How Q and Cash Flow Affect Investment without Frictions: An Analytic Explanation[1]

Andrew B. Abel
The Wharton School of the University of Pennsylvania
National Bureau of Economic Research

Janice C. Eberly
Kellogg School of Management, Northwestern University
National Bureau of Economic Research

August 2008, current version October 2010

[1]Earlier versions of this paper circulated under the title "Q Theory without Adjustment Costs & Cash Flow Effects without Financing Constraints." We thank Joao Gomes, Philipp Illeditsch, Richard Kihlstrom, Michael Michaux, Stavros Panageas, Julien Prat, Kjetil Storesletten, Ken West, and three referees for helpful comments, as well as seminar participants at Columbia University, the Federal Reserve Bank of Kansas City, New York University, Stanford University, University of California at San Diego, University of Chicago, University of Wisconsin, University of Southern California, the Penn Macro Lunch Group, the Workshop on Firms' Dynamic Adjustment in Bergamo, Italy, and the Society for Economic Dynamics meeting in Florence, Italy.

Abstract

We derive a closed-form solution for Tobin's Q in a stochastic dynamic framework. We show analytically that investment is positively related to Tobin's Q and cash flow, even in the absence of adjustment costs or financing frictions. Both Q and investment move in the same direction as expected revenue growth, so changes in expected revenue growth induce Q and investment to comove positively. Similarly, shocks to current cash flow, arising from shocks to the user cost of capital in our model, cause investment and cash flow per unit of capital to comove positively. Furthermore, we show that this alternative mechanism for the relationship among investment, Q, and cash flow delivers larger cash flow effects for smaller and faster-growing firms, as observed in the data. Moreover, the empirically small correlation between investment and Tobin's Q does not imply implausibly large adjustment costs in our model (since there are no adjustment costs). Calibrating the model generates values of Q similar to those in the data; investment is more sensitive to cash flow than it is to Q, and both responses are of empirically plausible magnitudes.

Regressions of investment on Tobin's Q and cash flow typically yield a small positive coefficient on Q and a significant positive coefficient on cash flow. The small coefficient on Q is interpreted as evidence of strongly convex adjustment costs, and the significant cash flow coefficient is interpreted as evidence of financing constraints facing firms. In this paper, we develop a simple neoclassical model without adjustment costs and without financing constraints. We show analytically that in this model the investment-capital ratio is positively related to Tobin's Q and is positively related to the ratio of cash flow to the capital stock, for a given value of Q_t, even though there are no financial constraints in the model. Also, the coefficient on Q can be small, even though there are no adjustment costs in the model. These effects arise in our model because Tobin's Q reflects expectations about future revenue growth, while cash flow reflects the user cost of capital, specifically the depreciation rate. Since both of these underlying variables (revenue growth and the depreciation rate) drive investment, investment will be correlated with both Q and cash flow.

Tobin's Q is the ratio of the market value of the firm to the replacement cost of the firm's capital stock. Research using Tobin's Q has proceeded in two directions that emphasize different sources of the rents that allow Q to exceed one. In the investment literature that uses Tobin's Q, capital investment is subject to a convex cost of adjustment, and rents accrue to the adjustment technology. In the industrial organization literature that uses Tobin's Q, adjustment costs are generally absent, and rents accrue primarily to market power.

Tobin (1969)[1] introduced Q as an empirical implementation of Keynes's (1936) notion that capital investment becomes more attractive as the value of capital increases relative to the cost of acquiring the capital. Neither Keynes nor Tobin provided a theoretic foundation for the Q theory of investment. Lucas and Prescott (1971) rigorously analyzed investment under uncertainty in the presence of convex costs of

[1] Brainard and Tobin (1968) introduced the idea that a firm's investment should be positively related to the ratio of its market value to the replacement value of its capital stock, though they did not use the letter Q to denote this ratio.

adjustment, and observed that the market value of capital can be an important element of the capital investment decision, though they did not explicitly make the link to Tobin's Q. The link between convex costs of adjustment and the Q theory of investment was made explicitly by Mussa (1977) in a deterministic framework and by Abel (1983) in a stochastic framework, but these papers focused on *marginal* Q – the ratio of the value of an additional unit of capital to its acquisition cost – rather than the concept of *average* Q introduced by Tobin. Hayashi (1982) bridged the gap between marginal Q and average Q by providing conditions under which marginal Q and average Q are equal. Specifically, marginal Q and average Q are equal for a competitive firm with a constant-returns-to-scale production function, provided that the adjustment cost function is linearly homogeneous in the rate of investment and the level of the capital stock. Much of the investment literature has adopted the assumption of perfect competition and constant returns to scale to take advantage of the equality of average Q and marginal Q in this case. Tobin's Q can exceed one in this case because rents accrue to the adjustment cost technology.

An important implication of Q-theoretic investment models based on convex adjustment costs is that (marginal) Q is a sufficient statistic for the rate of investment. However, many empirical studies have found that cash flow has a significant effect on investment, even if Q is included as an explanatory variable. This finding has been interpreted by Fazzari, Hubbard, and Petersen (1988) and others as evidence of financing constraints facing firms.

The industrial organization literature has used Tobin's Q as a measure of rents, primarily monopoly rents, earned by firms. Stigler (1960) showed that the market-to-book ratio, which is essentially Tobin's Q, has a strong rank correlation with measures of industry concentration. Lindenberg and Ross (1981) and Salinger (1984) formally derived the value of Q for a firm with monopoly power. Since these papers do not examine investment, and since they focus on monopoly power as the source of rents that allow Q to exceed one, they do not include convex adjustment costs. Moreover, since these derivations are confined to long-run equilibrium under certainty,[2] they are not able to account for any time-series variation in Q and thus cannot serve as the basis for analyzing Q as a regressor in an investment equation.

Our analysis draws on these two disparate literatures on Tobin's Q. We model the firm's revenue as a concave function of its capital stock. The concavity, which is the source of rents, can result from decreasing returns to scale or from monopoly rents, as emphasized in the industrial organization literature discussed above. However, unlike that literature, we analyze a firm in a dynamic stochastic environment, and we derive optimal investment behavior. In addition, we derive a closed-form expression for Tobin's Q and relate it to optimal investment.

The model we develop here also departs from standard models in the investment literature in several ways. First, the model does not incorporate convex adjustment costs, so marginal Q is identically equal to one. However, average Q, which is empirically observable, exceeds one, varies stochastically, and is positively related to the investment-capital ratio. In our model, there are no financing constraints – capital markets are perfect – yet investment is positively related to cash flow in addition to Q, consistent with findings in the empirical literature. Investment and cash flow (scaled by the capital stock) positively comove in our model because both react in the same direction to shocks to the user cost of capital. Furthermore, in our model this "cash flow effect" on investment is larger for smaller and faster-growing firms, as has been found empirically; it is usually argued that this differential cash flow effect across groups of firms strengthens the evidence of financing constraints, but in our model these differential effects exist in the absence of financial constraints or capital market imperfections. Finally, calibration of the model illustrates that, despite its simplicity, the values of Q and the partial derivatives of the investment-capital ratio with respect to Q and cash flow per unit of capital are close to values in the empirical literature.

[2] Footnote 9 in Lindenberg and Ross (1981) outlines an extension to the stochastic case, but does not derive a closed-form solution for Tobin's Q that has interesting stochastic variation.

Our finding of a positive cash flow effect on investment, for a given value of Q, despite perfect capital markets calls into question the interpretation of cash flow effects as evidence of financing constraints. Our analytic findings on this issue are consistent with, and complementary to, numerical findings in Gomes (2001), Cooper and Ejarque (2003), and Alti (2003). Gomes (2001) shows that, in his quantitative model, optimal investment is sensitive to both Tobin's Q and cash flow, whether or not a cost of external finance is present. Similarly, Cooper and Ejarque (2003) numerically solve a model with quadratic adjustment costs and a concave revenue function (consistent with monopoly power as in our analysis), and also find that investment is sensitive to both Tobin's Q and cash flow in the absence of financing constraints. In addition, they find that adding a fixed cost of access to capital markets does not improve the fit of the model and they conclude the cash flow sensitivity of investment reflects market power rather than financial constraints. Alti (2003) develops a continuous-time model of a firm facing quadratic adjustment costs and with a revenue function that is concave in the capital stock (again consistent with monopoly power) and subject to a multiplicative productivity shock. The logarithm of the shock reverts to an unknown mean and firms update their estimates of the mean by observing realizations of cash flow over time. For young firms, the estimate of the mean is noisy and Tobin's Q provides a noisy measure of long-run prospects that are important for investment.[3] For these firms especially, observations on cash flow provide important evidence that can change the estimate of the mean level of productivity and hence affect investment. Gomes (2001), Cooper and Ejarque (2003), and Alti (2003) numerically compute Tobin's Q because they cannot analytically solve for the value of the firm. A contribution of our paper is to provide a closed-form solution for the value of the firm and hence for Tobin's Q. Importantly, the closed-form solution allows a straightforward analytic description of the statistical relationship among investment, Tobin's Q, and cash flow.

[3] Similarly, Erickson and Whited (2000) find that when controlling for measurement error in a flexible way, the evidence for a cash flow effect on investment disappears in their sample.

Section 1 presents the stochastic environment in which the firm operates and derives the optimal capital stock at each point of time. The resulting optimal rate of investment is derived in Section 2. Section 3 derives the value of the firm and Tobin's Q. The relationship among investment, Tobin's Q, and cash flow is analyzed in Section 4, which includes, in Section 4.1, an analysis of the effects of firm size and growth on this relationship. Section 4.2 explores the impact of monopoly power on the value of Q and on the coefficients on Q and cash flow in investment regressions. In Section 5, we calibrate the model to examine the quantitative implications of the model for Tobin's Q and the effects on investment of Q and cash flow. Section 6 presents concluding remarks, and three appendices present derivations that would be distractions in the text.

1 The Decision Problem of the Firm

Consider a firm that uses capital, K_t, and labor, N_t, to produce nonstorable output, Y_t, at time t according to the production function

$$Y_t = A_t \left(K_t^{\gamma} N_t^{1-\gamma} \right)^s , \tag{1}$$

where A_t is productivity at time t, $0 < s \leq 1$ is the degree of returns to scale ($s = 1$ for constant returns to scale) and $0 < \gamma < 1$. The inverse demand function for the firm's output is

$$P_t = \left(\frac{Y_t}{h_t} \right)^{-\frac{1}{\varepsilon}} , \tag{2}$$

where $h_t > 0$ indicates the location of the demand curve and $\varepsilon > 1$ is the price elasticity of demand. The exogenous variables A_t and h_t evolve according to geometric Brownian motions. We show in Appendix A that the maximized value of revenue net of wages is

$$R_t = Z_t^{1-\alpha} K_t^{\alpha}, \tag{3}$$

where Z_t is a geometric average of A_t and h_t, and

$$\alpha \equiv \frac{\gamma s \left(1 - \frac{1}{\varepsilon}\right)}{1 - (1-\gamma) s \left(1 - \frac{1}{\varepsilon}\right)} > 0.$$

For a competitive firm with constant returns to scale ($\varepsilon = \infty$ and $s = 1$), $\alpha = 1$. However, if the firm has some monopoly power ($\varepsilon < \infty$) or if it faces decreasing returns to scale ($s < 1$), then $\alpha < 1$ and hence net revenue is a strictly concave function of the capital stock. This concavity implies that the firm will earn positive rents. Henceforth, we confine attention to the case with $\alpha < 1$.

Since Z_t is a geometric average of geometric Brownian motions, it also follows a geometric Brownian motion. We assume that the drift in this geometric Brownian motion is time-varying so

$$\frac{dZ_t}{Z_t} = \mu_t dt + \sigma dz. \tag{4}$$

If the growth rate μ_t were constant over time, the future growth prospects for the firm would always look the same, and, as we will show (see eqn. 29), there would be no time-series variation in the expected present value of the firm's future operating profits relative to current operating profits.

To introduce some interesting, yet tractable, variation in the firm's growth prospects, we assume that the process for Z_t follows a regime-switching process[4] in which a regime is defined by a constant value of the drift μ_t. A regime remains in force, that is, the drift remains constant, for a random length of time. A new regime, which is characterized by a new value of the drift, arrives with constant probability per unit of time, $\lambda \geq 0$. The new value of the drift is drawn from an unchanging distribution $F(\widetilde{\mu})$ with support in the interval $[\mu_L, \mu_H]$. The values of the drift are i.i.d. across regimes and are independent of the realizations of the other stochastic processes in the model. The value of the firm is finite and is increasing in contemporaneous operating profit for a given value of the capital stock if

$$E\left\{\frac{1}{r + \lambda - \widetilde{\mu}}\right\} > 0 \tag{5}$$

[4]Eberly, Rebelo, and Vincent (2008) find an important role for regime-switching in empirical investment equations. Specifically, they report that a "single-regime model ... cannot explain the role of lagged investment in investment regressions" but "the performance of the model can be greatly improved by a regime-switching component" for the exogenous stochastic process. (p. 11)

and

$$E\left\{\frac{\lambda}{r+\lambda-\widetilde{\mu}}\right\}<1, \tag{6}$$

where r is the discount rate of the firm.

Henceforth, we assume that the following condition holds.

Condition 1 $r > \mu_H$.

Condition 1, along with the fact that $\lambda \geq 0$, implies that equations (5) and (6) both hold.

The firm can purchase or sell capital instantaneously and frictionlessly, without any costs of adjustment, at a constant price that we normalize to one. Because there are no costs of adjustment, we can use Jorgenson's (1963) insight that the optimal path of capital accumulation can be obtained by solving a sequence of static decision problems using the concept of the user cost of capital. With the price of capital constant and equal to one, the user cost of capital, υ_t, is

$$\upsilon_t \equiv r + \delta_t, \tag{7}$$

where δ_t is the depreciation rate of capital. We will discuss the stochastic properties of δ_t later in this section. For the specific goal of studying the relationship between investment and Tobin's Q, we could simply assume that δ_t is constant. Variation in δ_t will be important when we examine variation in cash flow and its effect on investment.

At time t the firm chooses the capital stock K_t to maximize operating profit, π_t, which equals net revenue less the user cost of capital

$$\pi_t \equiv R_t - \upsilon_t K_t = Z_t^{1-\alpha} K_t^{\alpha} - \upsilon_t K_t. \tag{8}$$

Differentiating equation (8) with respect to K_t and setting the derivative equal to zero yields the optimal value of the capital stock

$$K_t = Z_t \left(\upsilon_t/\alpha\right)^{\frac{-1}{1-\alpha}}. \tag{9}$$

Substituting the optimal capital stock from equation (9) into equations (8) and (3), respectively, yields the optimal level of operating profit

$$\pi_t = (1-\alpha) Z_t (\upsilon_t/\alpha)^{\frac{-\alpha}{1-\alpha}} \tag{10}$$

and the optimal level of revenue (net of labor cost)

$$R_t = \frac{1}{1-\alpha}\pi_t. \tag{11}$$

Empirical investment equations often use a measure of cash flow, normalized by the capital stock, as an explanator of investment. Since R_t is defined as revenue net of labor costs, it is cash flow before investment expenditure. Let $c_t \equiv R_t/K_t$ be the cash flow before investment, normalized by the capital stock, and note, for later use, that

$$c_t = \frac{1}{1-\alpha}\frac{\pi_t}{K_t} = \frac{\upsilon_t}{\alpha}, \tag{12}$$

where the first equality follows from equation (11) and the second equality follows from equations (9) and (10).

Equations (10) and (11) together imply that an increase in the user cost of capital, υ_t, reduces cash flow, R_t. The concavity of the revenue function implies that although cash flow falls in response to an increase in the user cost of capital, it does not fall by as much as the optimal capital stock falls. Therefore, an increase in the user cost of capital will *increase* cash flow *per unit of capital*. Indeed, equation (12) indicates that cash flow per unit of capital is proportional to the user cost of capital.

It will be convenient to define a variable M_t that summarizes the effect of the user cost on the optimal capital stock and operating profit, and to specify the stochastic properties of M_t directly. In particular, define

$$M_t \equiv \left(\frac{\upsilon_t}{\alpha}\right)^{\frac{-\alpha}{1-\alpha}} = \left(\frac{r+\delta_t}{\alpha}\right)^{\frac{-\alpha}{1-\alpha}}. \tag{13}$$

We assume that M_t is a martingale and is independent of the parameters and realizations of the process for Z_t. The assumption that M_t is a martingale implies that δ_t is *not* a martingale. In particular, it implies that the depreciation rate is expected to grow over time.[5] For the sake of concreteness, we assume that M_t is a trendless geometric Brownian motion. Specifically, we assume that[6]

$$dM_t = \sigma_M M_t dz_M, \tag{14}$$

where $\sigma_M > 0$.

Use the definition of M_t in equation (13) to rewrite the expressions for the optimal capital stock and the optimal operating profit in equations (9) and (10), respectively, as

$$K_t = Z_t M_t^{\frac{1}{\alpha}} \tag{15}$$

and

$$\pi_t = (1-\alpha) Z_t M_t. \tag{16}$$

Equations (15), (16) and (11) indicate that the capital stock, operating profit, and revenue are all increasing functions of the contemporaneous values of Z_t and M_t and are independent of μ_t, conditional on Z_t and M_t. Thus, regardless of whether firm size is measured by the size of the capital stock, operating profit, or revenue, firm size is increasing in Z_t and M_t; however, because the firm can instantaneously adjust the capital stock without any adjustment costs, the optimal choice of K_t (and hence π_t) is a static decision and thus is independent of the expected growth rate μ_t. In addition,

[5] Note that for $\tau > 0$, $1 = E_t\left\{\frac{M_{t+\tau}}{M_t}\right\} = E_t\left\{\left(\frac{\upsilon_{t+\tau}}{\upsilon_t}\right)^{-\frac{\alpha}{1-\alpha}}\right\} > \left[E_t\left\{\frac{\upsilon_{t+\tau}}{\upsilon_t}\right\}\right]^{-\frac{\alpha}{1-\alpha}}$, where the first equality follows from the assumption that M_t is a martingale, the second equality follows from the definition of M_t, and the third (in)equality follows from Jensen's inequality. Therefore, $E_t\left\{\frac{\upsilon_{t+\tau}}{\upsilon_t}\right\} > 1$, which implies that $E_t\{\delta_{t+\tau}\} > \delta_t$ so that the depreciation rate is expected to grow over time. This implication is consistent with depreciation rates computed for U.S. private nonresidential fixed assets using the Bureau of Economic Analysis Fixed Asset Tables 4.1 and 4.4 and taking the ratio of annual depreciation to the beginning-of-year net stock of capital. The average depreciation rates for 1950-59, 1970-79, and 1997-2006 are, respectively, 6.3%, 7.1%, and 8.2%. For equipment and software, the corresponding figures are 13.7%, 14.2%, and 16.8%, and for structures the corresponding figures are 2.8%, 3.0% and 3.0%.

[6] Although the process in equation (14) implies that if the initial value of M_t is positive, it will remain positive, it does not ensure that δ_t is positive (it implies that $\delta_t > -r$).

equations (15) and (16) imply that $\frac{\pi_t}{K_t} = (1-\alpha) M_t^{-\frac{1-\alpha}{\alpha}}$ so that $\frac{\pi_t}{K_t}$ is a decreasing function of M_t. That is, an increase in δ_t, which reduces M_t, causes the optimal value of K_t to fall, and, because of the concavity of the revenue function (equation 3), the fall in K_t increases profit per unit of capital, $\frac{\pi_t}{K_t}$. Hence, for given Z_t, $\frac{\pi_t}{K_t}$ is larger for small firms than for large firms and, therefore, small firms on average have higher values of π_t/K_t than do large firms.[7]

In Section 2 we examine the firm's investment by analyzing the evolution of the optimal capital stock in equation (15). Then in Section 3 we use the expression for the optimal operating profit in equation (16) to compute the value of the firm.

2 Investment

Gross investment is the sum of net investment, dK_t, and depreciation $\delta_t K_t dt$. To calculate net investment divided by the capital stock, apply Ito's Lemma to the expression for the optimal capital stock in equation (15) and use the processes for Z_t and M_t in equations (4) and (14), respectively, to obtain[8]

$$\frac{dK_t}{K_t} = \mu_t dt + \frac{1}{2}\frac{1-\alpha}{\alpha^2}\sigma_M^2 dt + \sigma dz + \frac{1}{\alpha}\sigma_M dz_M. \tag{17}$$

[7]Let S be the size of the firm, and note from equations (15), (16) and (11) that if S is measured by the capital stock, operating profits, or revenue, then $S = Zg(M)$, where $g'(M) > 0$, and we have suppressed time subscripts. At a given point of time, the average of the cross-sectional distribution of π/K for a given value of S is $E\{\pi/K\} = (1-\alpha) E\left\{M^{-(1-\alpha)/\alpha}\right\} = (1-\alpha)E\{[g^{-1}(S/Z)]^{-(1-\alpha)/\alpha}\}$, which is strictly decreasing in S because $g^{-1}()$ is strictly increasing.

[8]The empirical literature generally focusses on gross investment rather than net investment, perhaps because net investment is more difficult to measure. Nevertheless, we point out some implications of the model for net investment. First, observe from equation (17) that net investment relative to capital stock, $\frac{dK_t}{K_t}$, is independent of cash flow per unit of capital, c_t. Second, $\frac{dK_t}{K_t}$ is positively correlated with dz_M, which means that over the short interval of time immediately following t, net investment is positively correlated with innovations to M_t and hence net investment is negatively correlated with innovations to the depreciation rate δ_t. However, because cash flow per unit of capital, c_t, is proportional to $r + \delta_t$, net investment is also negatively correlated with innovations to cash flow per unit of capital.

Adding the depreciation rate of capital, δ_t, to net investment per unit of capital in equation (17) yields gross investment per unit of capital

$$\frac{dI_t}{K_t} \equiv \frac{dK_t}{K_t} + \delta_t dt = (\mu_t + \delta_t)\, dt + \frac{1}{2}\frac{1-\alpha}{\alpha^2}\sigma_M^2 dt + \sigma dz + \frac{1}{\alpha}\sigma_M dz_M. \tag{18}$$

Investment is a linear function of the expected growth rate, μ_t, the depreciation rate, δ_t, and a mean-zero disturbance, $\sigma dz + \frac{1}{\alpha}\sigma_M dz_M$, that is independent of μ_t and δ_t. If μ_t and δ_t, as well as the investment-capital ratio, were observable, then we could use a linear regression to estimate the effects on the investment-capital ratio of μ_t and δ_t. In a sufficiently large sample, the coefficients on μ_t and δ_t would both equal one. However, μ_t is not observable and δ_t may not be well measured. We will show in later sections that movements in μ_t are reflected by movements in Tobin's Q, and movements in δ_t are reflected by movements in the firm's cash flow per unit of capital and in Tobin's Q. Thus, Tobin's Q and cash flow per unit of capital can help to explain investment statistically.

3 The Value of the Firm and Tobin's Q

The value of the firm, V_t, is the expected present value of current and future cash flows, *net* of the cost of current and future investment, and satisfies

$$rV_t dt = E_t \left\{ R_t dt - (dK_t + \delta_t K_t dt) + dV_t \right\}. \tag{19}$$

The left hand side of equation (19) is the required return on the firm over the interval of time from t to $t + dt$, and the right hand side of equation (19) is the expected return over this interval of time. The expected return on the right hand side consists of two components: (1) the expected net cash flow, which equals revenue, $R_t dt$, less gross investment, $dK_t + \delta_t K_t dt$, and (2) the expected capital gain or loss reflecting the change in the value of the firm.

The value of the firm depends on three state variables,[9] Z_t, M_t, and μ_t, so the value of the firm in equation (19) can be written as $V_t = V(Z_t, M_t, \mu_t)$. To express

[9] The capital stock is instantaneouly and costlessly adjustable, so it is not a state variable.

equation (19) in terms of the state variables, first use equations (11) and (16) to obtain

$$R_t = Z_t M_t, \tag{20}$$

and use equations (15) and (18) to obtain

$$E_t \{dK_t + \delta_t K_t dt\} = \left(\mu_t + \delta_t + \frac{1}{2}\frac{1-\alpha}{\alpha^2}\sigma_M^2\right) Z_t M_t^{\frac{1}{\alpha}} dt. \tag{21}$$

Then, to calculate $dV(Z_t, M_t, \mu_t)$, use Ito's Lemma, equations (4) and (14) for the evolution of Z_t and M_t, respectively, and the stochastic process for μ_t to obtain

$$\begin{aligned} E_t \{dV(Z_t, M_t, \mu_t)\} &= V_Z \mu_t Z_t dt + \frac{1}{2} V_{ZZ} \sigma^2 Z_t^2 dt + \frac{1}{2} V_{MM} \sigma_M^2 M_t^2 dt \\ &\quad + \lambda \left[E_t \{V(Z_t, M_t, \widetilde{\mu}) - V(Z_t, M_t, \mu_t)\}\right] dt. \end{aligned} \tag{22}$$

Equations (20), (21), and (22) allow us to rewrite equation (19) as

$$\begin{aligned} rV(Z_t, M_t, \mu_t) &= Z_t M_t - \left(\mu_t + \delta_t + \frac{1}{2}\frac{1-\alpha}{\alpha^2}\sigma_M^2\right) Z_t M_t^{\frac{1}{\alpha}} + V_Z \mu_t Z_t \\ &\quad + \frac{1}{2} V_{ZZ} \sigma^2 Z_t^2 + \frac{1}{2} V_{MM} \sigma_M^2 M_t^2 + \lambda \left[E_t \{V(Z_t, M_t, \widetilde{\mu}) - V(Z_t, M_t, \mu_t)\}\right]. \end{aligned} \tag{23}$$

In Appendix C, we show that a solution to equation (23) is

$$V(Z_t, M_t, \mu_t) = Z_t M_t^{\frac{1}{\alpha}} + \frac{\omega(1-\alpha) Z_t M_t}{r + \lambda - \mu_t} \tag{24}$$

where[10]

$$\omega \equiv \left[E\left\{\frac{r - \widetilde{\mu}}{r + \lambda - \widetilde{\mu}}\right\}\right]^{-1} \geq 1, \tag{25}$$

with strict inequality if $\lambda > 0$.

Here we present a heuristic derivation of the value function that has a simple economic interpretation. The value of the firm can be derived by viewing the firm as composed of two divisions – a capital-owning division that owns K_t at time t and a capital-operating division that produces and sells output at time t. Because capital

[10]Since $\omega^{-1} = E\left\{\frac{r-\widetilde{\mu}}{r+\lambda-\widetilde{\mu}}\right\} = 1 - E\left\{\frac{\lambda}{r+\lambda-\widetilde{\mu}}\right\}$, equation (5) implies that $\omega^{-1} \leq 1$, with strict inequality if $\lambda > 0$, and equation (6) implies that $\omega^{-1} > 0$. Therefore, since $0 < \omega^{-1} \leq 1$, $\omega \geq 1$, with strict inequality if $\lambda > 0$.

can be instantaneously and costlessly bought or sold at a price of one at time t, the value of the capital-owning division at time t is K_t. The value of the capital-operating division, which rents capital (in an amount equal to the amount of capital owned by the capital-owning division) to produce and sell output, is the present value of current and expected future operating profits. Therefore, the value of the firm at time t is

$$V_t = K_t + E_t \left\{ \int_t^\infty \pi_{t+\tau} e^{-r\tau} d\tau \right\}. \tag{26}$$

As a step toward calculating the present value in equation (26), use equation (16), the independence of M_t and Z_t, and the fact that M_t is a martingale to obtain

$$E_t \{\pi_{t+\tau}\} = (1-\alpha) M_t E_t \{Z_{t+\tau}\}. \tag{27}$$

Substituting equation (27) into equation (26) yields

$$V_t = K_t + (1-\alpha) M_t \int_t^\infty E_t \{Z_{t+\tau}\} e^{-r\tau} d\tau. \tag{28}$$

We show in Appendix B that the value of the integral on the right hand side of equation (28) is

$$\int_t^\infty E_t \{Z_{t+\tau}\} e^{-r\tau} d\tau = \frac{\omega}{r+\lambda-\mu_t} Z_t. \tag{29}$$

Note that when the arrival rate λ is zero, so that the growth rate of Z_t remains a constant value μ forever, $\omega = 1$ and the expected present value of the stream of $Z_{t+\tau}$ in equation (29) is simply $Z_t / (r-\mu)$. More generally, when the growth rate μ_t varies over time, a high value of μ_t implies a high value of the present value in equation (29).

The value of the firm can now be obtained by substituting equation (29) into equation (28), and recalling from equation (16) that $\pi_t = (1-\alpha) Z_t M_t$, to obtain

$$V_t = K_t + \frac{\omega \pi_t}{r+\lambda-\mu_t}. \tag{30}$$

Equation (30) is equivalent to equation (24) because equation (15) states that $K_t = Z_t M_t^{\frac{1}{\alpha}}$ and equation (16) states that $\pi_t = (1-\alpha) Z_t M_t$.

3.1 Tobin's Q

Tobin's Q is the ratio of the value of the firm to the replacement cost of the firm's capital stock. Since the price of capital is identically equal to one, the replacement cost of the firm's capital stock is simply K_t. Dividing the value of the firm in equation (30) by K_t yields

$$Q_t \equiv \frac{V_t}{K_t} = 1 + \frac{\omega}{r + \lambda - \mu_t} \frac{\pi_t}{K_t} > 1. \quad (31)$$

Tobin's Q is greater than one because the firm earns rents π_t. In the absence of rents, Tobin's Q would be identically equal to one because the firm can costlessly and instantaneously purchase and sell capital at a price of one.

The presence of positive rents π_t is sufficient to make Tobin's Q greater than one in our model. However, rents alone do not imply that Tobin's Q will vary over time for a firm. If $\lambda = 0$, so that μ_t, the expected growth rate of Z_t, is constant, and if the user cost υ_t is constant, so that $\frac{\pi_t}{K_t}$ is constant (see equation 12), then equation (31) shows that Tobin's Q is constant and greater than one. Thus, decreasing returns to scale or market power, as in Alti (2003) and Cooper and Ejarque (2001), are not sufficient to generate our results.[11] In fact, in this case, $\omega = 1$ (see equation 25) so equation (31) implies that $Q_t = 1 + \frac{1}{r-\mu} \frac{\pi_t}{K_t}$, which is equivalent to the expression derived by Salinger (1984) in a non-stochastic steady state.[12] Of course, in his special case, with constant values of Q_t and c_t, one cannot run a regression of the investment-capital ratio on Q_t and c_t.

To generate variation in Q_t and in cash flow per unit of capital, we have modeled both the growth rate μ_t and the user cost υ_t as stochastic, and equation (31) shows that Tobin's Q is an increasing function of the contemporaneous growth rate μ_t and

[11] Alti (2003) and Cooper and Ejarque (2001) generate time variation in rents with adjustment costs, whereas we allow time variation in growth prospects and eliminate adjustment costs.

[12] Salinger's equation (3) is, in our notation, $Q_t = 1 + \frac{R_t}{K_t} \frac{1}{\varepsilon} \frac{1}{r-\mu}$. Salinger derives his equation for a firm in long-run equilibrium with constant returns to scale, which has the effect of setting $\gamma = s = 1$ in the expression for α, so $\alpha = 1 - \frac{1}{\varepsilon}$. Substituting $1 - \alpha$ for $\frac{1}{\varepsilon}$, Salinger's equation (3) can be rewritten as $Q_t = 1 + \frac{R_t}{K_t} (1 - \alpha) \frac{1}{r-\mu} = 1 + \frac{1}{r-\mu} \frac{\pi_t}{K_t}$, where the second equality follows from equation (11).

an increasing function of the contemporaneous value of operating profit per unit of capital (which is proportional to cash flow per unit of capital). Recall from the discussion following equations (15) and (16) that the concavity of the revenue function implies that $\frac{\pi_t}{K_t}$ is higher on average for small firms than for large firms. Therefore, the value of Q_t in equation (31) is higher for small firms than for large firms.

The effect of the growth rate μ_t on Q_t illustrates the distinction in the finance literature between growth stocks and value stocks. Value stocks are identified as stocks with high book-to-market ratios, that is, stocks with low values of Q_t. By contrast, growth stocks are those with high values of Q_t, which is consistent with equation (31) because a high value of the expected growth rate μ_t implies a high value of Q_t.

To run a regression of the investment-capital ratio on Q_t and c_t, both Q_t and c_t must vary over time and they must not be perfectly correlated. To analyze the relationship between Q_t and c_t, use equation (12) to substitute $(1-\alpha)\, c_t$ for $\frac{\pi_t}{K_t}$ in equation (31) to obtain

$$Q_t = 1 + (1-\alpha)\,\omega \frac{c_t}{r + \lambda - \mu_t}. \tag{32}$$

Since c_t is proportional to the user cost υ_t (from equation 12), the user cost must be stochastic in order for c_t to be stochastic. Equation (32) shows that if the user cost is the only source of stochastic variation, so that, for instance, μ_t is constant, then Q_t would be perfectly correlated with c_t, and therefore Q_t and c_t cannot both be used as regressors. Indeed, stochastic variation in Z_t with a constant expected growth rate μ_t would not break the perfect correlation of Q_t and c_t. However, if both μ_t and υ_t vary stochastically, then Q_t and c_t are not perfectly correlated and both can be used as regressors. We analyze the impacts of Q_t and c_t on the investment-capital ratio in the next section.

4 The Effects of Tobin's Q and Cash Flow on Investment

Define ι_t to be the predictable component of the investment-capital ratio at time t in equation (18), so that, ignoring the constant $\frac{1}{2}\frac{1-\alpha}{\alpha^2}\sigma_M^2$,

$$\iota_t = \mu_t + \delta_t. \tag{33}$$

However, the growth rate μ_t is not observable and the depreciation rate δ_t may not be well measured. In this section, we show that the growth rate μ_t can be written as a function of Tobin's Q and cash flow per unit of capital c_t, and that the depreciation rate δ_t is linearly related to cash flow c_t. Thus, to the extent that Q_t and c_t reflect μ_t and δ_t, these variables can help account for movements in investment.

Multiply both sides of equation (32) by $r + \lambda - \mu_t$ and rearrange to obtain an expression for the growth rate μ_t in terms of the observable values of Tobin's Q and c_t

$$\mu_t = r + \lambda - (1-\alpha)\,\omega\frac{c_t}{Q_t - 1}. \tag{34}$$

To express $\iota_t = \mu_t + \delta_t$ as a function of Q_t and c_t, add δ_t to both sides of equation (34), and use the fact from equation (12) that $\alpha c_t = v_t$ and the definition of the user cost $v_t \equiv r + \delta_t$ to obtain

$$\iota\left(Q_t, c_t\right) \equiv \left(\alpha - \frac{(1-\alpha)\,\omega}{Q_t - 1}\right) c_t + \lambda. \tag{35}$$

We will analyze the effects of Q_t and c_t on investment by analyzing the effects of these variables on $\iota\left(Q_t, c_t\right)$.

First we analyze the effect of Tobin's Q on investment for a given level of cash flow c_t. Let $\beta_Q \equiv \partial\iota\left(Q_t, c_t\right)/\partial Q_t$ denote the response of the investment-capital ratio to an increase in Q_t. Partially differentiating $\iota\left(Q_t, c_t\right)$ with respect to Q_t yields

$$\beta_Q \equiv \frac{\partial\iota\left(Q_t, c_t\right)}{\partial Q_t} = \frac{(1-\alpha)\,\omega c_t}{\left(Q_t - 1\right)^2} > 0, \tag{36}$$

so that investment is an increasing function of Q_t. The positive relationship between investment and Tobin's Q in this model has some remarkable differences from the relationship in the standard convex adjustment cost framework. The positive relationship between investment and Q_t arises in the standard framework because of the convexity of the adjustment cost function. In a regression of the investment-capital ratio on Q_t and c_t, the coefficient on Q_t is the reciprocal of the second derivative of the adjustment cost function with respect to investment.[13] This estimated coefficient, which is the analogue of β_Q in equation (36), is typically quite small, which is usually interpreted to mean that adjustment costs are very large (or, more correctly, very convex). In the model we present here, investment is positively related to Q_t, that is, $\beta_Q > 0$, *even though there are no convex costs of adjustment.* In addition, it is quite possible for β_Q to be small, as we will show in the calibration in Section 5. Yet, in this model, without convex adjustment costs, the small value of β_Q cannot indicate large (or very convex) adjustment costs, as in standard interpretations.

Another remarkable difference from standard investment models based on convex capital adjustment costs is that the investment-capital ratio is related to average Q, $\frac{V_t}{K_t}$, rather than to marginal Q, $\frac{\partial V_t}{\partial K_t}$, which equals one in this model.[14] The relationship between investment and average Q in our model is noteworthy because average Q is observable, whereas marginal Q is not observable. The link between investment and Tobin's Q arises here because, even in the absence of adjustment costs, investment is a dynamic phenomenon. That is, investment is the growth of the capital stock plus depreciation, and the growth of the optimal capital stock depends on μ_t, the growth rate of Z_t. Since Q_t also depends on μ_t, it contains information about the growth of the capital stock.

[13]If the adjustment cost function is linearly homogeneous in I and K, it can be written as $C(I,K) = c\left(\frac{I}{K}\right)K$. The first-order condition for the optimal rate of investment is $1 + c'\left(\frac{I}{K}\right) = q$, where q is the marginal value of an additional unit of capital. Applying the implicit function theorem to this first-order condition yields $\frac{d(I/K)}{dq} = \frac{1}{c''(I/K)}$. Thus, the coefficient on (marginal) q in a regression of $\frac{I}{K}$ on q is the reciprocal of the second derivative of the adjustment cost function. A small value of the coefficient corresponds to a large value of $c''(I/K)$.

[14]Caballero and Leahy (1996) and Abel and Eberly (1998) analyze optimal investment in the presence of a fixed cost of investment and find that investment is related to average Q.

In the standard Q theory of investment, which is based on convex adjustment costs, the relation between investment and (marginal) Q is structural in the sense that it directly reflects the properties of the adjustment cost function. As we have pointed out, in a regression of the investment-capital ratio on Q in that framework, the coefficient on Q is an estimate of the reciprocal of the second derivative of the adjustment cost function with respect to investment. However, in the present framework, without convex adjustment costs, the coefficient on Tobin's Q is not structural.[15] That is, this coefficient does not represent any particular underlying cost or preference function, but instead is a reflection of the joint stochastic properties of various variables.

Equation (35) has another remarkable feature. Even after taking account of Q_t on the rate of investment, investment also depends on normalized cash flow c_t. Empirical studies of investment often find that the firm's cash flow per unit of capital is positively related to the rate of investment per unit of capital, even when a measure of Q is included as an explanator of investment. A typical empirical equation, starting from Fazzari, Hubbard, and Petersen (1988), would have the investment-capital ratio as the dependent variable, and Tobin's Q and c_t, the ratio of the firm's cash flow to its capital stock, as the independent variables. The finding of a positive cash flow effect is often interpreted as evidence of a financing constraint facing the firm.

To analyze the effect of cash flow on investment in our model, let $\beta_c \equiv \frac{\partial \iota(Q_t,c_t,)}{\partial c_t}$ denote the response of the investment-capital ratio to an increase in cash flow per unit of capital, c_t, holding Tobin's Q constant. Partially differentiate equation (35) with respect to c_t to obtain

$$\beta_c \equiv \frac{\partial \iota\,(Q_t, c_t)}{\partial c_t} = \alpha - \frac{(1-\alpha)\,\omega}{Q_t - 1}. \tag{37}$$

The sign of β_c in equation (37) depends on two competing factors. Gross investment is the sum of net investment and replacement investment. Replacement investment, relative to the capital stock, is simply the depreciation rate δ_t. Since

[15] In a different framework without convex adjustment costs, but with irreversibility of investment, Sargent (1980) has emphasized that the relationship between investment and Q is not structural.

$r + \delta_t = \alpha c_t$ for a firm choosing its capital stock optimally, a high value of c_t indicates a high value of δ_t and thus a high value of replacement investment. Working in the opposite direction is the effect of c_t on net investment. Net investment relative to the capital stock, $\frac{dK_t}{K_t}$, has an expected value (from equation 17) equal to $\left(\mu_t + \frac{1}{2}\frac{1-\alpha}{\alpha^2}\sigma_M^2\right) dt$. For a given value of Q_t, a high value of c_t indicates a low value of the growth rate μ_t because Q_t is an increasing function of both c_t and μ_t. Therefore, a high value of c_t indicates a low value of net investment. To summarize the effects on investment, the positive effect of cash flow on replacement investment is captured by α, the first term on the right hand side of equation (37), and the negative effect of cash flow on net investment is captured by $-\frac{(1-\alpha)\omega}{Q_t-1}$, the second term on the right hand side of equation (37).

Use equation (32) to substitute $\frac{(1-\alpha)\omega}{r+\lambda-\mu_t}c_t$ for $Q_t - 1$ in equation (37), then use (12) to substitute $\frac{\upsilon_t}{\alpha}$ for c_t, and use the definition of the user cost, $\upsilon_t \equiv r + \delta_t$, to obtain the following expression for β_c

$$\beta_c = \alpha\frac{\mu_t + \delta_t - \lambda}{r + \delta_t}. \tag{38}$$

Condition 2 $\delta_t + \mu_t > \lambda$ *for all* t.

Inspection of equation (38) reveals that Condition 2 is necessary and sufficient for $\beta_c > 0$ for all t.

Henceforth we will assume that Condition 2 holds so that $\beta_c > 0$. Although the traditional literature would interpret this positive relationship between cash flow and investment as evidence of a financing constraint, the positive effect arises in this model even though capital markets are perfect and there are *no financing constraints.*

The positive time-series relationship between investment and cash flow for a given firm operates through the user cost of capital, υ_t. As we discussed in Section 2, an increase in υ_t arising from an increase in the depreciation rate, δ_t, will increase gross investment relative to the capital stock. As is evident from equation (12), an increase in υ_t also increases the ratio of cash flow to the capital stock. Thus, the positive time-series association between cash flow and investment reflects the fact that each of these variables moves in the same direction in response to an increase in the user cost of capital.

4.1 The Effects of Firm Size and Growth on the Cash Flow Effect

The empirical literature on investment has found that cash flow has a more significant positive effect on investment for firms that are small or growing quickly. This finding has been interpreted as evidence that these firms face binding financial constraints, while large, slowly growing firms are either less constrained or financially unconstrained. This conclusion is perhaps appealing because it coheres well with the notion that small or rapidly growing firms do not have as much access to capital markets and external financing as large, slowly growing firms have. In this subsection we show that in our model, the effect of cash flow on investment, measured by β_c, is larger for firms that are small or rapidly growing than for large, slowly growing firms, which is consistent with empirical findings, even though there are no financial constraints in our model.

Inspection of equation (37) reveals that the cash flow coefficient β_c is increasing in Q_t. Recall from Section 3.1 that, on average, Q_t is decreasing in firm size and is increasing in the expected growth rate μ_t.[16] Therefore, the cash flow coefficient β_c is decreasing in firm size and increasing in the expected growth rate. These two predictions of the model are consistent with the empirical findings that cash flow coefficients are higher for small firms than for large firms and are higher for faster-growing firms than for slower-growing firms.

[16] An alternative demonstration of the model's prediction that β_c is increasing in μ_t is simply to observe that the right hand side of equation (38) is increasing in μ_t.

4.2 The Role of Monopoly Power

Our analysis so far has focussed on the behavior of a firm for given set of parameters. In this section, we briefly explore some comparative static implications for different parameter values. In particular, we consider firms with different degrees of monopoly power and examine the impact of monopoly power on Tobin's Q and the coefficients β_Q and β_c. We begin by expressing Q_t and β_Q in terms of exogenous parameters and variables facing the firm, which include α, r, ω, λ, δ_t, and μ_t. We have already expressed the cash flow coefficient, β_c, in terms of these exogenous parameters and variables (equation 38).

Use the fact that $c_t = \frac{r+\delta_t}{\alpha}$ to rewrite the expression for Q_t in equation (32) as

$$Q_t = 1 + \frac{1-\alpha}{\alpha}\omega\frac{r+\delta_t}{r+\lambda-\mu_t}. \tag{39}$$

Use the expression for Q_t in equation (39) and the fact that $c_t = \frac{r+\delta_t}{\alpha}$ to rewrite the coefficient β_Q in equation (36) as

$$\beta_Q = \frac{\alpha}{1-\alpha}\frac{(r+\lambda-\mu_t)^2}{(r+\delta_t)\,\omega}. \tag{40}$$

We will focus our attention on cross-firm differences in the price elasticity of demand, ε. Firms with lower values of ε have more monopoly power than firms with higher value of ε. The most direct impact on Q_t, β_Q, and β_c of the price elasticity of demand is through the parameter α, which is defined as $\alpha \equiv \frac{\gamma s\left(1-\frac{1}{\varepsilon}\right)}{1-(1-\gamma)s\left(1-\frac{1}{\varepsilon}\right)}$. It follows immediately from this definition that α is an increasing function of ε. Inspection of equations (39) reveals that Q_t is a decreasing function of α, for given values of r, ω, λ, δ_t, and μ_t. That is, Q_t is a decreasing function of the price elasticity of demand, ε, which means that Q_t is an increasing function of monopoly power. This prediction of our model is consistent with the empirical findings of Stigler (1964) and with the expressions for Q_t in Lindenberg and Ross (1981) and Salinger (1984). Indeed, the point of Lindenberg and Ross is that Q_t is higher for firms with higher monopoly power.

Inspection of equation (40) reveals that β_Q is an increasing function of α, for given values of r, ω, λ, δ_t, and μ_t. That is, the coefficient β_Q is predicted to smaller for firms with more monopoly power. Inspection of equation (38) reveals that the magnitude of the cash flow coefficient β_c is increasing in α. Under Condition 2, the cash flow coefficient is positive and is larger for firms with higher α. That is, the positive cash flow coefficient is predicted to be smaller for firms with more monopoly power.

So far we have focussed on the impact of the price elasticity ε, acting through its impact on α, on the values of Q_t, β_Q, and β_c, holding r, ω, λ, δ_t, and μ_t fixed. Although it is reasonable to hold r, λ, and δ_t fixed when ε changes, the values of μ_t and ω will potentially change when ε changes. Recall that μ_t is the drift, or expected growth rate of the exogenous variable Z_t, but Z_t is itself a composite variable. Specifically, Z_t is a geometric weighted average of the stochastic parameter, h_t, in the demand curve and the level of productivity, A_t, and the weights in the geometric average potentially depend on ε. Therefore, the expected growth rate μ_t potentially depends on ε, and since ω depends on the distribution of μ_t, it will also potentially depend on ε. In Appendix A, we show that under constant returns to scale in production, so that $s = 1$, the expected growth rate of Z_t is

$$\mu_t = \mu_{h,t} + (\varepsilon - 1)\,\mu_{A,t} + \frac{1}{2}(\varepsilon - 1)(\varepsilon - 2)\,\sigma_A^2, \tag{41}$$

where h_t and A_t evolve according to the geometric Brownian motions $\frac{dh_t}{h_t} = \mu_{h,t}dt + \sigma_h dz_h$ and $\frac{dA_t}{A_t} = \mu_{A,t}dt + \sigma_A dz_A$ and we assume that $(dz_h)(dz_A) = 0$.

First consider the special case in which productivity, A_t, is constant over time ($\mu_{A,t} \equiv 0$ and $\sigma_A = 0$) so that all of the movement in Z_t arises from movement in the demand parameter h_t. In this case, $\mu_t = \mu_{h,t}$ is invariant to ε, so ω is also invariant to ε.[17] Therefore, the impact of ε operates entirely through the effect on α described above: firms with higher monopoly power will have higher Q_t and lower coefficients β_Q and β_c.

To analyze the impact of ε when productivity follows a geometric Brownian mo-

[17] The invariance of μ_t to ε arises in this case because in the specification of the inverse demand curve in equation (2) the quantity of output demanded is proportional to h_t for a given price level.

tion, differentiate equation (41) with respect to ε to obtain

$$\frac{d\mu_t}{d\varepsilon} = \mu_{A,t} + \left(\varepsilon - \frac{3}{2}\right)\sigma_A^2. \tag{42}$$

In principle, $\frac{d\mu_t}{d\varepsilon}$ could be positive or negative and the impact of ε on μ_t and ω would need to be considered as well as the impact on α. We do not pursue this analysis here.

5 Calibration[18]

The model developed in this paper was designed to be simple enough to admit a closed-form solution for Tobin's Q and to permit easy analytic derivation of the coefficients β_Q and β_c. We have shown that β_Q is positive and that under Condition 2, β_c is also positive. In this section, we explore whether the *quantitative* values of Q, β_Q, and β_c produced by the model are plausible. Since the model is simple and frictionless, one might not expect the values of Q, β_Q, and β_c to necessarily be very close to the values typically found in empirical studies, but it would be reassuring if these values were of the same order of magnitude as their empirical counterparts. As we will show, the model-implied values of Q and especially β_Q and β_c are indeed close to the values found in empirical studies.

To calculate the values of Q, β_Q, and β_c, we need to specify $F(\widetilde{\mu})$, which is the distribution of growth rates of Z, as well as the values of the elasticity of net revenue with respect to capital, α, the discount rate, r, the Poisson arrival frequency λ of a new growth rate, and a typical value of the depreciation rate δ_t. (We do not need to specify values for the instantaneous standard deviations σ and σ_M because the values of Q, β_Q, and β_c are independent of these two parameters.) We assume that the distribution $F(\widetilde{\mu})$ is uniform on the interval $[a-b, a+b]$, where, of course, a is the mean of the distribution and $b > 0$ is the half-range.[19] Since net revenue is

[18] We thank Kjetil Storesletten for suggesting this quantitative analysis.

[19] We use this distribution to compute $\omega \equiv \left[E\left\{\frac{r-\widetilde{\mu}}{r+\lambda-\widetilde{\mu}}\right\}\right]^{-1}$. If $\widetilde{\mu}$ is uniformly dis-

proportional to $Z_t M_t$ and M_t is a martingale independent of Z_t, the average growth rate of revenue equals the average value of μ_t, which is a. Gomes (2001, Table 1) reports that the average growth rate of sales for firms in his sample is 0.036 per year, so we set $a = 0.036$. Provided that the instantaneous variation in M_t, which reflects instantaneous variation in δ_t, is sufficiently small, equation (18) implies that the average rate of gross investment relative to the capital stock is the average value of $\mu_t + \delta_t$. Gomes (2001, Table 1) reports that the average value of gross investment relative to the capital stock is 0.145, so the average value of $\mu_t + \delta_t$ is 0.145. Since we have set the average value of μ_t equal to 0.036, we choose the typical value of the depreciation rate to be $\delta_t = 0.145 - 0.036 = 0.109$. The discount rate, r, should be a risk-adjusted interest rate. Rather than choose a single specific value of r, we consider three values for r: 0.08, 0.10, and 0.14. The elasticity of net revenue with respect to capital, α, is smaller than one as a result of monopoly power and possibly also decreasing returns to scale. Since the values of Q, β_Q, and β_c are sensitive to the value of α, we consider five values of α, ranging from 0.5 to 0.9 in increments of 0.1. Our preferred value of α is 0.7, which is virtually identical to the value estimated by Cooper and Ejarque (2003, Table 2); Cooper and Haltiwanger (2006) estimate a value of α of 0.51 in plant-level data, while Burnside (1996) estimates a value of 0.8 at the industry level. We have less empirical guidance for the values of the half-range b and the Poisson parameter λ. Condition 1 requires $a + b = \mu_H < r$, so $b < r - a$. Since $a = 0.036$ and we consider values of r as low as 0.08, b must be less than $0.080 - 0.036 = 0.044$. We consider two values of b: 0.040 and 0.001. In choosing a value for λ, we make sure that λ satifies Condition 2. Specifically, λ must be less than $\delta_t + \mu_t$. Since μ_t can take on values as low as $a - b =$ -0.004, and since $\delta_t = 0.109$, $\delta_t + \mu_t$ can be as low as $0.109 - 0.004 = 0.105$. Therefore, we confine attention to values of λ smaller than 0.105. Specifically, we consider two values of λ: 0.10 and 0.05.

tributed on $[a-b, a+b]$, then $E\left\{\frac{r-\widetilde{\mu}}{r+\lambda-\widetilde{\mu}}\right\} = \int_{a-b}^{a+b} \frac{1}{2b} \frac{r-\widetilde{\mu}}{r+\lambda-\widetilde{\mu}} d\widetilde{\mu} = \frac{1}{2b} \int_{a-b}^{a+b} \left[1 - \frac{\lambda}{r+\lambda-\widetilde{\mu}}\right] d\widetilde{\mu} = \frac{1}{2b} \left(\widetilde{\mu} + \lambda \ln\left(r + \lambda - \widetilde{\mu}\right)\right)\Big|_{a-b}^{a+b} = 1 + \frac{\lambda}{2b} \ln \frac{r+\lambda-a-b}{r+\lambda-a+b}$.

Table 1 reports the values of Q, β_Q, and β_c for various combinations of the parameters. The table is divided into three panels. In Panel 1, which we will call the baseline panel, $\lambda = 0.05$, $a = 0.036$, $b = 0.040$, and $\delta_t = 0.109$. Panel 2 differs from Panel 1 only in that λ is 0.10 in Panel 2 rather than 0.05 as in Panel 1. Panel 3 differs from Panel 1 only in that b is 0.001 in Panel 3 rather than 0.040 as in Panel 1.

First consider the baseline in Panel 1. Each row corresponds to a different value of α. Comparing different rows, it is clear that α has an important impact on Q, β_Q, and β_c. As discussed earlier, our preferred value of α is 0.7, and the rows of each panel corresponding to $\alpha = 0.7$ are shaded for emphasis. Let's focus on the shaded row in Panel 1. The values of Q range from 2.99 when $r = 0.08$ to 2.04 when $r = 0.14$. These values of Q are very much in line with the range of values in studies cited by Cooper and Ejarque in their Table 1. Specifically, they report sample-wide average values of Q ranging from a low of 1.56 in Gomes (2001) to 2.95 in Gilchrist and Himmelberg (1995).[20] Again confining attention to the shaded row in Panel 1, β_Q ranges from 0.05 to 0.15. These small positive positive are close to the range of findings summarized by Cooper and Ejaque, which extends from essentially zero (but positive) to 0.06. The values of β_c in the shaded row in Panel 1 range from 0.27 to 0.35 and in all cases the value of β_c is greater than the corresponding value of β_Q. These values of β_c are within the range reported by Cooper and Ejarque (from 0.14 to 0.46); in all of the studies reported by Cooper and Ejarque the cash flow coefficient is larger than the coefficent on Q, as in the shaded row in Panel 1.

Panel 2 illustrates the impact of doubling the parameter λ to 0.10 from its value of 0.05 in Panel 1. When $\alpha = 0.7$, this doubling of the value of λ has only a minor impact on Q, causes β_Q to increase somewhat, and causes β_c fall to about half its value in Panel 1. Nevertheless, for $r = 0.08$ and $r = 0.10$, the value of β_c in the shaded row of Panel 2 remains larger than the corresponding value of β_Q. Panel

[20] In Fazzari, Hubbard and Petersen's (1988) sub-sample of firms that pay low dividends, the average value of Q is even higher at 3.8.

Panel 1: $\lambda = 0.05$; $a = 0.036$, $b = 0.040$; $\delta_t = 0.109$										
		Q			β_Q			β_c		
		r			r			r		
		0.08	0.10	0.14	0.08	0.10	0.14	0.08	0.10	0.14
	0.5	5.65	4.38	3.42	0.02	0.03	0.06	0.25	0.23	0.19
	0.6	4.10	3.26	2.61	0.03	0.05	0.10	0.30	0.27	0.23
α	0.7	2.99	2.45	2.04	0.05	0.08	0.15	0.35	0.32	0.27
	0.8	2.16	1.85	1.61	0.08	0.13	0.25	0.40	0.36	0.31
	0.9	1.52	1.38	1.27	0.18	0.30	0.57	0.45	0.41	0.34

Panel 2: $\lambda = 0.10$; $a = 0.036$, $b = 0.040$; $\delta_t = 0.109$										
		Q			β_Q			β_c		
		r			r			r		
		0.08	0.10	0.14	0.08	0.10	0.14	0.08	0.10	0.14
	0.5	5.58	4.37	3.42	0.03	0.05	0.08	0.12	0.11	0.09
	0.6	4.05	3.25	2.62	0.05	0.07	0.13	0.14	0.13	0.11
α	0.7	2.96	2.45	2.04	0.07	0.11	0.20	0.17	0.15	0.13
	0.8	2.14	1.84	1.61	0.13	0.19	0.34	0.19	0.17	0.14
	0.9	1.51	1.37	1.27	0.28	0.44	0.76	0.21	0.19	0.16

Panel 3: $\lambda = 0.05$; $a = 0.036$, $b = 0.001$; $\delta_t = 0.109$										
		Q			β_Q			β_c		
		r			r			r		
		0.08	0.10	0.14	0.08	0.10	0.14	0.08	0.10	0.14
	0.5	5.30	4.27	3.39	0.02	0.03	0.06	0.25	0.23	0.19
	0.6	3.86	3.18	2.60	0.03	0.05	0.10	0.30	0.27	0.23
α	0.7	2.84	2.40	2.03	0.05	0.08	0.15	0.35	0.32	0.27
	0.8	2.07	1.82	1.60	0.09	0.14	0.26	0.40	0.36	0.31
	0.9	1.48	1.36	1.27	0.20	0.31	0.58	0.45	0.41	0.34

Figure 1:

3 illustrates impact of the half-range b, which equals 0.040 in Panel 1 and is only 0.001 in Panel 3. This near-elimination of variation in μ_t reduces Q by a modest amount, increases β_Q only very slightly and has no impact at all on β_c.[21] Overall, the results in Table 1 indicate that the model's implications for the values of Q, β_Q, and β_c, especially when $\alpha = 0.7$, are remarkably close to their empirical counterparts reported in the literature.

6 Concluding Remarks

This paper presents a new explanation for the empirical relationship between investment and Tobin's Q. Traditional explanations of this relationship are based on convex costs of adjusting the capital stock. In this paper, we have assumed that there are no such adjustment costs that drive a wedge between the purchase price of capital and the market value of installed capital. Instead, the wedge between the market value of a firm and the replacement cost of its capital stock is based on rents accruing to market power or to decreasing returns to scale in the production function. The presence of these rents implies that Tobin's Q exceeds one.

Beyond showing that Q exceeds one, we showed that the investment-capital ratio is positively related to Tobin's Q, which is a measure of average Q, rather than to marginal Q, as in the convex adjustment cost literature. This departure from the adjustment cost literature is particularly important because average Q is readily observable, whereas marginal Q is not directly observable. In the empirical investment literature, relatively small responses of investment to Q have been taken as evidence of strongly convex adjustment costs; here there are no adjustment costs at all, and yet the response of investment to Q can be small.

In addition to being consistent with a positive relationship between investment and Tobin's Q, the model in this paper can account for the positive effect of cash flow on investment, even when Q is included as an explanator of investment. The common

[21] Inspection of equation (38) reveals that β_c is independent of the distribution $F(\widetilde{\mu})$.

interpretation of the positive cash flow effect on investment is that it is evidence of financing constraints facing firms. However, the model in this paper has perfect capital markets without financing constraints, and yet cash flow can have a positive effect on investment, even after taking account of the effect of Q on investment. Therefore, contrary to the common interpretation, and consistent with Gomes (2001), Cooper and Ejarque (2003), and Alti (2003), a positive cash flow effect on investment need not be evidence of a financing constraint.

The empirical literature has recognized that the investment regression may be misspecified or mismeasured, leading to spurious cash flow effects. One strategy to address these potentially spurious effects is to split the sample into *a priori* financially constrained and unconstrained firms. Typically, smaller and faster growing firms, which are often classified *a priori* as financially constrained, are found to have larger cash flow effects. The same pattern of cash flow effects emerges in our model, even though there are no financing constraints in the model, which calls into question this interpretation of the empirical findings.

The model in this paper is, by design, very simple and stylized. In order for the ratio of cash flow to the capital stock to exhibit time-series variation, the user cost of capital must vary over time, and we induced this variation by allowing the depreciation rate to vary stochastically. In order for Tobin's Q to exhibit time-series variation that is not perfectly correlated with contemporaneous cash flow per unit of capital, we assumed that the growth rate of Z_t varies stochastically over time according to a regime-switching model. We eliminated adjustment costs from the current analysis, not because we believe they are irrelevant for an empirical investment model, but rather because they are extraneous to the effects we examine here. The goal of the current paper is to articulate the relationship among investment, Tobin's Q, and cash flow. Empirical findings regarding these relationships have been used to detect the presence of adjustment costs and financing constraints, and to evaluate their importance for investment. Even when these adjustment costs and financing constraints are eliminated, however, we show that investment remains sensitive to both Tobin's Q and cash flow. Indeed, despite the simple analytic structure of our model, it can very closely match the empirical magnitude of Tobin's Q as well as the values of the coefficients on Q and cash flow typically estimated in investment regressions.

An avenue for future research would be to introduce richer and more realistic processes for the various exogenous variables facing the firm. Another direction would be to introduce delivery or gestation lags in the capital investment process. In ongoing research (Abel and Eberly, 2005), we endogenize the growth in technology, summarized here by an exogenous parameter, and similar effects arise in that framework.

References

[1] Abel, Andrew B., "Optimal Investment Under Uncertainty," American Economic Review, 73:1 (March 1983), 228-233.

[2] Abel, Andrew B. and Janice C. Eberly, "The Mix and Scale of Factors with Irreversibility and Fixed Costs of Investment," Carnegie-Rochester Conference Series on Public Policy, 48 (1998), 101-135.

[3] Abel, Andrew B. and Janice C. Eberly, "Investment, Valuation, and Growth Options," revised October 2005.

[4] Alti, Aydogan, "How Sensitive is Investment to Cash Flow When Financing is Frictionless?" The Journal of Finance, 58:2 (April 2003), 707-722.

[5] Brainard, William and James Tobin, "Pitfalls in Financial Model Building," American Economic Review, 58:2 (May 1968), pp. 99-122.

[6] Burnside, Craig, "Production Function Regressions, Returns to Scale and Externalities," Journal of Monetary Economics, 37:2 (April 1996), 177-201.

[7] Caballero, Ricardo and John V. Leahy, "Fixed Costs: The Demise of Marginal q," National Bureau of Economic Research Working Paper No. W5508, March 1996.

[8] Cooper, Russell and Joao Ejarque, "Financial Frictions and Investment: Requiem in Q," Review of Economic Dynamics, 6:4 (October 2003), 710-728.

[9] Cooper, Russell and John Haltiwanger, "On the Nature of Capital Adjustment Costs," Review of Economic Studies, 73:3 (July 2006), 611-633.

[10] Eberly, Janice, Sergio Rebelo and Nicolas Vincent, "Investment and Value: A Neoclassical Benchmark," National Bureau of Economic Research Working Paper No. 13866, March 2008.

[11] Erickson, Timothy and Toni M. Whited, "Measurement Error and the Relationship between Investment and 'q'," Journal of Political Economy, 108:5 (October 2000), 1027-1057.

[12] Fazzari, Steven, R. Glenn Hubbard and Bruce Petersen, "Finance Constraints and Corporate Investment," Brookings Papers on Economic Activity, 1:1988, 141-195.

[13] Gilchrist, Simon and Charles Himmelberg, "Evidence on the Role of Cash Flow for Investment," Journal of Monetary Economics, 36:3 (December 1995), 541-572.

[14] Gomes, Joao F., "Financing Investment," American Economic Review, 91:5 (December 2001), 1263-1285.

[15] Jorgenson, Dale W., "Capital Theory and Investment Behavior," American Economic Review, Papers and Proceedings, 53:2 (May 1963), 247-259.

[16] Keynes, John Maynard, The General Theory of Employment, Interest, and Money, The Macmillian Press, Ltd., 1936.

[17] Lindenberg, Eric B. and Stephen A. Ross, "Tobin's Q Ratio and Industrial Organization," The Journal of Business, 54:1. (January 1981), 1-32.

[18] Lucas, Robert E., Jr. and Edward C. Prescott, "Investment Under Uncertainty," Econometrica, 39:5 (September 1971), 659-681.

[19] Mussa, Michael, "External and Internal Adjustment Costs and the Theory of Aggregate and Firm Investment," Economica, 44:174 (May 1977), 163-178.

[20] Salinger, Michael A., "Tobin's q, Unionization, and the Concentration-Profits Relationship," The RAND Journal of Economics, 15:2 (Summer 1984), 159-170.

[21] Sargent, Thomas J., "'Tobin's q' and the Rate of Investment in General Equilibrium," Carnegie-Rochester Conference Series on Public Policy, 12 (Spring 1980), 107-154.

[22] Stigler, George J., "A Theory of Oligopoly," Journal of Political Economy, 72, 1 (February 1964), 44-61.

[23] Tobin, James, "A General Equilibrium Approach to Monetary Theory," Journal of Money, Credit, and Banking, 1:1 (February 1969), 15-29.

A Appendix: Net Revenue of the Firm

At time t, the firm chooses labor, N_t, to maximize revenue net of labor costs,

$$R_t = P_t Y_t - wN_t, \tag{A.1}$$

where w is the wage rate, which, for simplicity, we assume is constant. Substitute the production function from equation (1) and the inverse demand function from equation (2) into equation (A.1) to write net revenue as

$$R_t = g_t N_t^{\nu} - wN_t, \tag{A.2}$$

where $g_t \equiv h_t^{\frac{1}{\varepsilon}} A_t^{1-\frac{1}{\varepsilon}} K_t^{\gamma s\left(1-\frac{1}{\varepsilon}\right)}$ and $\nu \equiv (1-\gamma)\, s \left(1-\frac{1}{\varepsilon}\right)$. Differentiating equation (A.2) with respect to N_t and setting the derivative equal to zero yields $\nu g_t N_t^{\nu-1} = w$, which can be used to write net revenue as

$$R_t = \frac{1-\nu}{\nu} wN_t = (1-\nu)\, \nu^{\frac{\nu}{1-\nu}} w^{-\frac{\nu}{1-\nu}} g_t^{\frac{1}{1-\nu}}. \tag{A.3}$$

Define

$$\alpha \equiv \frac{\gamma s \left(1-\frac{1}{\varepsilon}\right)}{1-(1-\gamma)\, s\left(1-\frac{1}{\varepsilon}\right)} > 0. \tag{A.4}$$

Substitute the definition of g_t into equation (A.3) and use the definition of α to obtain

$$R_t = \chi \left[h_t^{\frac{1}{\varepsilon}} A_t^{1-\frac{1}{\varepsilon}}\right]^{\frac{1}{1-\nu}} K_t^{\alpha}, \tag{A.5}$$

where $\chi \equiv (1-\nu)\, \nu^{\frac{\nu}{1-\nu}} w^{-\frac{\nu}{1-\nu}}$. Use the fact that $1-\alpha = \frac{1-s\left(1-\frac{1}{\varepsilon}\right)}{1-\nu}$ to rewrite equation (A.5) as

$$R_t = \left[\chi^{\frac{1}{1-\alpha}} \left(h_t^{\frac{1}{\varepsilon}} A_t^{1-\frac{1}{\varepsilon}}\right)^{\frac{1}{1-s\left(1-\frac{1}{\varepsilon}\right)}}\right]^{1-\alpha} K_t^{\alpha}, \tag{A.6}$$

so $R_t = Z_t^{1-\alpha} K_t^{\alpha}$ where

$$Z_t \equiv \chi^{\frac{1}{1-\alpha}} \left(h_t^{\frac{1}{\varepsilon}} A_t^{1-\frac{1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon - s\varepsilon + s}}. \tag{A.7}$$

Since Z_t is an isoelastic function of h_t and A_t (with different, but constant elasticities, with respect to these variables), the growth rate of Z_t is a weighted average of the growth rates of h_t and A_t, with the weights equal to the corresponding elasticities.

A.1 Constant Returns to Scale ($s = 1$)

Set $s = 1$ in the definition of Z_t in equation (A.7) to obtain

$$Z_t \equiv \chi^{\frac{1}{1-\alpha}} A_t^{\varepsilon - 1} h_t. \tag{A.8}$$

Suppose that productivity, A_t, and the demand curve parameter, h_t, evolve according to the following independent geometric Brownian motions

$$\frac{dA_t}{A_t} = \mu_{A,t} dt + \sigma_A dz_A \tag{A.9}$$

$$\frac{dh_t}{h_t} = \mu_{h,t} dt + \sigma_h dz_h. \tag{A.10}$$

Apply Ito's Lemma to equation (A.8) and use equations (A.9) and (A.10) to compute $\mu_t \equiv \frac{1}{dt} E_t \left\{ \frac{dZ_t}{Z_t} \right\}$ as

$$\mu_t = \mu_{h,t} + (\varepsilon - 1)\, \mu_{A,t} + \frac{1}{2} (\varepsilon - 1)(\varepsilon - 2)\, \sigma_A^2. \tag{A.11}$$

B Appendix: Expected Present Value of a Stream with Variable Drift

Let $P(\mu_t, Z_t) = p(\mu_t) Z_t$, where $p(\mu_t) \equiv E_t \left\{ \int_0^\infty \frac{Z_{t+\tau}}{Z_t} e^{-r\tau} d\tau \right\}$. Let $p(\mu_t, T)$ be the expected value of $p(\mu_t)$ conditional on the assumption that the growth rate of Z_t remains equal to μ_t until time $t+T$, and that a new value of the growth rate is drawn from the unconditional distribution at time $t+T$. Therefore,

$$p(\mu_t, T) = \int_0^T e^{-(r-\mu_t)\tau} d\tau + e^{-(r-\mu_t)T} E_t \left\{ \int_T^\infty \frac{Z_{t+\tau}}{Z_{t+T}} e^{-r(\tau - T)} d\tau \right\}. \tag{B.1}$$

Evaluating the first integral on the right hand side of equation (B.1) and rewriting the second integral yields

$$p(\mu_t, T) = \frac{1 - e^{-(r-\mu_t)T}}{r - \mu_t} + e^{-(r-\mu_t)T} E_t \left\{ \int_0^\infty \frac{Z_{t+T+\tau}}{Z_{t+T}} e^{-r\tau} d\tau \right\}. \tag{B.2}$$

Let p^* be the expectation of $p(\mu_t)$ when μ_t is drawn from its unconditional distribution, so that equation (B.2) can be written as

$$p(\mu_t, T) = \frac{1 - e^{-(r-\mu_t)T}}{r - \mu_t} + e^{-(r-\mu_t)T} p^*. \tag{B.3}$$

The density of T is

$$f(T) = \lambda e^{-\lambda T} \tag{B.4}$$

and

$$p(\mu_t) = \int_0^\infty p(\mu_t, T) f(T) \, dT. \tag{B.5}$$

Substituting equations (B.3) and (B.4) into equation (B.5) yields

$$p(\mu_t) = \int_0^\infty \left[\frac{1 - e^{-(r-\mu_t)T}}{r - \mu_t} + e^{-(r-\mu_t)T} p^*\right] \lambda e^{-\lambda T} dT. \tag{B.6}$$

Equation (B.6) can be rewritten as

$$p(\mu_t) = \frac{1}{r - \mu_t}\left[\int_0^\infty \left[1 + (rp^* - \mu_t p^* - 1) e^{-(r-\mu_t)T}\right] \lambda e^{-\lambda T} dT\right]. \tag{B.7}$$

Evaluating the integral in equation (B.7) yields

$$p(\mu_t) = \frac{1}{r - \mu_t}\left[1 + (rp^* - \mu_t p^* - 1) \frac{\lambda}{r + \lambda - \mu_t}\right], \tag{B.8}$$

which can be rearranged to yield

$$p(\mu_t) = \frac{1 + \lambda p^*}{r + \lambda - \mu_t}. \tag{B.9}$$

Since $p^* = E\{p(\widetilde{\mu})\}$, take the unconditional expectation of both sides of equation (B.9) to obtain

$$p^* = E\left\{\frac{1}{r + \lambda - \widetilde{\mu}}\right\}(1 + p^* \lambda), \tag{B.10}$$

which implies

$$p^* = \omega E\left\{\frac{1}{r + \lambda - \widetilde{\mu}}\right\}, \tag{B.11}$$

where

$$\omega \equiv \left[E\left\{\frac{r - \widetilde{\mu}}{r + \lambda - \widetilde{\mu}}\right\}\right]^{-1}. \tag{B.12}$$

As we show in footnote 10, equations (5) and (6) together imply that $\omega \geq 1$, and equation (5) then is needed for $p^* > 0$.

Use equation (B.11) to obtain

$$1 + \lambda p^* = 1 + \omega E \left\{ \frac{\lambda}{r + \lambda - \widetilde{\mu}} \right\} = 1 + \omega \left[1 - E \left\{ \frac{r - \widetilde{\mu}}{r + \lambda - \widetilde{\mu}} \right\} \right] = \omega. \tag{B.13}$$

Substituting equation (B.13) into equation (B.9) yields

$$p(\mu_t) = \frac{\omega}{r + \lambda - \mu_t}, \tag{B.14}$$

Therefore,

$$P(\mu_t, Z_t) = \frac{\omega}{r + \lambda - \mu_t} Z_t. \tag{B.15}$$

C Appendix: Verification of the Solution to Equation (23)

Suppose that the value of the firm is given by equation (24), which we repeat here

$$V_t(Z_t, M_t, \mu_t) = Z_t M_t^{\frac{1}{\alpha}} + \frac{(1-\alpha)\,\omega Z_t M_t}{r + \lambda - \mu_t}. \tag{C.1}$$

Use Ito's Lemma and the facts that the right hand side of equation (C.1) is linear in Z_t and that M_t is a martingale to obtain

$$\begin{aligned} E_t\{dV_t\} &= \mu_t Z_t M_t^{\frac{1}{\alpha}} dt + \frac{1}{2}\frac{1-\alpha}{\alpha^2}\sigma_M^2 Z_t M_t^{\frac{1}{\alpha}} dt \\ &\quad + (1-\alpha)\,\omega Z_t M_t \left(\frac{1}{r + \lambda - \mu_t}\mu_t + \lambda \left[E_t\left\{ \frac{1}{r + \lambda - \widetilde{\mu}} \right\} - \frac{1}{r + \lambda - \mu_t} \right] \right) dt. \end{aligned} \tag{C.2}$$

Use the facts that $E_t\left\{\frac{\lambda}{r+\lambda-\widetilde{\mu}}\right\} = 1 - E_t\left\{\frac{r-\widetilde{\mu}}{r+\lambda-\widetilde{\mu}}\right\}$ and $\frac{\mu_t - \lambda}{r+\lambda-\mu_t} = -1 + \frac{r}{r+\lambda-\mu_t}$ to rewrite equation (C.2) as

$$\begin{aligned} E_t\{dV_t\} &= \mu_t Z_t M_t^{\frac{1}{\alpha}} dt + \frac{1}{2}\frac{1-\alpha}{\alpha^2}\sigma_M^2 Z_t M_t^{\frac{1}{\alpha}} dt \\ &\quad + (1-\alpha)\,\omega Z_t M_t \left(\frac{r}{r + \lambda - \mu_t} - E_t\left\{ \frac{r - \widetilde{\mu}}{r + \lambda - \widetilde{\mu}} \right\} \right) dt. \end{aligned} \tag{C.3}$$

Use equation (C.3) along with equations (20) and (21) to obtain

$$E_t \{R_t dt - (dK_t + \delta_t K_t dt) + dV_t\} = \left[\begin{array}{c} Z_t M_t - \delta_t Z_t M_t^{\frac{1}{\alpha}} \\ + (1-\alpha)\,\omega Z_t M_t \left(\frac{r}{r+\lambda-\mu_t} - E_t \left\{ \frac{r-\widetilde{\mu}}{r+\lambda-\widetilde{\mu}} \right\} \right) \end{array} \right] dt. \tag{C.4}$$

Use the definition of ω in equation (25) to substitute ω^{-1} for $E\left\{\frac{r-\widetilde{\mu}}{r+\lambda-\widetilde{\mu}}\right\}$ in equation (C.4) to obtain

$$E_t \{R_t dt - (dK_t + \delta_t K_t dt) + dV_t\} = \left[\alpha Z_t M_t - \delta_t Z_t M_t^{\frac{1}{\alpha}} + \frac{r(1-\alpha)\,\omega Z_t M_t}{r+\lambda-\mu_t} \right] dt. \tag{C.5}$$

Add and subtract $r Z_t M_t^{\frac{1}{\alpha}}$ on the right hand side of equation (C.5) to obtain

$$E_t \{R_t dt - (dK_t + \delta_t K_t dt) + dV_t\} = \left[\begin{array}{c} \left(\alpha M_t - (r+\delta_t) M_t^{\frac{1}{\alpha}} \right) Z_t \\ + r \left(\frac{(1-\alpha)\omega Z_t M_t}{r+\lambda-\mu_t} + Z_t M_t^{\frac{1}{\alpha}} \right) \end{array} \right] dt. \tag{C.6}$$

Use the definition $M_t \equiv \left(\frac{v_t}{\alpha}\right)^{\frac{-\alpha}{1-\alpha}} = \left(\frac{r+\delta_t}{\alpha}\right)^{\frac{-\alpha}{1-\alpha}}$ to show that

$$\alpha M_t - (r+\delta_t) M_t^{\frac{1}{\alpha}} = 0. \tag{C.7}$$

Substitute equation (C.7) into equation (C.6) to obtain

$$E_t \{R_t dt - (dK_t + \delta_t K_t dt) + dV_t\} = r \left(\frac{(1-\alpha)\,\omega Z_t M_t}{r+\lambda-\mu_t} + Z_t M_t^{\frac{1}{\alpha}} \right) dt. \tag{C.8}$$

Finally, use equation (C.1) to rewrite the right hand side of equation (C.8) so that

$$E_t \{R_t dt - (dK_t + \delta_t K_t dt) + dV_t\} = r V_t dt,$$

which shows that equation (24) is a solution to equation (23).