lobby to engage in corruption and choose $\frac{\beta b}{\alpha+\beta}>0$ which an honest politician would choose, again a contradiction. This completes the proof of uniqueness of an equilibrium if (5) holds.

If (5) does not hold, then the citizens are indifferent between all politicians, and thus reelect the incumbent for sure. Hence, the first-period problems of both politicians's types and SIG are identical to their second-period problems, and therefore there is no corruption either.

Part 2. This follows immediately from (15).

Part 3. This is established as part of the proof of Part 1. ■

Proof of Proposition 3. We first need to prove that for small W, $q<\frac{\beta}{\alpha+\beta}b$. The bias q is given by (18). If W is small, then

$$\begin{aligned}
H+R &= W+(1-\mu)\,\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2+(\chi+\mu-\mu\chi)\left(\frac{\beta^2b^2}{\alpha+\beta}-K\right)\\
&< \alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2+\frac{\beta^2b^2}{\alpha+\beta}<\frac{2\beta^2b^2}{\alpha+\beta}<2\beta b^2.
\end{aligned}$$

Hence,

$$q < \frac{1}{2(\alpha+\beta)} 2\beta b^2 \frac{2}{5b} = \frac{2}{5}\frac{\beta b}{\alpha+\beta} < \frac{\beta b}{\alpha+\beta}.$$

Suppose now that $W \geq 4\alpha b^2$. Take $\sigma = \frac{W}{4\alpha b}$, then Assumption 1 holds, and thus

$$q = \frac{1}{2(\alpha+\beta)}(H+R)\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\Delta^2}{2\sigma^2}}.$$

Now, $\Delta \equiv c - h$ is bounded by b, so the exponent is close to 1 for W (and thus σ) large. The ratio $\frac{H+R}{\sigma}$ tends to $4\alpha b$ as W increases. Consequently, for any $\varepsilon > 0$ we can find W large enough so that

$$q > (1-\varepsilon)\frac{1}{2(\alpha+\beta)}4\alpha b\frac{1}{\sqrt{2\pi}},$$

which exceeds $\frac{\beta b}{\alpha+\beta}$, if ε is sufficiently small and $\frac{\alpha}{\beta} > \frac{4}{3} > \sqrt{\frac{\pi}{2}}$. ■

Proof of Proposition 4. Part 1. Define

$$\begin{aligned}
P(\alpha,\beta,\mu,\chi,b,W,K,p) &\equiv 2\alpha p - Hf\left(\frac{\beta b}{2(\alpha+\beta)} + \frac{p}{2(\alpha+\beta)}\left(\frac{\beta H - \alpha R}{H}\right)\right),\\
Q(\alpha,\beta,\mu,\chi,b,W,K,q) &\equiv 2(\alpha+\beta)q - (H+R)f\left(\frac{\beta b}{2(\alpha+\beta)} + \frac{q}{2\alpha}\left(\frac{\beta H - \alpha R}{H+R}\right)\right),
\end{aligned}$$

where H and R are defined in the text. Since the argument of f is positive and thus f' is negative, we have

$$\begin{aligned}
\frac{\partial P}{\partial p} &= 2\alpha - \frac{\beta H - \alpha R}{2(\alpha+\beta)}f'(\cdot) = 2\alpha + \frac{\beta H - \alpha R}{2(\alpha+\beta)}\left|f'(\cdot)\right|\\
&> 2\alpha - \frac{\alpha}{2(\alpha+\beta)}\frac{\beta^2 b^2}{\alpha+\beta}\frac{5/4}{(W/\alpha)+b^2} > 2\alpha - \frac{5}{8}\alpha > 0,\\
\frac{\partial Q}{\partial q} &= 2(\alpha+\beta) - \frac{\beta H - \alpha R}{2\alpha}f'(\cdot) = 2(\alpha+\beta) + \frac{\beta H - \alpha R}{2\alpha}\left|f'(\cdot)\right|\\
&> 2(\alpha+\beta) - \frac{1}{2}\frac{\beta^2 b^2}{\alpha+\beta}\frac{5/4}{(W/\alpha)+b^2} > 2(\alpha+\beta) - \frac{5}{8}(\alpha+\beta) > 0.
\end{aligned}$$

Now we differentiate P and Q with respect to W. Using (17), we have

$$\begin{aligned}
\frac{\partial P}{\partial W} &= -f(\cdot) - \frac{\alpha p R}{2(\alpha+\beta)H}f'(\cdot) = -f(\cdot) - \frac{R}{2}\frac{f(\cdot)}{2(\alpha+\beta)}f'(\cdot)\\
&< -f(\cdot) - \frac{\beta^2 b^2}{\alpha+\beta}\frac{f(\cdot)}{4(\alpha+\beta)}f'(\cdot) < 0,
\end{aligned}$$

which implies $\frac{dp}{dW} = -\frac{\partial P}{\partial W} / \frac{\partial P}{\partial p} > 0$. Similarly, using (18), we have

$$\frac{\partial Q}{\partial W} = -f(\cdot) - \frac{q(\alpha+\beta)R}{2(H+R)} f'(\cdot) = -f(\cdot) - \frac{R}{2}\frac{f(\cdot)}{2\alpha} f'(\cdot) < 0,$$

and hence, $\frac{dq}{dW} > 0$.

Part 2. Differentiate P and Q with respect to K. We have

$$\begin{aligned}
\frac{\partial P}{\partial K} &= -H\left(\frac{p}{2(\alpha+\beta)}\frac{\alpha(\chi+\mu-\mu\chi)}{H}\right) f'(\cdot) > 0, \\
\frac{\partial Q}{\partial K} &= (\chi+\mu-\mu\chi) f(\cdot) - \left(\frac{q}{2\alpha}\frac{H(\alpha+\beta)}{H+R}(\chi+\mu-\mu\chi)\right) f'(\cdot) > 0,
\end{aligned}$$

so $\frac{dp}{dK} < 0$ and $\frac{dq}{dK} < 0$.

Part 3. Differentiate P and Q with respect to χ:

$$\begin{aligned}
\frac{\partial P}{\partial \chi} &= H\left(\frac{p}{2(\alpha+\beta)}\frac{\alpha}{H}(1-\mu)\left(\frac{\beta^2 b^2}{\alpha+\beta} - K\right)\right) f'(\cdot) < 0, \\
\frac{\partial Q}{\partial \chi} &= -(1-\mu)\left(\frac{\beta^2 b^2}{\alpha+\beta} - K\right) f(\cdot) + \left(\frac{q}{2\alpha}\frac{H(\alpha+\beta)}{H+R}(1-\mu)\left(\frac{\beta^2 b^2}{\alpha+\beta} - K\right)\right) f'(\cdot) < 0,
\end{aligned}$$

therefore, $\frac{dp}{d\chi} > 0$ and $\frac{dq}{d\chi} > 0$.

Part 4. Differentiate P and Q with respect to μ. We have

$$\begin{aligned}
\frac{\partial P}{\partial \mu} &= \alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2 f(\cdot) + H\left(\frac{\alpha p}{2(\alpha+\beta)}\frac{d(R/H)}{d\mu}\right) f'(\cdot) \\
&= f(\cdot)\left(\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2 - \frac{1}{4(\alpha+\beta)}\left(\begin{array}{c}(1-\chi)\left(\frac{\beta^2 b^2}{\alpha+\beta}-K\right)\left(W+(1-\mu)\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2\right) \\ +(\chi+\mu-\mu\chi)\left(\frac{\beta^2 b^2}{\alpha+\beta}-K\right)\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2\end{array}\right)\left|f'(\cdot)\right|\right) \\
&> f(\cdot)\left(\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2 - \frac{1}{4(\alpha+\beta)}\left(\frac{\beta^2 b^2}{\alpha+\beta}\left(W+\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2\right) + \frac{\beta^2 b^2}{\alpha+\beta}\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2\right)\frac{(5/4)\alpha}{W+\alpha b^2}\right) \\
&> f(\cdot)\left(\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2 - \frac{1}{4(\alpha+\beta)}\left(\frac{5}{4}\alpha\frac{\beta^2 b^2}{\alpha+\beta} + \frac{5}{4}\alpha\frac{\beta^2 b^2}{\alpha+\beta}\right)\right) > \frac{3}{8} f(\cdot)\,\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2 > 0, \\
\frac{\partial Q}{\partial \mu} &= \left(\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2 - (1-\chi)\left(\frac{\beta^2 b^2}{\alpha+\beta}-K\right)\right) f(\cdot) - (H+R)\left(\frac{q}{2\alpha}\frac{d\left(\frac{\beta H-\alpha R}{H+R}\right)}{d\mu}\right) f'(\cdot) \\
&= \left(\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2 - (1-\chi)\left(\frac{\beta^2 b^2}{\alpha+\beta}-K\right)\right) f(\cdot) \\
&\quad -\frac{q}{2\alpha}\frac{\alpha+\beta}{H+R}\left(R\alpha\left(\frac{\beta}{\alpha+\beta}b\right)^2 + H(1-\chi)\left(\frac{\beta^2}{\alpha+\beta}b^2-K\right)\right)\left|f'(\cdot)\right|.
\end{aligned}$$

We thus have $\frac{dp}{d\mu} < 0$, while the sign of $\frac{dq}{d\mu}$ may be ambiguous. Now, the ratio $\frac{|f'(\Delta/2)|}{f(\Delta/2)} = \frac{\Delta}{2\sigma^2}$. Let us prove that

$$\left(\alpha\left(\frac{\beta b}{\alpha+\beta}\right)^2 - (1-\chi)\left(\frac{\beta^2 b^2}{\alpha+\beta}-K\right)\right) - \frac{q}{2\alpha}\frac{\alpha+\beta}{H+R}\left(R\alpha\left(\frac{\beta}{\alpha+\beta}b\right)^2 + H(1-\chi)\left(\frac{\beta^2}{\alpha+\beta}b^2-K\right)\right)\frac{\Delta}{2\sigma^2} \tag{A3}$$

is increasing in χ. This is obviously true for the first term. Next,

$$\begin{aligned}
\frac{d\left(R\alpha\left(\frac{\beta}{\alpha+\beta}b\right)^2 + H(1-\chi)\left(\frac{\beta^2}{\alpha+\beta}b^2-K\right)\right)}{d\chi} &= (1-\mu)\left(\frac{\beta^2}{\alpha+\beta}b^2-K\right)\alpha\left(\frac{\beta}{\alpha+\beta}b\right)^2 \\
&- \left(W+(1-\mu)\alpha\left(\frac{\beta}{\alpha+\beta}b\right)^2\right)\left(\frac{\beta^2}{\alpha+\beta}b^2-K\right) < 0.
\end{aligned}$$

Finally, we prove that $\Delta \equiv c-h$ is decreasing in χ. To do so, we can combine (15) and (16) to find the equation on the equilibrium value of Δ:

$$2\alpha\left((\alpha+\beta)\Delta - b\beta\right) - (H\beta - R\alpha) f\left(\frac{\Delta}{2}\right) = 0.$$

The left-hand side of this expression is increasing in Δ, since

$$
\begin{aligned}
2\alpha(\alpha+\beta)+\frac{1}{2}(H\beta-R\alpha)\left|f'(\cdot)\right| &> 2\alpha(\alpha+\beta)-\frac{1}{2}\frac{\beta^2}{\alpha+\beta}b^2\alpha\frac{(5/4)\,\alpha}{W+\alpha b^2}\\
&\geq \frac{1}{8}\alpha\frac{16\alpha^2+11\beta^2+32\alpha\beta}{\alpha+\beta}>0.
\end{aligned}
$$

It is also increasing in χ, because R is increasing in χ. Hence, $\frac{d\Delta}{d\chi}<0$. This proves that the second term in (A3) decreases in absolute value (and it is positive), hence, (A3) is increasing in χ. Therefore, $\frac{\partial Q}{\partial\mu}$ may be positive only for χ above some threshold, and may be negative only for χ below some threshold. This shows that $\frac{dq}{d\mu}>0$ for $\chi<\bar{\chi}$ and $\frac{dq}{d\mu}<0$ for $\chi>\bar{\chi}$ for some $\bar{\chi}$. ■

Proof of Proposition 5. Part 1. Differentiate P with respect to b. We have

$$
\begin{aligned}
\frac{\partial P}{\partial b} &= -2(1-\mu)\alpha\left(\frac{\beta}{\alpha+\beta}\right)^2 bf(\cdot)+H\left(\frac{\alpha p}{2(\alpha+\beta)}\frac{d(R/H)}{db}-\frac{\beta}{2(\alpha+\beta)}\right)f'(\cdot)\\
&= -2(1-\mu)\alpha\left(\frac{\beta}{\alpha+\beta}\right)^2 bf(\cdot)\\
&\quad +H\left(\frac{1}{H^2}\alpha p(\chi+\mu-\mu\chi)\left(\frac{\beta}{\alpha+\beta}\right)^2\left(W+(1-\mu)\frac{\alpha}{\alpha+\beta}K\right)-\frac{\beta}{2(\alpha+\beta)}\right)bf'(\cdot)\\
&< -2(1-\mu)\alpha\left(\frac{\beta}{\alpha+\beta}\right)^2 bf(\cdot)+\frac{\beta\left(W+(1-\mu)\alpha\left(\frac{\beta}{\alpha+\beta}b\right)^2\right)}{2(\alpha+\beta)}\left|f'(\cdot)\right|.
\end{aligned}
$$

To show that it is negative, it suffices to show that the following is positive:

$$
4(1-\mu)\alpha\left(\frac{\beta}{\alpha+\beta}\right)b-\left(W+(1-\mu)\alpha\left(\frac{\beta}{\alpha+\beta}b\right)^2\right)\frac{\Delta}{2\sigma^2}.
$$

In the proof of Proposition 2, we showed that $\frac{\Delta}{\sigma}<2\frac{\beta}{\alpha+\beta}$, hence it suffices to prove that

$$
4(1-\mu)\alpha b-\left(W+(1-\mu)\alpha\left(\frac{\beta}{\alpha+\beta}b\right)^2\right)\frac{1}{\sigma}.
$$

Since $\sigma>b$ by Assumption 1, this expression is unambiguously positive for W small. For such W, $\frac{\partial P}{\partial b}<0$, and thus $\frac{dp}{db}>0$.

Part 2. P and Q depend on σ only through $f\left(\frac{\Delta}{2}\right)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\Delta^2}{8\sigma^2}}$. We have

$$\frac{d\left(\frac{1}{\sigma}e^{-\frac{\Delta^2}{8\sigma^2}}\right)}{d\sigma}=\frac{1}{\sigma^4}e^{-\frac{\Delta^2}{8\sigma^2}}\left(\frac{\Delta^2}{4}-\sigma^2\right).$$

This is negative, since $\sigma>b>\frac{\Delta}{2}$ (the last inequality was proved in the proof of Proposition 2). Consequently, $f\left(\frac{\Delta}{2}\right)$ is decreasing in σ, and thus $\frac{\partial P}{\partial\sigma}>0$ and $\frac{\partial Q}{\partial\sigma}>0$. Hence, $\frac{dp}{d\sigma}<0$ and $\frac{dq}{d\sigma}<0$. ■

Proof of Proposition 6.

Suppose that K increases to $\frac{\beta^2b^2}{\alpha+\beta}$. This implies that the payoff of the lobby in the second period approaches $-\beta b^2$ (honest politicians choose $x_2=0$, and the lobby's utility from corrupt politicians is only marginally higher than it would be if bribing does not happen and the politician chooses 0). Now, we showed in the proof of Proposition 2 that in the first period, honest politicians would be populist, and corrupt politicians would choose $\tilde{c}<h$ if bribing fails (threat point); moreover, as $K\to\frac{\beta^2b^2}{\alpha+\beta}$, p remains bounded away from 0, as follows from (17). Hence, the first-period utility of the lobby is in this case worse than if bribing was impossible. The exact same argument also applies when χ increases to 1. ■

Proof of Proposition 7. Let us denote the probability of having an honest politician in the second period (without term limits) by λ. Notice that the equilibrium probability of reelection of an honest politician is $F\left(\frac{c-h}{2}\right)$, and the probability of reelection of a right-wing politician is $F\left(\frac{h-c}{2}\right)=1-F\left(\frac{c-h}{2}\right)$. Consequently,

$$\begin{aligned}\lambda &= \mu F\left(\frac{c-h}{2}\right)+\mu\left(1-F\left(\frac{c-h}{2}\right)\right)\mu+(1-\mu)\left(1-\left(1-F\left(\frac{c-h}{2}\right)\right)\right)\mu\\ &= \mu\left(\mu+2(1-\mu)F\left(\frac{c-h}{2}\right)\right),\\ 1-\lambda &= (1-\mu)\left(1-\mu\left(2F\left(\frac{c-h}{2}\right)-1\right)\right)\end{aligned}$$

Intuitively, the politician in the second period is honest if (a) an honest politician is reelected, (b) honest politician is not reelected, but another honest one comes instead, and (c) corruptible politician is replaced by an honest one. Therefore, without (hard) term limits social welfare is

$$V^n=-\mu h^2-(1-\mu)c^2-(1-\mu)\left(1-\mu\left(2F\left(\frac{c-h}{2}\right)-1\right)\right)\left(\frac{\beta}{\alpha+\beta}b\right)^2,$$

and social welfare with term limits is

$$V^t = -2(1-\mu)\left(\frac{\beta}{\alpha+\beta}b\right)^2.$$

We therefore have

$$V^n - V^t = -\mu h^2 - (1-\mu)c^2 + (1-\mu)\left(1+\mu\left(2F\left(\frac{c-h}{2}\right)-1\right)\right)\left(\frac{\beta}{\alpha+\beta}b\right)^2. \tag{A4}$$

First we show that this expression is strictly increasing in W starting from $W=0$, which will establish that for W sufficiently close to 0, an increase in W makes term limits less likely to increase social welfare. Differentiating with respect to W, we have

$$d\frac{\left(V^n - V^t\right)}{dW} = -2\mu h\frac{dh}{dW} - 2(1-\mu)c\frac{dc}{dW} + (1-\mu)\mu f\left(\frac{c-h}{2}\right)\left(\frac{\beta}{\alpha+\beta}b\right)^2\left(\frac{dc}{dW}-\frac{dh}{dW}\right).$$

When $W=0$, (15) implies that $(1-\mu)f\left(\frac{c-h}{2}\right)\left(\frac{\beta}{\alpha+\beta}b\right)^2 = -2h$. Substituting for this, we have

$$\begin{aligned} d\frac{\left(V^n - V^t\right)}{dW} &= -2\mu h\frac{dh}{dW} - 2(1-\mu)c\frac{dc}{dW} - 2\mu h\left(\frac{dc}{dW}-\frac{dh}{dW}\right) \\ &= -2c\frac{dc}{dW}. \end{aligned}$$

From Proposition 4, $\frac{dc}{dW} < 0$, and moreover, when $W=0$, $c>0$ which establishes the desired result.

We next show that for μ close to 1, the comparison of term limits to no term limits depends on whether $W>0$. Recall that $p \equiv |h|$ and $\Delta \equiv c-h$, and rewrite (A4) as

$$V^n - V^t = -\mu p^2 - (1-\mu)(\Delta - p)^2 + (1-\mu)\left(1+\mu\left(2F\left(\frac{\Delta}{2}\right)-1\right)\right)\left(\frac{\beta}{\alpha+\beta}b\right)^2. \tag{A5}$$

Notice that (17) implies that as $\mu \to 1$, p tends to the solution of the equation

$$p = \frac{1}{2\alpha}Wf\left(\frac{\beta}{2(\alpha+\beta)}b + \frac{p}{2(\alpha+\beta)}\left(\beta - \frac{\alpha}{W}\left(\frac{\beta^2}{\alpha+\beta}b^2 - K\right)\right)\right),$$

which is positive and unique, in the case $W > 0$. If $W = 0$, then p tends to 0, and in such a way that $\frac{p}{1-\mu}$ tends to a solution to

$$\frac{p}{1-\mu} = \frac{1}{2}\left(\frac{\beta}{\alpha+\beta}b\right)^2 f\left(\frac{\beta}{2(\alpha+\beta)}b - \frac{1}{2}\frac{p}{1-\mu}\left(1 - \frac{K(\alpha+\beta)}{\beta^2 b^2}\right)\right),$$

which is finite and positive. In either case, Δ tends to the solution of

$$2\alpha(\alpha+\beta)\left(\Delta - \frac{\beta b}{\alpha+\beta}\right) = \left(\beta W - \alpha\left(\frac{\beta^2}{\alpha+\beta}b^2 - K\right)\right) f\left(\frac{\Delta}{2}\right),$$

which is unique and positive (indeed, if $\Delta = 0$, then left-hand side is $-2\alpha\beta b$, and the right-hand side is at least $-\alpha\frac{\beta^2}{\alpha+\beta}b^2\frac{2}{b} = -2\alpha\beta b\frac{\beta}{\alpha+\beta}$.

These results already imply that if $W > 0$, then for μ sufficiently close to 1, (A5) is negative.

Suppose that $W = 0$. To get that (A5) is positive, it suffices to prove that

$$-(\Delta - p)^2 + \left(1 + \mu\left(2F\left(\frac{\Delta}{2}\right) - 1\right)\right)\left(\frac{\beta}{\alpha+\beta}b\right)^2 > 0 \tag{A6}$$

However, this exceeds

$$\left(\frac{\beta}{\alpha+\beta}b\right)^2 - (\Delta-p)^2 = \left(\frac{\beta}{\alpha+\beta} - (\Delta - p)\right)\left(\frac{\beta}{\alpha+\beta} + \Delta + p\right).$$

But since $\Delta - p = c$ and $c < \frac{\beta}{\alpha+\beta}b$, (A6) is unambiguously positive, and thus so it is (A5) provided that $W > 0$ and μ sufficiently close to 1. ■

Proof of Proposition 8.

Proceeding as in Section 3, we find the following equilibrium conditions on the first-period policy choices of the two types of politicians:

$$\begin{aligned}
-2\alpha h - Hf\left(\frac{c-h}{2} - \sigma^2\frac{\ln\eta}{c-h}\right) &= 0,\\
-2\alpha c - 2\beta(c-b) - (H+R)f\left(\frac{h-c}{2} - \sigma^2\frac{\ln\eta}{c-h}\right) &= 0.
\end{aligned}$$

For a normal distribution,

$$f\left(\frac{c-h}{2} - \sigma^2\frac{\ln\eta}{c-h}\right)/f\left(\frac{h-c}{2} - \sigma^2\frac{\ln\eta}{c-h}\right) = \eta,$$

and this allows us to obtain the condition for the equilibrium value of Δ:

$$\Delta = b\frac{\beta}{\alpha+\beta} - \frac{H\left(\alpha - \eta\left(\alpha+\beta\right)\right) + R\alpha}{2\alpha\left(\alpha+\beta\right)\eta} f\left(\frac{\Delta}{2} - \sigma^2 \frac{\ln \eta}{\Delta}\right).$$

One can check that derivative of the right-hand side with respect to Δ is bounded away from 1 uniformly in η, provided that σ is high enough; this implies the existence and uniqueness of an equilibrium. The condition for the populist bias, $p = |h|$, is now the following:

$$2\alpha p - Hf\left(\frac{\frac{\beta}{\alpha+\beta}b - \frac{H(\alpha-(\alpha+\beta)\eta)+R\alpha}{H\eta(\alpha+\beta)}p}{2} - \sigma^2 \frac{\ln \eta}{\frac{\beta}{\alpha+\beta}b - \frac{H(\alpha-(\alpha+\beta)\eta)+R\alpha}{H\eta(\alpha+\beta)}p}\right) = 0. \tag{A7}$$

Let us rewrite this as $A\left(p,\eta\right) = 0$, and study $\frac{dp}{d\eta}$.

Our first observation is that $p \to 0$ as $\sigma \to \infty$ uniformly in η; indeed, (A7) implies

$$p < \frac{H}{2\sqrt{2\pi}\alpha}\frac{1}{\sigma}.$$

Moreover, the same is true for $\frac{p}{\eta}$, and we have $\frac{1}{\eta} f\left(\frac{x}{2} - \sigma^2 \frac{\ln \eta}{x}\right) = f\left(\frac{x}{2} + \sigma^2 \frac{\ln \eta}{x}\right)$ for any x, and therefore

$$\frac{p}{\eta} < \frac{H}{2\sqrt{2\pi}\alpha}\frac{1}{\sigma}.$$

We can thus pick σ large enough, so that $\frac{\beta}{\alpha+\beta}b - \frac{H(\alpha-(\alpha+\beta)\eta)+R\alpha}{H\eta(\alpha+\beta)}p \in \left(\frac{1}{2}\frac{\beta}{\alpha+\beta}b, \frac{3}{2}\frac{\beta}{\alpha+\beta}b\right)$.

Differentiating A with respect to p establishes that it is an increasing function. In particular,

$$\frac{\partial A}{\partial p} = 2\alpha - H\frac{z}{\sigma^2}\frac{H\left(\alpha - \left(\alpha+\beta\right)\eta\right) + R\alpha}{H\eta\left(\alpha+\beta\right)} \frac{\frac{1}{2} + \sigma^2 \ln \eta}{\left(\frac{\beta}{\alpha+\beta}b - \frac{H(\alpha-(\alpha+\beta)\eta)+R\alpha}{H\eta(\alpha+\beta)}p\right)^2} f\left(z\right),$$

where

$$z = \frac{\frac{\beta}{\alpha+\beta}b - \frac{H(\alpha-(\alpha+\beta)\eta)+R\alpha}{H\eta(\alpha+\beta)}p}{2} - \sigma^2 \frac{\ln \eta}{\frac{\beta}{\alpha+\beta}b - \frac{H(\alpha-(\alpha+\beta)\eta)+R\alpha}{H\eta(\alpha+\beta)}p}.$$

But $f\left(z\right) < k\frac{1}{\sigma}e^{-(\ln \eta)^2}$ for some constant k and σ large enough, and this implies that $\frac{\partial A}{\partial p}$ tends to 2α uniformly in η as σ becomes larger (and thus it is also bounded away from 0).

Let us now show that z is decreasing in η (this will enable us to study how A depends on η). We have

$$\frac{\partial z}{\partial \eta}=\frac{1}{2}\frac{\alpha\left(H+R\right)p}{H\left(\alpha+\beta\right)\eta^{2}}-\sigma^{2}\frac{H\eta\left(\alpha+\beta\right)}{\beta bH\eta-\left(H\left(\alpha-\left(\alpha+\beta\right)\eta\right)+R\alpha\right)p}\frac{1}{\eta}\left(1-\frac{\alpha\left(H+R\right)p}{\beta bH\eta-\left(H\left(\alpha-\left(\alpha+\beta\right)\eta\right)+R\alpha\right)p}\right).$$

Multiplying both sides by η and using that p and p/η will be arbitrarily small for large σ, we notice that only the second term will matter, and it will be negative. Hence, z is decreasing in η.

The previous result implies that $\frac{\partial A}{\partial \eta}<0$ if $z>0$ and $\frac{\partial A}{\partial \eta}>0$ if $z<0$. This, in turn, means that $\frac{dp}{d\eta}>0$ if $z>0$ and $\frac{dp}{d\eta}>0$ if $z<0$. We have already shown that for a fixed p, z is decreasing in η. To complete the proof, we need to show that if p is given by (A7), then $\frac{\partial z}{\partial \eta}$ can only change its sign from positive to negative as η increases. Indeed, suppose, to obtain a contradiction, that $\frac{\partial z}{\partial \eta}$ changes its sign from negative to positive at some $\bar{\eta}$. Then at this $\bar{\eta}$, we must have $\frac{\partial z}{\partial \eta}=0$, and therefore $\frac{dp}{d\eta}=0$. But this means that even though p varies in the neighborhood of $\bar{\eta}$, this does not contribute to $\frac{\partial z}{\partial \eta}$, and it is still negative. However, this contradicts our assertion, and therefore $\frac{\partial z}{\partial \eta}$ can only change its sign from positive to negative.

Notice that for η close to 0, $z>0$, for η large enough, $z<0$, and for $\eta=1$, $z>0$. Therefore, there is $\eta^{*}>1$ such that $\frac{dp}{d\eta}>0$ if and only if $\eta<\eta^{*}$. This proves that $\frac{dp}{d\nu}>0$ if and only if $\nu<\nu^{*}$, where $\nu^{*}=\frac{\eta^{*}\mu}{1-\mu+\eta^{*}\mu}>\mu$.

The proof that $q=\left|c-\frac{\beta}{\alpha+\beta}b\right|$ is increasing in ν for $\nu<\nu^{**}$ and increasing in ν for $\nu>\nu^{**}$, where $\nu^{**}<\mu$, is similar and is omitted. ■

Proof of Proposition 9. First, note that given the normal distribution of z, (22) is equivalent to

$$\mu^{l}\left(1-\exp\left(-\frac{1}{\sigma^{2}}\left(h-c^{l}\right)\left(s-\frac{c^{l}+h}{2}\right)\right)\right)+\mu^{r}\left(1-\exp\left(\frac{1}{\sigma^{2}}\left(c^{r}-h\right)\left(s-\frac{c^{r}+h}{2}\right)\right)\right)\geq 0. \tag{A8}$$

Next the problem of an honest politician can be written as

$$\begin{aligned}&\max_{x\in\mathbb{R}}-\alpha x^{2}-\beta\left(x+b\right)^{2}+\left(W-\frac{\alpha\beta}{\alpha+\beta}b^{2}-K\right)\pi\left(x\right)\\&-\left[\mu\beta b^{2}-\mu^{l}\left(\frac{\alpha\beta}{\alpha+\beta}b^{2}+\left(\chi+\frac{\alpha}{\alpha+\beta}\right)\frac{\beta^{2}}{\alpha+\beta}b^{2}+\left(1-\chi\right)K\right)-\mu^{r}\frac{\beta\left(\alpha+4\beta\right)}{\alpha+\beta}b^{2}\right]\left(1-\pi\left(x\right)\right).\end{aligned}$$

Similarly, for a right-wing incument, the bargaining problem is

$$
\begin{aligned}
&\max_{x\in\mathbb{R}} -\alpha x^2 - \beta\left(x-b\right)^2 + \left(W - \frac{\alpha\beta}{\alpha+\beta}b^2 - K\right)\pi\left(x\right) \\
&-\left[\mu\beta b^2 - \mu^r\left(\frac{\alpha\beta}{\alpha+\beta}b^2 + \left(\chi + \frac{\alpha}{\alpha+\beta}\right)\frac{\beta^2}{\alpha+\beta}b^2 + \left(1-\chi\right)K\right) - \mu^l\frac{\beta\left(\alpha+4\beta\right)}{\alpha+\beta}b^2\right]\left(1-\pi\left(x\right)\right),
\end{aligned}
$$

The first-order conditions for the bargaining problems of left and right wing incumbents are

$$
-2\alpha x - 2\beta\left(x+b\right) - \begin{pmatrix} \left(W - \frac{\alpha\beta}{\alpha+\beta}b^2 - K\right) + \mu\beta b^2 \\ -\mu^l\left(\frac{\alpha\beta}{\alpha+\beta}b^2 + \left(\chi+\frac{\alpha}{\alpha+\beta}\right)\frac{\beta^2}{\alpha+\beta}b^2 + \left(1-\chi\right)K\right) \\ -\mu^r\frac{\beta(\alpha+4\beta)}{\alpha+\beta}b^2 \end{pmatrix}\left(f\left(s_r - x\right) - f\left(s_l - x\right)\right) = 0. \tag{A9}
$$

and

$$
-2\alpha x - 2\beta\left(x-b\right) - \begin{pmatrix} \left(W - \frac{\alpha\beta}{\alpha+\beta}b^2 - K\right) + \mu\beta b^2 \\ -\mu^r\left(\frac{\alpha\beta}{\alpha+\beta}b^2 + \left(\chi+\frac{\alpha}{\alpha+\beta}\right)\frac{\beta^2}{\alpha+\beta}b^2 + \left(1-\chi\right)K\right) \\ -\mu^l\frac{\beta(\alpha+4\beta)}{\alpha+\beta}b^2 \end{pmatrix}\left(f\left(s_r - x\right) - f\left(s_l - x\right)\right) = 0. \tag{A10}
$$

In equilibrium, (23), (A9), and (A10) must hold for $x = h, c^{c^l}, c^r$, respectively. This implies that the following three conditions characterize an equilibrium

$$\begin{aligned}
- \quad & 2\alpha h - \left(W + (1-\mu)\,\alpha \left(\frac{\beta}{\alpha+\beta} b \right)^2 \right) && \text{(A11)} \\
\times \quad & \left(f\left(s_r - h\right) - f\left(s_l - h\right) \right) = 0 \\
- \quad & 2\alpha l - 2\beta \left(c^l + b \right) - \begin{pmatrix} \left(W - \frac{\alpha\beta}{\alpha+\beta} b^2 - K \right) + \mu\beta b^2 \\ -\mu^l \begin{pmatrix} \frac{\alpha\beta}{\alpha+\beta} b^2 \\ + \left(\chi + \frac{\alpha}{\alpha+\beta} \right) \frac{\beta^2}{\alpha+\beta} b^2 \\ + (1-\chi) K \end{pmatrix} \\ -\mu^r \frac{\beta(\alpha+4\beta)}{\alpha+\beta} b^2 \end{pmatrix} && \text{(A12)} \\
\times \quad & \left(f\left(s_r - c^l\right) - f\left(s_l - c^l\right) \right) = 0, \\
- \quad & 2\alpha r - 2\beta \left(c^r - b \right) - \begin{pmatrix} \left(W - \frac{\alpha\beta}{\alpha+\beta} b^2 - K \right) + \mu\beta b^2 \\ -\mu^r \begin{pmatrix} \frac{\alpha\beta}{\alpha+\beta} b^2 \\ + \left(\chi + \frac{\alpha}{\alpha+\beta} \right) \frac{\beta^2}{\alpha+\beta} b^2 \\ + (1-\chi) K \end{pmatrix} \\ -\mu^l \frac{\beta(\alpha+4\beta)}{\alpha+\beta} b^2 \end{pmatrix} && \text{(A13)} \\
\times \quad & \left(f\left(s_r - c^r\right) - f\left(s_l - c^r\right) \right) = 0.
\end{aligned}$$

Let $H \subset \mathbb{R}^3$ be the (open) set defined by

$$\left(h, c^l, c^r \right) \in H \Longleftrightarrow c^l < h < c^r.$$

We will first prove that the set of signals s that satisfy (A8) is a (closed) interval $[s_l, s_r]$ such that $-\infty < s_l < \frac{c^l+h}{2}$ and $\frac{h+c^r}{2} < s_r < +\infty$ whenever $\left(h, c^l, c^r\right) \in H$. Indeed, as s becomes close to $-\infty$, the first term becomes negative, and arbitrarily large in absolute value (since the exponent tends to $+\infty$), while the second term tends to 0, so the left-hand side of (A8) is negative. Likewise, as s becomes large and positive, the first term tends to 0 and the second becomes large and negative, so the left-hand side is negative. At the same time, if we pick $s = \frac{c^l+h}{2}$ or $s = \frac{h+c^r}{2}$, then one term is positive and the other is zero, so the left-hand side is positive. It now suffices to prove that the left-hand side of (A8) is a concave function of s. This follows by observing that the derivative with respect to s,

$$\mu^l \frac{1}{\sigma^2}\left(h-c^l\right)\exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s-\frac{c^l+h}{2}\right)\right)-\mu^r\frac{1}{\sigma^2}\left(c^r-h\right)\exp\left(\frac{1}{\sigma^2}\left(c^r-h\right)\left(s-\frac{h+c^r}{2}\right)\right),$$

is a decreasing function of s.

We have thus shown that for all $\left(h,c^l,c^r\right)\in H$, there are exactly two different solutions to the equation

$$\mu^l\left(1-\exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s-\frac{c^l+h}{2}\right)\right)\right)+\mu^r\left(1-\exp\left(\frac{1}{\sigma^2}\left(c^r-h\right)\left(s-\frac{h+c^r}{2}\right)\right)\right)=0; \tag{A14}$$

we can denote the lesser of them as $s_l\left(h,c^l,c^r\right)$ and the greater as $s_r\left(h,c^l,c^r\right)$.

Denote $a=\frac{\beta}{\alpha+\beta}b$. For each $\rho\in\left(0,\frac{1}{3}\right)$, let $H_\rho\subset\mathbb{R}^3$ be the compact set given by

$$\left(h,c^l,c^r\right)\in H_\rho \Longleftrightarrow \begin{cases} -\rho a\le h\le \rho a \\ -\left(1+\rho\right)a\le c^l\le-\left(1-\rho\right)a \\ \left(1-\rho\right)a\le c^r\le\left(1+\rho\right)a \end{cases}.$$

Let us prove that the functions $\frac{\partial s}{\partial x}\frac{s-y}{\sigma^3}\exp\left(-\frac{(s-y)^2}{2\sigma^2}\right)$, where $s\in\{s_l,s_r\}$, and $x,y\in\left\{h,c^l,c^r\right\}$ in all possible combinations (18 totally) are each $o\left(1\right)$ as $\sigma\to\infty$, provided that $\left(h,c^l,c^r\right)\in H_\rho$ for ρ chosen below. We choose ρ in the following way. Consider the function

$$Q\left(a,s\right)=\frac{1-\exp\left(-a\left(s+\frac{a}{2}\right)\right)}{1-\exp\left(-a\left(s-\frac{a}{2}\right)\right)}-\exp\left(-a\left(s+\frac{a}{2}\right)\right)$$

for $s\in\left(\frac{a}{2},\infty\right)$. One can verify that for all $s\in\left(\frac{a}{2},\infty\right)$,

$$Q\left(a,s\right)\ge 2e^{-a^2}\sqrt{e^{a^2}-1}. \tag{A15}$$

To see this, notice that $\lim_{s\to\frac{a}{2}+0}Q\left(a,s\right)=+\infty$ and $\lim_{s\to+\infty}Q\left(a,s\right)=1$. In addition, if $e^{a^2}<2$, then $s_0=\frac{\ln\left(e^{-\frac{1}{2}a^2}-\sqrt{1-e^{-a^2}}\right)}{-a}$ is a local minimum on $\left(\frac{a}{2},+\infty\right)$, and $Q\left(a,s_0\right)=2e^{-a^2}\sqrt{e^{a^2}-1}$. Since $2e^{-a^2}\sqrt{e^{a^2}-1}\le 1$ for all a (with equality achieved if $e^{a^2}=2$), we have that $Q\left(a,s\right)$ is bounded from below by $2e^{-a^2}\sqrt{e^{a^2}-1}$ for all a, so (A15) holds.

Consider now the function

$$\bar{Q}(a,s,\rho) = \frac{1-2\rho}{1+2\rho}\frac{1-\exp\left(-a(1-2\rho)\left(s+\frac{a(1-2\rho)}{2}\right)\right)}{1-\exp\left(-a(1+2\rho)\left(s-\frac{a(1-2\rho)}{2}\right)\right)} - \exp\left(-a(1-2\rho)\left(s+\frac{a(1-2\rho)}{2}\right)\right).$$

By continuity, we can choose $\rho^* > 0$ such that for all $\rho \in [0,\rho^*]$ and for all $s > \frac{a(1-2\rho)}{2}$,

$$Q(a,s,\rho) \geq e^{-a^2}\sqrt{e^{a^2}-1}.$$

Next, for $\rho \in [0,\rho^*]$ and $\sigma \geq 1$ consider the function

$$\hat{Q}(a,s,\rho,\sigma) = \bar{Q}\left(\frac{a}{\sigma},\frac{s}{\sigma},\rho\right) = \frac{1-2\rho}{1+2\rho}\frac{1-\exp\left(-\frac{1}{\sigma^2}a(1-2\rho)\left(s+\frac{a(1-2\rho)}{2}\right)\right)}{1-\exp\left(-\frac{1}{\sigma^2}b(1+2\rho)\left(s-\frac{b(1-2\rho)}{2}\right)\right)} - \exp\left(-\frac{1}{\sigma^2}a(1-2\rho)\left(s+\frac{a(1-2\rho)}{2}\right)\right).$$

Applying the previous result, we immediately get that for $s > \frac{a}{2}$,

$$\hat{Q}(a,s,\rho,\sigma) \geq e^{-\left(\frac{a}{\sigma}\right)^2}\sqrt{e^{\left(\frac{a}{\sigma}\right)^2}-1} \geq e^{-\left(\frac{a}{\sigma}\right)^2}\sqrt{\left(\frac{a}{\sigma}\right)^2} \geq e^{-a^2}\frac{a}{\sigma}$$

(where we used the fact that $e^x > 1+x$ for all $x > 0$ and that $\sigma \geq 1$).

Consider now the following function:

$$\tilde{Q}\left(h,c^l,c^r,s,\sigma\right) = \frac{c^r-h}{h-c^l}\frac{1-\exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s-\frac{c^l+h}{2}\right)\right)}{1-\exp\left(-\frac{1}{\sigma^2}\left(c^r-h\right)\left(s-\frac{h+c^r}{2}\right)\right)} - \exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s-\frac{c^l+h}{2}\right)\right),$$

defined on $H \cap \left\{s > \frac{h+c^r}{2}\right\}$. We can observe that if $\left(h,c^l,c^r\right) \in H_\rho$, then the four values, $h-c^l, -\left(c^l+h\right), c^r-h, h+c^r$ lie on $[a(1-2\rho), a(1+2\rho)]$, which implies

$$\tilde{Q}\left(h,c^l,c^r,s,\sigma\right) \geq \hat{Q}(b,s,\rho,\sigma) \geq e^{-a^2}\frac{a}{\sigma}.$$

We are now ready to estimate $\frac{\partial s}{\partial x}\frac{s-y}{\sigma^3}\exp\left(-\frac{(s-y)^2}{2\sigma^2}\right)$. Suppose, for example, that $s = s_r$. Since s_r is the larger root of (A14), consider the function

$$L(s) = \mu^l\left(1-\exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s-\frac{c^l+h}{2}\right)\right)\right)+\mu^r\left(1-\exp\left(\frac{1}{\sigma^2}\left(c^r-h\right)\left(s-\frac{h+c^r}{2}\right)\right)\right);$$

we then have

$$\frac{\partial s}{\partial x} = -\frac{\partial L/\partial x}{\partial L/\partial s}.$$

As argued above, $\partial L/\partial s$ is negative at $s = s_r$, so consider

$$\left|\frac{\partial L}{\partial s}\right| = \mu^r\frac{1}{\sigma^2}\left(c^r-h\right)\exp\left(\frac{1}{\sigma^2}\left(c^r-h\right)\left(s-\frac{c^r+h}{2}\right)\right)-\mu^l\frac{1}{\sigma^2}\left(h-c^l\right)\exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s-\frac{c^l+h}{2}\right)\right).$$

Consider the following two possibilities separately.

If $\mu^r \geq 6\mu^l$ (so $\mu^r \geq \frac{6}{7}(1-\mu)$ and $\mu^l \leq \frac{1}{7}(1-\mu)$), then (since $s_r > \frac{h+c^r}{2} > \frac{c^l+h}{2}$)

$$\begin{aligned}\left|\frac{\partial L}{\partial s}\right| &> \frac{1}{\sigma^2}\left(\mu^r\left(c^r-h\right)-\mu^l\left(h-l\right)\right)\\ &> \frac{1}{\sigma^2}\left(\mu^r\frac{a}{3}-\mu^l\frac{5a}{3}\right) \geq \frac{a}{\sigma^2}\frac{1-\mu}{21}.\end{aligned}$$

Otherwise, if $\mu^r < 6\mu^l$ (so $\mu^l > \frac{1}{7}(1-\mu)$), then, substituting for μ^r/μ^l from (A14), we get

$$\begin{aligned}\left|\frac{\partial L}{\partial s}\right| &= \mu^l\frac{1}{\sigma^2}\left(h-c^l\right)\left(\frac{\mu^r\left(c^r-h\right)\exp\left(\frac{1}{\sigma^2}\left(c^r-h\right)\left(s-\frac{h+c^r}{2}\right)\right)}{\mu^l\left(h-c^l\right)}-\exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s-\frac{c^l+h}{2}\right)\right)\right)\\ &= \mu^l\frac{1}{\sigma^2}\left(h-c^l\right)\times\left(-\frac{\left(c^r-h\right)\exp\left(\frac{1}{\sigma^2}\left(c^r-h\right)\left(s-\frac{h+c^r}{2}\right)\right)}{\left(h-c^l\right)}\frac{1-\exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s-\frac{c^l+h}{2}\right)\right)}{1-\exp\left(\frac{1}{\sigma^2}\left(c^r-h\right)\left(s-\frac{c^r+h}{2}\right)\right)}\right.\\ &\quad\left.-\exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s-\frac{c^l+h}{2}\right)\right)\right)\\ &= \mu^l\frac{1}{\sigma^2}\left(h-c^l\right)\left(\frac{c^r-h}{h-c^l}\frac{1-\exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s-\frac{c^l+h}{2}\right)\right)}{1-\exp\left(-\frac{1}{\sigma^2}\left(c^r-h\right)\left(s-\frac{h+c^r}{2}\right)\right)}-\exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s-\frac{c^l+h}{2}\right)\right)\right)\\ &= \mu^l\frac{1}{\sigma^2}\left(h-c^l\right)\tilde{Q}\left(h,c^l,c^r,s,\sigma\right).\end{aligned}$$

Consequently, if $\left(h,c^l,c^r\right)\in H_\rho$, then

$$\left|\frac{\partial L}{\partial s}\right| \geq \frac{1}{7}(1-\mu)\frac{1}{\sigma^2}\frac{1}{3}ae^{-a^2}\frac{a}{\sigma} \geq \frac{1-\mu}{21\sigma^3}a^2e^{-a^2}.$$

This implies that, given $\sigma \geq 1$ and $a^2 e^{-a^2} \leq 1$, that in both cases

$$\left|\frac{\partial L}{\partial s}\right| \geq \frac{1-\mu}{21\sigma^3} a^2 e^{-a^2}. \tag{A16}$$

It is straightforward to check (or invoke the symmetry argument) that inequality (A16) would hold for $s = s_{c^l}$ as well.

Consider now the derivatives

$$\begin{aligned}
\frac{\partial L}{\partial h} &= \frac{1}{\sigma^2}(s-h)\left(\mu^l \exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s-\frac{c^l+h}{2}\right)\right)+\mu^r \exp\left(\frac{1}{\sigma^2}(c^r-h)\left(s-\frac{h+c^r}{2}\right)\right)\right), \\
\frac{\partial L}{\partial l} &= -\mu^l \frac{1}{\sigma^2}\left(s-c^l\right)\exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s-\frac{c^l+h}{2}\right)\right), \\
\frac{\partial L}{\partial c^r} &= -\mu^r \frac{1}{\sigma^2}(s-c^r)\exp\left(\frac{1}{\sigma^2}(c^r-h)\left(s-\frac{h+c^r}{2}\right)\right).
\end{aligned}$$

We have

$$\begin{aligned}
\exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s_r-\frac{c^l+h}{2}\right)\right) &< 1, \\
\exp\left(\frac{1}{\sigma^2}(c^r-h)\left(s_l-\frac{h+c^r}{2}\right)\right) &< 1,
\end{aligned}$$

and from (A14) we also have

$$\exp\left(-\frac{1}{\sigma^2}\left(h-c^l\right)\left(s_l-\frac{c^l+h}{2}\right)\right) < \frac{\mu^l+\mu^r}{\mu^l}, \tag{A17}$$

$$\exp\left(\frac{1}{\sigma^2}(c^r-h)\left(s_r-\frac{h+c^r}{2}\right)\right) < \frac{\mu^l+\mu^r}{\mu^r}. \tag{A18}$$

Consequently, for any $s \in \{s_l, s_r\}$ and $x \in \left\{h, c^l, c^r\right\}$, we have

$$\left|\frac{\partial L}{\partial x}\right| \leq 2(1-\mu)\frac{1}{\sigma^2}|s-x|. \tag{A19}$$

To proceed, consider the term $|s-x|$. Notice that (A17) and (A18) imply that for σ large enough

$$\begin{aligned} s_l &> -2\frac{\sigma^2}{a}\ln\frac{\mu^l+\mu^c}{\mu^l}, \\ s_h &< -2\frac{\sigma^2}{a}\ln\frac{\mu^l+\mu^c}{\mu^c}. \end{aligned}$$

Let us define $k_l(\sigma) \equiv \frac{s_l}{\sigma^2}$ and $k_r(\sigma) \equiv \frac{s_r}{\sigma^2}$; they are bounded away from $\pm\infty$. From (A14) we obtain that as $\sigma \to \infty$, $k_l(\sigma)$ and $k_r(\sigma)$ tend to the two solutions of the equation

$$\mu^l\left(1-\exp\left(-\left(h-c^l\right)k\right)\right)+\mu^r\left(1-\exp\left(\left(c^r-h\right)k\right)\right),$$

which are also bounded away from 0 for $\left(h,c^l,c^r\right) \in H_\rho$ for $\rho < \hat{\rho}$ for some $\hat{\rho}$ small enough. Let $\rho = \min(\rho^*, \hat{\rho})$. Therefore, there exist positive constants τ_1 and τ_2 such that whenever σ is large enough,

$$\tau_1 < \frac{|s_l|}{\sigma^2}, \frac{|s_h|}{\sigma^2} < \tau_2. \tag{A20}$$

It is now straightforward to see that for σ large enough, we have $|s-y| > \frac{|s|}{2}$, and thus the following holds:

$$\begin{aligned} \exp\left(-\frac{(s-y)^2}{2\sigma^2}\right) &\leq \exp\left(-\frac{s^2}{4\sigma^2}\right) \\ &\leq \exp\left(-\frac{\sigma^2\tau_1}{4}\right) < \frac{1}{\sigma^3}. \end{aligned}$$

Hence, for large σ:

$$\begin{aligned} \left|\frac{\partial s}{\partial x}\frac{s-y}{\sigma^3}\exp\left(-\frac{(s-y)^2}{2\sigma^2}\right)\right| &= \left|\frac{\partial L/\partial x}{\partial L/\partial s}\right|\frac{|s-y|}{\sigma^3}\exp\left(-\frac{(s-y)^2}{2\sigma^2}\right) \\ &\leq \frac{2(1-\mu)\frac{1}{\sigma^2}|s-x||s-y|}{\frac{1-\mu}{21\sigma^3}a^2e^{-a^2}\sigma^3}\exp\left(-\frac{(s-y)^2}{2\sigma^2}\right) \\ &\leq \frac{42}{a^2e^{-a^2}\sigma^2}\frac{s^2}{4}\frac{1}{\sigma^3} \\ &\leq \frac{42}{4a^2e^{-a^2}\sigma^2}(\tau_2)^2\sigma^4\frac{1}{\sigma^3} < \frac{\tau}{\sigma} \end{aligned}$$

for some constant τ. Remember that this has been proved for $\left(h,c^l,c^r\right) \in H_\rho$.

Now consider

$$Z = \begin{pmatrix} Z_h \\ Z_l \\ Z_r \end{pmatrix} = \begin{pmatrix} h(s_l, s_r) \\ c^l(s_l, s_r) \\ c^r(s_l, s_r) \end{pmatrix}$$

defined as the functions introduced in (A11) – (A13). We will prove that there exists $\rho \in \left(0, \frac{1}{3}\right)$ such that for σ large enough, mapping A given by

$$A(h, c^l, c^r) = Z\left(h\left(s_l(h, c^l, c^r), s_r(h, c^l, c^r)\right), c^l\left(s_l(h, c^l, c^r), s_r(h, c^l, c^r)\right), c^r\left(s_l(h, c^l, c^r), s_r(h, c^l, c^r)\right)\right)$$

maps H into H_ρ and is a contraction on H_ρ. First, clearly H_ρ is mapped into H_ρ. Consider next the Jacobian of mapping A. It consists of derivatives of the kind $\frac{\partial Z_x\left(x\left(s_l(h,c^l,c^r), s_r(h,c^l,c^r)\right)\right)}{\partial y}$ for $x, y \in \left\{h, c^l, c^r\right\}$. Consider, for example, $x = y = c^l$. The function Z_l is obtained from (A12). Denote

$$\begin{aligned} R_l\left(c^l, Z_l\right) &= -2\alpha Z_l - 2\beta(Z_l + b) \\ &- \begin{pmatrix} \left(W - \frac{\alpha\beta}{\alpha+\beta}b^2 - K\right) + \mu\beta b^2 \\ -\mu^l\left(\frac{\alpha\beta}{\alpha+\beta}b^2 + \left(\chi + \frac{\alpha}{\alpha+\beta}\right)\frac{\beta^2}{\alpha+\beta}b^2 + (1-\chi)K\right) \\ -\mu^r\frac{\beta(\alpha+4\beta)}{\alpha+\beta}b^2 \end{pmatrix}\left(f\left(s_r(h,l,r) - Z_l\right) - f\left(s_l(h,l,r) - Z_l\right)\right). \end{aligned}$$

We have

$$\frac{\partial R_l}{\partial Z_l} = -2\alpha + H\left(\frac{s_r - Z_l}{\sqrt{2\pi}\sigma^3}\exp\left(-\frac{(s_h - Z_l)^2}{2\sigma^2}\right) - \frac{s_l - Z_l}{\sqrt{2\pi}\sigma^3}\exp\left(-\frac{(s_l - Z_l)^2}{2\sigma^2}\right)\right)$$

for some constant H, so for large σ, $\left|\frac{\partial R_l}{\partial Z_l}\right| > \alpha$. Likewise,

$$\frac{\partial R_l}{\partial c^l} = -H\left(\frac{\partial s_r}{\partial c^l}\frac{s_r - Z_l}{\sqrt{2\pi}\sigma^3}\exp\left(-\frac{(s_r - Z_l)^2}{2\sigma^2}\right) - \frac{\partial s_l}{\partial c^l}\frac{s_l - Z_l}{\sqrt{2\pi}\sigma^3}\exp\left(-\frac{(s_l - Z_l)^2}{2\sigma^2}\right)\right);$$

if σ is large enough, then $\left|\frac{\partial R_l}{\partial c^l}\right| < \frac{\alpha}{2}$. This already implies that $\frac{\partial Z_l}{\partial c^l} < \frac{1}{2}$; the same may be proved in a similar way for the other derivatives. This implies that mapping A is a contraction on H_ρ for σ large enough.

We have thus proved that for σ large enough, there exists a unique equilibrium $(h, c^l, c^r) \in H_\rho$. However, for σ large enough, A maps any element of H into H_ρ, so for large σ, there may be no other fixed points of mapping A, and therefore no other monotonic equilibria. It remains to prove that there are no non-monotonic equilibria. However, it is quite easy to see that for σ high enough, politicians' best responses will lie arbitrarily close to $0, -\frac{\beta}{\alpha+\beta}b, \frac{\beta}{\alpha+\beta}b$ for honest and the two corruptible types, respectively, so there will be no non-monotonic equilibria. This completes the proof. ■

Proof of Proposition 10. **Part 1.** If $\mu^l = \mu^c$, then mapping A maps symmetric triples $(-x, 0, x)$ to similar triples. As any such sequence converges to the equilibrium because A is a contraction for σ large enough, this property holds in the equilibrium as well. This also implies $s_l + s_r = 0$. Now, inequalities $c^l > -\frac{\beta}{\alpha+\beta}b$ and $c^r < \frac{\beta}{\alpha+\beta}b$ follow from (A12) and (A13).

Part 2. The equilibrium values of h, c^l, c^r are given by the equation

$$\begin{pmatrix} h \\ c^l \\ c^r \end{pmatrix} - \begin{pmatrix} Z_h\left(s_l(h,c^l,c^r), s_r(h,c^l,c^r)\right) \\ Z_l\left(s_l(h,c^l,c^r), s_r(h,c^l,c^r)\right) \\ Z_r\left(s_l(h,c^l,c^r), s_r(h,c^l,c^r)\right) \end{pmatrix} = 0.$$

Now suppose that W increases. To differentiate the implicit function, notice first that if σ is sufficiently large, then the derivatives of Z with respect to any of h, c^l, c^r are arbitrarily close to 0, and thus the matrix of derivatives of the left-hand side with respect to h, c^l, c^r is close to unit matrix. To determine the signs, it therefore suffices to differentiate Z_h, Z_l, Z_r with respect to W. Since $s_l(h, c^l, c^r)$ and $s_r(h, c^l, c^r)$ do not depend on W explicitly, we only need to look at the explicit appearances of W on the left-hand sides of (A11)–(A13). These depend on the signs of $f\left(s_r(h,c^l,c^r) - x\right) - f\left(s_l(h,c^l,c^r) - x\right)$ for $x \in \left\{h, c^l, c^r\right\}$.

If σ is large enough, then the derivatives of $f(s-x)$ for $s \in \{s_l, s_h\}$ and $x \in \left\{c^l, h, c^r\right\}$ with respect to μ^l and μ^c are negligible. Now, both s_l and s_h are increasing in δ, as follows from (A14). More precisely, we need to write the equations for the equilibrium values of s_l and s_h, notice that the $Z_x(s_l, s_h)$ has an arbitrarily small derivative with respect to δ, and therefore only the direct inclusion of δ in (A14) through μ^l and μ^c should matter. This, together with $s_l < h < s_h$, implies that h decreases in δ. This implies that for all δ, $f(s_h - c) - f(s_l - c) > 0$, and therefore c is increasing in δ. As for c^l, $f\left(s_h - c^l\right) - f\left(s_l - c^l\right) < 0$ in the neighborhood of $\delta = 0$, and therefore c^l is increasing in that neighborhood. However, as δ increases enough so that μ^l is sufficiently close to 0, then s_l will tend to $-\infty$ whereas s_h will remain finite. This means that for such δ, $f\left(s_h - c^l\right) - f\left(s_l - c^l\right) > 0$, and c^l will increase in δ.

Part 3. The proof is similar to the proof of Part 2 and is omitted. ■

Proof of Proposition 12. The second-period problems are identical to the main case. In the first period, the maximization problems of honest politicians and corrupt politicians with the lobby, respectively, are now given by

$$\max_{x\in\mathbb{R}} -\alpha x^2 + \delta\left(W\pi(x) - (1-\mu)\,\alpha\left(\frac{\beta}{\alpha+\beta}b\right)^2(1-\pi(x))\right),$$

$$\max_{x\in\mathbb{R}}\max_{x\in\mathbb{R}}\left\{\begin{array}{c} -\alpha x^2 - \beta(x-b)^2 + \delta\left(W - \frac{\alpha\beta}{\alpha+\beta}b^2 - K\right)\pi(x) \\ -\delta(1-\mu)\left(\frac{\alpha\beta}{\alpha+\beta}b^2 + \left(\chi + \frac{\alpha}{\alpha+\beta}\right)\frac{\beta^2}{\alpha+\beta}b^2 + (1-\chi)K\right)(1-\pi(x)) - \delta\mu\beta b^2(1-\pi(x)) \end{array}\right\}.$$

Proceeding as before, the equilibrium is characterized by the two first-order conditions:

$$\begin{aligned} -2\alpha h - \delta H f\left(\frac{h-c}{2}\right) &= 0, \\ -2\alpha c - 2\beta(c-b) - \delta(H+R)f\left(\frac{h-c}{2}\right) &= 0, \end{aligned}$$

Therefore, $p = |h|$ and $q = \left|c - \frac{\beta}{\alpha+\beta}b\right|$ satisfy equations analogous to (17) and (18). As in the proofs of Proposition 4, we get that p and q increase in δ: $\frac{\partial p}{\partial \delta} > 0$ and $\frac{\partial q}{\partial \delta} > 0$. ■