Investment, Valuation, and Growth Options*

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Abstract

We develop a model in which the opportunity for a firm to upgrade its technology to the frontier (at a cost) leads to growth options in the firm's value; that is, a firm's value is the sum of value generated by its current technology plus the value of the option to upgrade. Variation in the technological frontier leads to variation in firm value that is unrelated to current cash flow and investment, though variation in firm value anticipates future upgrades and investment. We simulate this model and show that, consistent with the empirical literature, in situations in which growth options are important, regressions of investment on Tobin's Q and cash flow yield small positive coefficients on Q and larger coefficients on cash flow. We also show that growth options increase the volatility of firm value relative to the volatility of cash flow.

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1 Introduction

A firm's value should measure the expected present value of future payouts to claimholders. This insight led Keynes (1936), Brainard and Tobin (1968), and Tobin (1969) to the ideas underlying Q theory–that the market value of installed capital (relative to uninstalled capital) summarizes the incentive to invest. This insight, while theoretically compelling, has met with mixed empirical success. Although empirical studies typically find that investment is correlated with Tobin's Q, the effect of Tobin's Q on investment is sometimes weak and often dominated by the direct effect of cash flow on investment. Moreover, the measured volatility of firms' market values greatly exceeds the volatility of the fundamentals that they supposedly summarize, creating the "excess volatility" puzzle documented by Leroy and Porter (1981), Shiller (1981), and West (1988).

While these findings might be interpreted as irrationality in valuation, or as evidence that the stock market is a "sideshow" for real investment and value, we show that these phenomena can arise in an optimizing model with growth options. We develop a model in which the firm has a standard production function, with frictionless use of factor inputs (capital and labor). In the standard model, the level of productivity is generally assumed to evolve exogenously. However, we model the firm's level of technology as an endogenous variable chosen optimally by the firm. Specifically, the frontier level of technology evolves exogenously over time, and the firm can choose to adopt the frontier level of technology whenever it chooses to do so. Since the adoption of the frontier level of technology is costly, it will be optimal to upgrade the technology to the frontier at discretely-spaced points of time.

The salient feature of this simple structure is the generation of "growth options" in the value of the firm. These "growth options" generate value for the firm in addition to the present value of cash flows from the firm's current technology. Even though the frontier technology generally differs from the firm's current level of technology, and thus does not affect current cash flows, the firm has the option to adopt the frontier level of technology whenever it chooses. The value of this option fluctuates as the frontier technology fluctuates according to its own exogenous stochastic process. Importantly, these fluctuations in the value of the growth options are independent of current cash flow, thereby causing fluctuations in the firm's value that are unrelated to its current cash flow.

Since the firm's investment in physical capital is frictionless, it depends only on current conditions, which are summarized by current cash flow. Thus, during intervals of time between consecutive technology upgrades, investment is closely related to cash flow, and is independent of Tobin's Q, given the value of cash flow. However, when the firm upgrades its technology to the frontier, it undertakes a burst of investment in technology and in physical capital. We will show that the value of the firm, and thus Tobin's Q, rise as the frontier technology improves and the firm approaches a time at which it will upgrade to the frontier. Thus, a high value of Tobin's Q is associated with the prospect of burst of investment in the near future, thereby generating a positive correlation between investment and Tobin's Q in discretely sampled data.

Investment regressions including both Tobin's Q and cash flow are often used as a diagnostic of the Q theory of investment and as a test for financing constraints. In the model presented here, both Q and cash flow are correlated with investment, but there are no adjustment costs (as there would be in Q theory) and no financing constraints. By simulating the current model, allowing for discretely sampled data and also for time aggregation, we show that growth options can result in a small regression coefficient on Q and a large effect of cash flow on investment. The former is often interpreted as an indicator of large capital adjustment costs–while in the current model there are no adjustment costs at all. Similarly, following Fazzari, Hubbard, and Peterson (1988), a positive coefficient on cash flow, when controlling for Q in an investment regression, is often interpreted as evidence of financing constraints. However, our model is constructed without any capital market imperfections, so the cash flow effect on investment is not evidence of a financing constraint.

Growth options cause fluctuations in firm valuation that are not matched by current variation in cash flows. Instead, these fluctuations are driven by variation in the frontier technology. This independent variation in the value of growth options thus has the potential to generate "excess volatility" in firm valuation relative to its fundamental cash flows. Such excess volatility has been empirically documented at least since Leroy and Porter (1981), who examined equity prices relative to earnings, and Shiller (1981), who examined equity prices relative to dividends. Both of these studies required stationarity of the underlying processes, an assumption that was relaxed by West (1988), who also found excess volatility of equity prices relative to dividends. We examine the extent to which excess volatility characterizes the simulated data generated from our model.

We begin Section 2 by laying out the model. The first part of this section examines the firm's static choice of capital and labor, given the level of technology, and the second part of the section tackles the more difficult problem of choosing when to upgrade the level of technology. A valuable by-product of solving the upgrade problem is an explicit expression for the value of the firm. The value of the firm is the numerator of Tobin's Q. The denominator of Tobin's Q is the replacement cost of the firm's total capital stock, which comprises both physical capital and technology. We calculate Tobin's Q in Section 3. Then in Section 4 we turn our attention to investment in physical capital and in technology, deriving explicit expressions for investment expenditures between consecutive upgrades and for investment expenditures associated with adopting new technology and increasing the capital stock at the times of upgrades. We begin our simulation analysis in Section 5 by showing various features of the simulated data and then running regressions of investment on Tobin's Q and cash flow. We analyze the issue of excessively volatile firm valuation in Section 6 and present concluding remarks in Section 7.

2 A Model of the Firm with Growth Options

Consider a firm that uses physical capital and labor to produce nonstorable output. Both physical capital and labor are freely and instantaneously adjustable. Total factor productivity for the firm is determined by the level of technology in use by the firm. The firm can adjust the level of technology whenever it chooses to pay the cost of adopting a new technology. Because technology is a productive resource that is useful in producing output over sustained periods of time, we will treat technology as a type of capital. Thus, the firm has two types of capital: physical capital and technology. If we view technology as software, which can be a disembodied enhancement to the productivity of physical capital, then the treatment of software expenditures as an investment expenditure is consistent with the current treatment of software expenditures in the National Income and Product Accounts by the Bureau of Economic Analysis. More generally, however, expenditures on technology to enhance productivity are not limited to software. They can represent any expenditures on a second type of physical capital that enhances the productivity of the first type of physical capital.

We solve for the optimal behavior of the firm in two steps. Since physical capital is costlessly adjustable, we first solve for optimal choice of physical capital and the resulting operating profit, for a given level of the firm's technology. Once these values are derived in Section 2.1, we analyze the firm's technology upgrade decisions in Section 2.2. We then solve for the value of a firm that has access to the frontier technology and upgrades its technology optimally.

2.1 Operating Profits and Static Optimization

Suppose that the firm uses physical, K_t, and labor, N_t, to produce nonstorable output, y_t, at time t according to a production function that is homogeneous of degree s in K_t and N_t. Specifically, assume that the production function is

$$y_t = A_t^* \left(K_t^{\alpha} N_t^{1-\alpha}\right)^s, \tag{1}$$

where A_t^* is total factor productivity at time t and $0 < \alpha < 1$. Under constant returns to scale, $s = 1$. With competitive markets for capital and labor, α is the share of capital in factor income under constant returns to scale.

The demand curve for the firm's output is $y_t = h_t P_t^{-\varepsilon}$ where P_t is the price of output, the price elasticity of demand is $\varepsilon > 1$, and h_t is a parameter that locates the demand curve. At time t, the firm chooses labor, N_t, to maximize revenue net of labor costs, $R_t = P_t y_t - w_t N_t$ where w_t is the wage rate at time t. Use the production function and the demand curve to rewrite the expression for net revenue as

$$R_t = h_t^{\frac{1}{\varepsilon}} \left[A_t^* K_t^{\alpha s} N_t^{(1-\alpha)s}\right]^{1-\frac{1}{\varepsilon}} - w_t N_t. \tag{2}$$

Differentiating the right hand side of equation (2) with respect to N_t and setting the derivative equal to zero yields the optimal level of labor

$$N_t = \left[(1-\alpha)\, s \left(1 - \frac{1}{\varepsilon}\right) w_t^{-1} h_t^{\frac{1}{\varepsilon}} \left(A_t^* K_t^{\alpha s}\right)^{1-\frac{1}{\varepsilon}}\right]^{\frac{1}{1-(1-\alpha)s\left(1-\frac{1}{\varepsilon}\right)}}. \tag{3}$$

Substitute the optimal value of N_t from equation (3) into equation (2) to obtain

$$R_t = \left(A_t Y_t\right)^{1-\gamma} K_t^{\gamma}, \tag{4}$$

where

$$A_t \equiv A_t^{*\frac{\varepsilon-1}{\varepsilon-\varepsilon s+s}}, \tag{5}$$

$$Y_t \equiv \chi^{\frac{1}{1-\gamma}} w_t^{-\frac{s(1-\alpha)(\varepsilon-1)}{\varepsilon-\varepsilon s+s}} h_t^{\frac{1}{\varepsilon-\varepsilon s+s}}, \tag{6}$$

where

$$\chi \equiv \left[1-(1-\alpha)\, s\left(1-\frac{1}{\varepsilon}\right)\right]\left[(1-\alpha)\, s\left(1-\frac{1}{\varepsilon}\right)\right]^{\frac{(1-\alpha)\gamma}{\alpha}}$$

and

$$\gamma \equiv \frac{\alpha s \left(1 - \frac{1}{\varepsilon}\right)}{1 - (1-\alpha)\, s \left(1 - \frac{1}{\varepsilon}\right)}.$$

The expression for net revenue in equation (4) is the firm's cash flow before taking account of expenditures associated with the acquisition of physical capital or technology. In this expression, we introduced the variables A_t and Y_t, defined in equations (5) and (6), respectively, because it will be convenient to express R_t as a linearly homogeneous function of the product $A_t Y_t$ and K_t. For expositional convenience, we will henceforth refer to A_t as the level of technology, though it is a homogeneous function of the level of productivity, rather than simply equal to productivity. For instance, with constant returns to scale ($s = 1$), $A_t = A_t^{*\varepsilon - 1}$. The variable Y_t depends on both the wage rate, w_t, and the location of the demand curve, h_t. For expositional convenience, we will henceforth refer to Y_t as the level of demand. Under constant returns to scale ($s = 1$), Y_t is, in fact, strictly proportional to the demand parameter, h_t, when the wage rate w_t is constant. We want to restrict attention to cases in which $\gamma < 1$, which is equivalent to $\left(1 - \frac{1}{\varepsilon}\right) s < 1$. Thus, if the firm has decreasing returns to scale ($s < 1$), then $\gamma < 1$. If the firm has constant returns to scale ($s = 1$), then provided that the firm has some monopoly power ($\varepsilon < \infty$), $\gamma < 1$.

Define the user cost factor as $u_t \equiv r + \delta_t - \mu_p$, where r is the discount rate, δ_t is the depreciation rate of physical capital at time t,[1] and p_t is the purchase/sale price of physical capital, which grows deterministically at rate μ_p. Operating profits, which are net revenue minus the user cost of physical capital, are given by

$$\pi_t = (A_t Y_t)^{1-\gamma} K_t^{\gamma} - u_t p_t K_t, \tag{7}$$

where $u_t p_t$ is the user cost of a unit of physical capital. Maximizing operating profits

[1] We allow the depreciation rate to be stochastic to motivate the stochastic user cost of capital. Specifically, since the user cost factor is $u_t \equiv r + \delta_t - \mu_p$, the increment to the user cost factor, u_t, equals the increment to the depreciation rate, $du_t = d\delta_t$.

in equation (7) with respect to K_t yields the optimal physical capital stock[2]

$$K_t = \frac{A_t X_t}{u_t p_t} \frac{\gamma}{1-\gamma}, \tag{8}$$

and the optimized value of operating profits

$$\pi_t = A_t X_t, \tag{9}$$

where

$$X_t \equiv Y_t \left(\frac{\gamma}{u_t p_t} \right)^{\frac{\gamma}{1-\gamma}} (1-\gamma) \tag{10}$$

summarizes the non-technology factors affecting operating profits. We assume that Y_t follows a geometric Brownian motion and that u_t follows a driftless geometric Brownian motion with instantaneous variance σ_u^2.[3] Therefore, X_t follows a geometric Brownian motion

$$dX_t = mX_t dt + sX_t dz_X, \tag{11}$$

where the drift, m, and instantaneous variance, s^2, depend on the drifts and instantaneous variances and covariances of the underlying processes for Y_t, u_t, and p_t.[4]

[2] Differentiating the right-hand side of equation (7) with respect to K_t, and setting the derivative equal to zero yields

$$\gamma \left(\frac{A_t Y_t}{K_t} \right)^{1-\gamma} = u_t p_t. \tag{*}$$

Solving this first-order condition for the optimal capital stock yields

$$K_t = A_t Y_t \left(\frac{\gamma}{u_t p_t} \right)^{\frac{1}{1-\gamma}}. \tag{**}$$

Substituting equation (**) into the operating profit function in equation (7) yields optimized operating profits

$$\pi_t = u_t p_t K_t \left(\frac{1-\gamma}{\gamma} \right) = A_t Y_t \left(\frac{\gamma}{u_t p_t} \right)^{\frac{\gamma}{1-\gamma}} (1-\gamma). \tag{***}$$

Use the definition of X_t in equation (10) to rewrite equation (**) as equation (8) and equation (***) as equation (9).

[3] By assuming that u_t follows a driftless geometric Brownian motion, we are implicitly assuming that the depreciation rate evolves according to $d\delta_t = \left(r + \delta_t - \mu_p\right) \sigma_u dz_u$.

[4] If Y_t, u_t, and p_t are geometric Brownian motions, then the composite term X_t also follows a geometric Brownian motion. Specifically, let the instantaneous drift of the process for Y_t be μ_Y and its instantaneous variance be σ_Y^2. Then given our specification of the processes for u_t and p_t, $m \equiv \mu_Y - \frac{\gamma}{1-\gamma} \left[\mu_p - \frac{1}{2} \frac{\sigma_u^2}{1-\gamma} + \rho_{Yu} \sigma_Y \sigma_u \right]$ and $sdz_X = \sigma_Y dz_Y - \frac{\gamma \sigma_u}{1-\gamma} dz_u$, where $\rho_{Yu} \equiv \frac{1}{dt} E\left(dz_Y dz_u\right)$ is the correlation between the shocks to Y_t and u_t. In addition, $s^2 = \sigma_Y^2 + \left(\frac{\gamma \sigma_u}{1-\gamma} \right)^2 - 2 \frac{\gamma}{1-\gamma} \rho_{Yu} \sigma_Y \sigma_u$; $s\rho_{Xu} = \rho_{Yu} \sigma_Y - \frac{\gamma}{1-\gamma} \sigma_u$; and $s\rho_{X\hat{A}} = \rho_{Y\hat{A}} \sigma_Y - \frac{\gamma}{1-\gamma} \rho_{u\hat{A}} \sigma_u$, where $\rho_{ij} \equiv \frac{1}{dt} E\left(dz_i dz_j\right)$.

Since, in the next section, we will examine the relationship between investment and cash flow, note that the firm's cash flow before investment expenditure is given by $C_t \equiv (A_t Y_t)^{1-\gamma} K_t^{\gamma}$. Equations (7), (9) and equation (**) in footnote (2) imply that

$$C_t \equiv \frac{\pi_t}{1-\gamma} = \frac{A_t X_t}{1-\gamma}. \tag{12}$$

It will be convenient to work with the ratio of cash flow, C_t, to the replacement cost of the physical capital stock, $p_t K_t$,

$$c_t \equiv \frac{C_t}{p_t K_t} = \frac{u_t}{\gamma} = \frac{1}{\gamma}\left(r + \delta_t - \mu_p\right), \tag{13}$$

which is proportional to the user cost factor when the capital stock is optimally chosen.[5]

2.2 Optimal Upgrades and the Value of the Firm

The optimal value of the physical capital stock and the resulting values of the operating profit, π_t, and cash flow, C_t, are conditional on the level of installed technology, A_t. In subsection 2.1, we treated the value of A_t as given. Now we will treat A_t as a choice variable of the firm. Specifically, the firm can choose to upgrade A_t to the frontier technology, $\widehat{A}_t$, which evolves exogenously according to the geometric Brownian motion

$$d\widehat{A}_t = \mu \widehat{A}_t dt + \sigma \widehat{A}_t dz_{\widehat{A}}. \tag{14}$$

The instantaneous correlation between the innovations to X_t and $\widehat{A}_t$ is $\rho_{X\widehat{A}} \equiv \frac{1}{dt}E(dz_X dz_{\widehat{A}})$, and we assume that $\mu > \frac{1}{2}\sigma^2$.[6]

The cost of upgrading to the frontier technology, $\widehat{A}_t$, at time t, is $\theta \widehat{A}_t X_t$, where $\theta \geq 0$ is a constant. Because upgrading incurs a fixed cost (the cost depends only on exogenous variables and is independent of the size of the upgrade), it will not be optimal to upgrade continuously. The firm optimally determines discrete times τ_j, $j = 0, 1, 2, ...$ at which to upgrade.

To analyze the upgrade decision, begin with a firm that does not own any physical capital. This firm owns the technology, A_t, but rents the services of physical capital at each point in time, paying a user cost of $u_t p_t$ per unit of physical capital at time

[5] The definition of the cash flow-to-capital ratio, along with the first equality in equation (***) in footnote 2, yields $c_t \equiv \frac{C_t}{p_t K_t} = \frac{u_t p_t K_t}{\gamma p_t K_t} = \frac{u_t}{\gamma}$.

[6] The assumption that $\mu > \frac{1}{2}\sigma^2$ guarantees that the expected first passage time to the upgrade threshold is finite. We also assume initial conditions X_0, $\widehat{A}_0, u_0, p_0 > 0$.

t. The value of this firm is the expected present value of operating profits less the cost of any future technology upgrades. Let $\Psi\left(A_t, X_t, \widehat{A}_t\right)$ be the expected present value of operating profits, net of upgrade costs, from time t onward, so

$$\Psi\left(A_t, X_t, \widehat{A}_t\right) = \max_{\{\tau_j\}_{j=1}^{\infty}} E_t \left\{ \int_0^{\infty} A_{t+s} X_{t+s} e^{-rs} ds - \sum_{j=1}^{\infty} \theta \widehat{A}_{\tau_j} X_{\tau_j} e^{-r(\tau_j - t)} \right\}, \quad (15)$$

where $\widehat{A}_{\tau_j}$ is the value of the available frontier technology when the upgrade occurs at time τ_j. We require that (1) $r - m > 0$, so that a firm that never upgrades has finite value; (2) $r - m - \mu - \rho_{X\widehat{A}} s\sigma > 0$, so that a firm that continuously maintains $A_t = \widehat{A}_t$ has a value that is bounded from above;[7] and (3) $(r - m)\theta < 1$, so that the upgrade cost is not large enough to prevent the firm from ever upgrading.[8]

In order to calculate the present value of optimal operating profits, we first calculate the value of the firm when it is not upgrading, and then use the boundary conditions that hold when the firm upgrades its technology. The required return on the firm, $r\Psi_t$, must equal current operating profits plus its expected capital gain. When the firm is not upgrading its technology, A_t is constant, so the equality of the required return and the expected return can be written as (omitting time subscripts)

$$\begin{aligned} r\Psi &= \pi + E(d\Psi) \quad (16) \\ &= AX + mX\Psi_X + \frac{1}{2}s^2 X^2 \Psi_{XX} + \mu \widehat{A} \Psi_{\widehat{A}} + \frac{1}{2}\sigma^2 \widehat{A}^2 \Psi_{\widehat{A}\widehat{A}} + \rho_{X\widehat{A}} s\sigma X \widehat{A} \Psi_{X\widehat{A}}. \end{aligned}$$

Direct substitution verifies that the following function satisfies the partial differential equation in equation (16)

$$\Psi\left(A_t, X_t, \widehat{A}_t\right) = \frac{A_t X_t}{r - m} + B A_t X_t \left(\frac{\widehat{A}_t}{A_t}\right)^{\phi}, \quad (17)$$

where B is an unknown constant and the parameter $\phi > 1$ is the positive root[9,10] of

[7] The condition $r - m - \mu - \rho_{X\widehat{A}} s\sigma > 0$ imples that even if the firm could maintain $A_t = \widehat{A}_t$ for all t without facing any upgrade costs, its value would be finite. Therefore, the value of a firm that faces upgrade costs would be bounded from above if it maintained $A_t = \widehat{A}_t$ for all t.

[8] See footnote 12 for the properties of the upgrade trigger $\overline{a}$.

[9] Notice that $f(0) = r - m > 0$, $f(1) = r - m - \mu - \rho_{X\widehat{A}} s\sigma > 0$, and $f''(\zeta) < 0$, so that the positive root of this equation exceeds one.

[10] An additional term including the negative root of the quadratic equation also enters the general solution to the differential equation. However, the negative exponent would imply that the firm's value goes to infinity as the frontier technology approaches zero. We set the unknown constant in this term equal to zero and eliminate this term from the solution.

the quadratic equation

$$f(\zeta) \equiv r - m - (\mu + \rho_{X\widehat{A}} s\sigma - \frac{1}{2}\sigma^2)\zeta - \frac{1}{2}\sigma^2\zeta^2 = 0. \tag{18}$$

The boundary conditions imposed at times of technological upgrading determine the constant B and the rule for optimally upgrading to the new technology. The first boundary condition is the value-matching condition, which requires that at the time of the upgrade, the value of the firm increases by the amount of the fixed cost. Formally this requires

$$\Psi\left(\widehat{A}_{\tau_j}, X_{\tau_j}, \widehat{A}_{\tau_j}\right) - \Psi\left(A_{\tau_j}, X_{\tau_j}, \widehat{A}_{\tau_j}\right) = \theta \widehat{A}_{\tau_j} X_{\tau_j}, \tag{19}$$

where $\Psi\left(A_{\tau_j}, X_{\tau_j}, \widehat{A}_{\tau_j}\right)$ is the value of the firm evaluated at the current (pre-upgrade) technology, A_{τ_j}, and $\Psi\left(\widehat{A}_{\tau_j}, X_{\tau_j}, \widehat{A}_{\tau_j}\right)$ is the value of the firm immediately after upgrading to the frontier technology. Substitute the proposed value of the firm from equation (17) into equation (19) and simplify to obtain the boundary condition in terms of the relative technology variable $a \equiv \frac{\widehat{A}}{A}$ and the unknown constant B:

$$\frac{a-1}{r-m} - aB\left(a^{\phi-1} - 1\right) = \theta a \quad \text{for } a = \overline{a}, \tag{20}$$

where $\overline{a}$ is the trigger value of $a \equiv \frac{\widehat{A}}{A}$ associated with a technological upgrade. The value-matching condition thus reduces to a nonlinear equation in the relative technology a and the unknown constant B. The value-matching condition holds with equality when a equals the trigger value $\overline{a}$.

The second boundary condition requires that the value of the firm is maximized with respect to the choice of τ_j, the upgrade time. Formally, this requires[11]

$$\Psi_A\left(A_{\tau_j}, X_{\tau_j}, \widehat{A}_{\tau_j}\right) = \frac{X_{\tau_j}}{r-m} + (1-\phi) B X_{\tau_j} \left(\frac{\widehat{A}_{\tau_j}}{A_{\tau_j}}\right)^{\phi} = 0, \tag{21}$$

which implies that

$$\frac{1}{r-m} + (1-\phi) B a^{\phi} = 0 \quad \text{for } a = \overline{a}. \tag{22}$$

[11] This boundary condition can be expressed in a more familiar way by noting that the value of the firm, $\Psi\left(A_t, X_t, \widehat{A}_t\right)$, in equation (15) is proportional to X_t and is a linearly homogeneous function of A_t and $\widehat{A}_t$. Therefore, the value of the firm, $\Psi\left(A_t, X_t, \widehat{A}_t\right)$, can be rewritten as $\widehat{A}_t X_t \psi(a_t)$. The value matching condition is $\widehat{A}_t X_t \psi(1) - \widehat{A}_t X_t \psi(\overline{a}) = \widehat{A}_t X_t \theta$, which simplifies to $\psi(1) - \psi(\overline{a}) = \theta$. The second boundary condition is simply $\psi'(\overline{a}) = 0$. Equation (17) implies that $\psi(a) = \frac{a^{-1}}{r-m} + Ba^{\phi-1}$, so $\psi'(\overline{a}) = 0$ implies $-\frac{a^{-2}}{r-m} + (\phi-1) Ba^{\phi-2} = 0$, which is equivalent to equation (22).

The second boundary condition also reduces to a nonlinear equation that depends on the relative technology a and the unknown constant B. This condition holds when $a = \overline{a}$, that is, when an upgrade from the current value of A_t to the available frontier technology $\widehat{A}_t$ occurs. Solving equation (22) for B yields

$$B = \frac{\overline{a}^{-\phi}}{(\phi - 1)(r - m)} > 0, \tag{23}$$

where $\overline{a}$ is the threshold value of the relative technology a_t at which an upgrade is optimally undertaken. Substituting the expression for B from equation (23) into equation (20) yields a single nonlinear equation characterizing the threshold for optimal upgrades

$$g(a;\theta) \equiv a - 1 - \frac{1 - a^{1-\phi}}{\phi - 1} - a\theta(r - m) = 0 \text{ for } a = \overline{a}. \tag{24}$$

Notice that this expression depends only on the relative technology, $a \equiv \frac{\widehat{A}}{A}$, and constant parameters. Therefore, the relative technology a must have the same value whenever the firm upgrades its technology; we defined this boundary value above as $\overline{a}$, so $g(\overline{a};\theta) = 0$. It is straightforward to verify that $\overline{a} \geq 1$, with strict inequality when $\theta > 0$ and that $\frac{d\overline{a}}{d\theta} > 0$ when $\theta > 0$.[12] The firm upgrades A_t to the available technology when $\widehat{A}_t$ reaches a sufficiently high value; specifically, the firm upgrades when $\widehat{A}_t = \overline{a} \times A_t \geq A_t$. The size of the increase in A_t that is needed to trigger an upgrade, i.e., $\overline{a}$, is an increasing function of the fixed cost parameter θ.

Substituting equation (23) into the value of the firm in equation (17) yields

$$\Psi\left(A_t, X_t, \widehat{A}_t\right) = \frac{A_t X_t}{r - m} H\left(\frac{a_t}{\overline{a}}\right) > \frac{A_t X_t}{r - m}. \tag{25}$$

where

$$H\left(\frac{a_t}{\overline{a}}\right) \equiv 1 + \frac{1}{\phi - 1}\left(\frac{a_t}{\overline{a}}\right)^{\phi} > 1. \tag{26}$$

The value of the firm in equation (25) is the product of two terms: (1) the expected present value of operating profits evaluated along the path of no future upgrades,

[12] To see that $\overline{a} \geq 1$, use $\phi > 1$ and $(r - m)\theta < 1$ to note that $\lim_{a \to 0} g(a;\theta) > 0$, $g(1;\theta) = -\theta(r - m) < 0$, $\lim_{a \to \infty} g(a;\theta) > 0$, and $g''(a;\theta) > 0$. Thus $g(a;\theta)$ is a convex function of a with two distinct positive roots, $0 < \underline{a} < 1 < \overline{a}$, when $\theta > 0$, with $\frac{\partial g(\underline{a};\theta)}{\partial a} < 0$ and $\frac{\partial g(\overline{a};\theta)}{\partial a} > 0$. The smaller root, $\underline{a} < 1$, can be ruled out since it implies that the firm reduces the value of its technology when it changes technology. Since $\frac{\partial g(\overline{a};\theta)}{\partial \theta} = -(r - m)a < 0$, the implicit function theorem implies that $\frac{d\overline{a}}{d\theta} > 0$ when $\theta > 0$. When $\theta = 0$ there is a unique positive value of a that solves equation (24); specifically, $\overline{a} = 1$ when $\theta = 0$.

$\frac{A_t X_t}{r-m}$; and (2) $H\left(\frac{a_t}{\overline{a}}\right) > 1$, which captures the value of growth options associated with expected future technological upgrades. If the frontier technology were permanently unavailable, so that the firm would have to maintain the current level of technology, A_t, forever, then the value of the firm would simply be $\frac{A_t X_t}{r-m}$. However, since the firm has the option to adopt the frontier technology, the value of the firm exceeds $\frac{A_t X_t}{r-m}$ by the multiplicative factor $H\left(\frac{a_t}{\overline{a}}\right) > 1$. Since $H'\left(\frac{a_t}{\overline{a}}\right) > 0$, the value of the firm is increasing in the relative value of the frontier technology, $a_t \equiv \frac{\widehat{A}_t}{A_t}$.

As noted above, $\Psi\left(A_t, X_t, \widehat{A}_t\right)$ gives the value of a firm that never owns physical capital but rents the services of physical capital at each point of time. The value of a firm that owns physical capital K_t and technology A_t at time t is simply equal to the sum of $p_t K_t$ and $\Psi\left(A_t, X_t, \widehat{A}_t\right)$. Thus, letting V_t be the value of the firm at time t, equation (25) implies that

$$V_t = p_t K_t + \frac{A_t X_t}{r-m} H\left(\frac{a_t}{\overline{a}}\right). \tag{27}$$

We will relate the value of the firm to its cash flow by using the optimal capital stock in equation (8) and the definition of cash flow, C_t, in equation (12) to obtain

$$V_t = C_t \left[\frac{\gamma}{u_t} + \frac{1-\gamma}{r-m} H\left(\frac{a_t}{\overline{a}}\right)\right]. \tag{28}$$

The value of the firm is proportional to the optimal cash flow, C_t, with the time-varying factor of proportionality being a decreasing function of the user cost factor u_t and an increasing function of the relative technology a_t. When the relative technology is high, say near $\overline{a}$, the firm is likely to upgrade its technology in the near future. The prospect of an imminent upgrade is reflected in the current value of the firm V_t.

3 Tobin's Q

Tobin's Q is the ratio of the value of the firm, V_t, to the replacement cost of its capital, which comprises both physical capital, K_t, and the level of technology, A_t. We have already calculated the numerator of Tobin's Q, i.e., the value of the firm, in equation (27). Now we turn to the denominator of Tobin's Q, which is the replacement cost of the firm's capital stock. Because the physical capital stock, K_t, can be instantaneously adjusted, the replacement cost of the firm's physical capital at time t is $p_t K_t$, where p_t is the purchase/sale price per unit of physical capital. Because the firm adopts new technology only at discretely spaced points in time and the cost per unit of technology changes over time, the replacement cost of technology is different from its historical (i.e., original acquisition) cost. In effect, the adoption cost per unit of techology at date t is θX_t. For example, suppose that the technology in place at time t, A_t, was adopted at some earlier date $\tau < t$, when the frontier level of technology was $\widehat{A}_\tau$. In this case, $A_t = \widehat{A}_\tau$. The historical cost of the technology is $\theta \widehat{A}_\tau X_\tau$, or θX_τ per unit of technology. However, if the same level of technology were adopted at date t, the adoption cost would be $\theta \widehat{A}_\tau X_t$, or θX_t per unit of technology. Thus, the replacement of the firm's technology at date t is $\theta \widehat{A}_\tau X_t = \theta A_t X_t$.

The replacement cost of the total capital stock is the sum of $p_t K_t$, the replacement cost of the physical capital stock, and $\theta A_t X_t$, the replacement cost of the technology. Using the expression for the optimal capital stock in equation (8), and recalling from equation (13) that $c_t = \frac{u_t}{\gamma}$, we can write the replacement cost of the total capital stock as

$$p_t K_t + \theta A_t X_t = \left[1 + (1-\gamma)\,\theta c_t\right] p_t K_t. \tag{29}$$

The right hand side of equation (29) includes c_t, which is defined in equation (13) as the ratio of cash flow to $p_t K_t$, the replacement cost of the physical capital stock. However, since empirical work usually involves the ratio of cash flow to the replacement cost of the total capital stock, we will define c_t^* to be this ratio. Specifically,[13]

$$c_t^* \equiv \frac{C_t}{\left[1 + (1-\gamma)\,\theta c_t\right] p_t K_t} = \frac{c_t}{1 + (1-\gamma)\,\theta c_t}, \tag{30}$$

where the right hand side of equation (30) is obtained using equations (13) and (29).

Tobin's Q is the ratio of the value of the firm to the replacement cost of the total capital stock. To calculate Tobin's Q, divide the value of the firm in equation (27) by the replacement cost of the total capital stock in equation (29), and use equations (12), (13), (30) and the second equation in footnote 13 to obtain

$$\begin{aligned} Q_t &\equiv \frac{V_t}{\left[1 + (1-\gamma)\,\theta c_t\right] p_t K_t} \\ &= 1 + (1-\gamma)\left[H\left(\frac{a_t}{\bar{a}}\right) - (r-m)\,\theta\right]\frac{c_t^*}{r-m} > 1, \end{aligned} \tag{31}$$

where the inequality follows from $1-\gamma > 0$, $H() > 1$, and our previous assumptions that $r - m > 0$ and $(r-m)\,\theta < 1$. Tobin's Q exceeds 1 because of the rents represented by the operating profits, π_t. It is an increasing function of the value

[13] In subsequent derivations, it is helpful to note that equation (30) implies $c_t = \frac{c_t^*}{1-(1-\gamma)\theta c_t^*}$, which further implies that $\frac{1}{1+(1-\gamma)\theta c_t} = 1 - (1-\gamma)\,\theta c_t^*$.

of the frontier technology relative to the installed technology, measured by a_t. In addition, Tobin's Q is an increasing linear function of c_t^*.

Technological upgrades occur when the level of the frontier technology, $\widehat{A}_t$, becomes high enough relative to the installed technology, A_t, to compensate for the cost of upgrading to the frontier. The ratio of the frontier technology to the installed technology, a_t, is a sufficient statistic for the upgrade decision. If a_t is below the threshold value, $\overline{a}$, the firm does not upgrade. When a_t reaches $\overline{a}$, the firm upgrades its technology to the frontier. The frontier technology, and hence a_t, are unobservable to an outside observer. Tobin's Q, however, provides an observable indicator of a_t that can help predict the timing of technology upgrades and the associated purchase of physical capital. Equation (31) shows that Q_t is an increasing function of a_t. Therefore, since the expected time until the next upgrade is a decreasing function of a_t, the expected time until the next upgrade is decreasing in Q_t. That is, high values of Tobin's Q predict imminent technology upgrades and the associated investment in physical capital.

4 Investment

The firm's capital investment expenditures consist of what we will call *investment gulps*[14] and *continuous investment*. Investment gulps take place at the points of time at which the firm upgrades its technology to the frontier technolgy. When the firm adopts the frontier level of technology, its level of technology jumps upward, and hence the marginal product of capital jumps upward. As a result of the jump in the marginal product of capital, the optimal capital stock jumps upward, so the firm takes of "gulp" of physical investment in addition to making an expenditure on technology. During intervals of time between consecutive technology upgrades, the optimal level of capital varies continuously over time. The firm continuously maintains its physical capital stock equal to its optimal level by undertaking continuous investment during these intervals of time. In this section, we calculate optimal investment gulps and the associated expenditures to upgrade technology and optimal continuous investment.

First consider investment gulps. Recall that the firm upgrades its technology to the frontier technology at optimally chosen dates τ_j, $j = 0, 1, 2, ...$ The increase in

[14]Hindy and Huang (1993) use the term "gulps" of consumption to describe jumps in the cumulative stock of consumption. We borrow their term to apply to jumps in the stock of capital, which is the cumulation of past (net) investment.

the physical capital stock that accompanies the upgrade at time τ_j is calculated using the expression for the optimal capital stock in equation (8) to obtain[15]

$$\frac{K_{\tau_j^+}}{K_{\tau_j^-}} = \frac{A_{\tau_j^+}}{A_{\tau_j^-}} = \frac{\widehat{A}_{\tau_j}}{\widehat{A}_{\tau_{j-1}}} = \overline{a}. \tag{32}$$

When the firm upgrades its technology, its physical capital stock, K_t, jumps instantly by a factor $\overline{a}$. Total investment expenditures at the instant of an upgrade comprise the gulp of physical capital, $(\overline{a}-1)\,p_{\tau_j}K_{\tau_j^-}$, and the expenditure to upgrade the technology, $\theta\widehat{A}_{\tau_j}X_{\tau_j}$. Equations (8) and (13) imply that the total investment expenditures at the instant of an upgrade are

$$(\overline{a}-1)\,p_{\tau_j}K_{\tau_j}^- + \theta\widehat{A}_{\tau_j}X_{\tau_j} = \left[\overline{a}-1+(1-\gamma)\,\overline{a}\theta c_{\tau_j}\right]p_{\tau_j}K_{\tau_j^-}. \tag{33}$$

Let ι_t denote the ratio of total investment expenditures at time t to the replacement cost of the total capital stock at time t. Thus ι_{τ_j} is the ratio of total investment expenditures at the instant of the upgrade at time τ_j to the replacement cost of the total capital stock. We calculate this ratio by dividing equation (33) by the replacement cost of the capital stock immediately before the upgrade, which, from equation (29) is $\left[1+(1-\gamma)\,\theta c_{\tau_j}\right]p_{\tau_j}K_{\tau_j^-}$, and use the the second equation in footnote 13 to obtain

$$\iota_{\tau_j} = \overline{a} - \frac{1}{1+(1-\gamma)\,\theta c_{\tau_j}} = \overline{a}-1+(1-\gamma)\,\theta c^*_{\tau_j}. \tag{34}$$

Investment expenditures at the instant when technology is upgraded are an increasing linear function of normalized cash flow, c^*_t.

Continuous investment, which takes place during intervals of time between consecutive technology upgrades, consists only of investment in physical capital because technology remains constant during intervals of time between consecutive upgrades. Investment in physical capital is the sum of net investment expenditures, $p_t dK_t$, and depreciation, $p_t\delta_t K_t dt$. Net investment in physical capital is obtained by calculating the change in the optimal physical capital stock by applying Ito's Lemma to the expression for the optimal capital stock in equation (8). During an interval of time between consecutive upgrades, $dA_t = 0$ so

$$\frac{dK_t}{K_t} = \frac{dX_t}{X_t} - \frac{du_t}{u_t} + (\sigma_u^2 - \rho_{Xu}s\sigma_u - \mu_p)dt. \tag{35}$$

[15] The superscript "+" on τ_j denotes the instant of time immediately following τ_j, and the superscript "−" denotes the instant of time immediately preceding τ_j.

Use equation (35), along with equation (11) and the assumption that $\frac{du_t}{u_t} = \sigma_u dz_u$, to calculate gross investment in physical capital at any time between technology upgrades as

$$p_t dK_t + p_t \delta_t K_t dt = \left[(\delta_t + m + \sigma_u^2 - \rho_{Xu} s\sigma_u - \mu_p) dt + s dz_X - \sigma_u dz_u \right] p_t K_t. \quad (36)$$

Next calculate the ratio of investment expenditures to the total capital stock by dividing equation (36) by the replacement cost of the capital stock, $[1 + (1-\gamma)\,\theta c_t]\, p_t K_t$, using equation (13) to substitute $\gamma c_t - r$ for $\delta_t - \mu_p$, and rearranging, using the second equation in footnote 13, to obtain[16]

$$\iota_t = \left(\left[\gamma + (1-\gamma)\,\theta\Gamma \right] c_t^* - \Gamma \right) dt + \frac{c_t^*}{c_t} \left[s dz_X - \sigma_u dz_u \right], \quad (37)$$

where $\Gamma \equiv r - m - \sigma_u^2 + \rho_{Xu} s\sigma_u$ is constant. The parameter Γ will be positive if and only if the replacement cost of the physical capital stock, $p_t K_t$, for a firm that never upgrades its technology, grows (on average) at a rate that is less than the discount rate r.[17] Henceforth, we confine attention to this case, so that $\Gamma > 0$.

Over finite intervals of time, investment is composed of continuous investment and possibly also gulps of investment and the associated expenditures to upgrade the technology. Now consider regressing the investment-capital ratio during an interval of time on variables that are known as of the beginning of the interval, such as Tobin's Q and cash flow per unit of capital during the preceding period. Equation (37) shows that continuous investment during an interval of time is the sum of a component that is known at the beginning of the interval, and a component that is uncorrelated with information at the beginning of the interval. Specifically, the drift term in equation (37), $\left(\left[\gamma + (1-\gamma)\,\theta\Gamma \right] c_t^* - \Gamma \right) dt$, is a linear function of normalized cash flow,

[16] The notation for the investment-capital ratio in this equation is non-standard in the literature using continuous-time stochastic models, but is more familiar to readers of the empirical investment literature. The right hand side of equation (37) contains innovations to Brownian motions, dz_X and dz_u, which have infinite variation, so a more standard continuous-time notation for the left hand side of this equation would be $d\iota$ rather than ι. Nevertheless, we use ι to represent investment-capital ratio.

[17] Equation (8) implies that the replacement cost of the physical capital stock is $p_t K_t = \frac{A_t X_t}{u_t} \frac{\gamma}{1-\gamma}$. If the firm never upgrades its technology, then A_t is constant. Ito's Lemma implies that $\frac{d(p_t K_t)}{p_t K_t} = \frac{dX_t}{X_t} - \frac{du_t}{u_t} - \frac{dX_t}{X_t}\frac{du_t}{u_t} + \left(\frac{du_t}{u_t^2} \right)^2$. Use equation (11) and $\frac{du_t}{u_t} = \sigma_u dz_u$ to obtain $\frac{d(p_t K_t)}{p_t K_t} = \left(m - \rho_{Xu} s\sigma_u + \sigma_u^2 \right) dt + s dz_X - \sigma_u dz_u$. Therefore, if the firm never upgrades its technology, the drift in $p_t K_t$ is $m - \rho_{Xu} s\sigma_u + \sigma_u^2$. If this drift is less than the discount rate r, then $\Gamma > 0$.

c_t^*. To the extent that c_t^* is positively serially correlated, this component of continuous investment during an interval of time will be positively correlated with normalized cash flow in the previous interval. The innovation in equation (37), $\frac{c_t^*}{c_t}[sdz_X - \sigma_u dz_u]$, is uncorrelated with any information available before the beginning of the interval. Interestingly, continuous investment is independent of Tobin's Q, given c_t^*. Thus, in a regression of continuous investment on Tobin's Q and normalized cash flow in the previous period, we would expect a zero coefficient on Q and a positive coefficient on lagged normalized cash flow.

It might appear from equation (34) that the investment-capital ratio associated with upgrades is also a linear function of c_t^* and independent of Q. While it is true that the *magnitude* of the investment-capital ratio at upgrade dates τ_j is independent of Q (for given c_t^*), the probability that an investment gulp will occur during an interval of time is an increasing function of Q at the beginning of the interval. As we have discussed, the relative technology a_t is unobservable, but (see equation 31) Q_t is an increasing of a_t. Thus, a high value of Q indicates that a_t is near the trigger value $\overline{a}$, and thus that an upgrade in the near future is likely. Therefore, the value of Q at the beginning of an interval can help indicate that an investment gulp will take place during that interval. Hence, both Tobin's Q at the beginning of the interval and normalized cash flow from the previous interval will help to explain investment expenditures arising from investment gulps and the associated technology upgrades.

Discrete-time data on investment expenditure by firms contain both continuous investment and investment gulps with the associated technology upgrades. For the reasons we have just discussed, the investment-capital ratio during an interval of time should be positively related to Tobin's Q at the beginning of the interval and to normalized cash flow in the previous period. The next step is to generate values of the investment-capital ratio, Tobin's Q and normalized cash flow from the model and use these simulated data to run regressions of the investment-capital ratio on Tobin's Q and lagged normalized cash flow.

5 Investment, Tobin's Q, and Cash Flow: Simulation Results

In this section we quantitatively examine the effects of Q and normalized cash flow on the investment-capital ratio. We simulate the model by first choosing a baseline set of parameters. We solve for the optimal upgrade threshold, $\overline{a}$, given these parameters and then, for each firm independently, generate a quarterly series of normally-distributed values for each of the random variables, u, $\widehat{A}$, and Y, in the model.[18] We generate a simulated panel of data, corresponding to 500 firms over 80 quarters (roughly the size of the Compustat data set often used in empirical work). To generate heterogeneity among otherwise identical firms, we draw the initial value relative technology, a_t, for each firm from the steady-state distribution of a_t.[19] Using the solution for $\overline{a}$ and the exogenous path of $\widehat{A}$, we solve for optimal upgrades and the path of the installed technology, A. We also calculate the composite variable X to summarize the non-technology components of operating profits, and then solve for the variables of interest: the physical capital stock, the level of technology, investment, cash flow, firm value, and Tobin's Q.

5.1 Features of the Model

Table 1 reports basic features of the model under various parameter configurations. The first row (labelled "none") reports the features for the baseline parameters; the remaining rows report the features of the model as we change one parameter value at a time from the baseline. In the baseline, the value of $\overline{a}$ is 1.5632, which means that a firm will maintain its currrently installed level of technology, A_t, until the frontier level of technology, $\widehat{A}_t$, is 56.32% more productive than the currently installed technology. Given the geometric Brownian motion for $\widehat{A}_t$ in equation (14), this value of $\overline{a}$ implies that the mean time between successive technology upgrades (shown in the second column) is 15.5383 years. However, the distribution of the time between successive upgrades is, evidently, quite skewed. The median time between successive upgrades

[18] Instead of generating one normally distributed value per quarter for each variable, we divide each quarter into 60 intervals and generate a normally distributed shock for each interval. The reason for using finer intervals of time is to avoid the following problem: Suppose that during a quarter the continuous path of a_t rises above the trigger $\overline{a}$ and then returns below $\overline{a}$ and remains below $\overline{a}$ at the end of the quarter. If we viewed the path of a_t only at the end of each quarter, we would have missed the fact that a_t reached the trigger $\overline{a}$ during the quarter, and thus we would have missed the investment gulp and the associated expenditure to upgrade the technology. Dividing each quarter into 60 intervals substantially mitigates this potential problem.

[19] We limit our simulation to ex ante identical firms in order to explore the ex post variation generated by the mechanisms of our model, rather than imposing a priori hetergeneity on the simulated sample. We should also note that variation in firm scale would not affect our findings, since the model is homogeneous and thus scale-free.

Table 1: Features of the Model					
		Time between upgrades		Q at upgrade	
deviation from baseline	$\overline{a}$	Mean	Median	Before	After
none	1.5632	15.5383	1.8883	4.2891	3.1282
$\theta = 0.25$	1.3502	10.4432	0.8903	4.3715	3.5113
$\sigma = 0.30$	1.5480	5.1410	2.4025	3.5592	2.6775
$\mu_Y = 0.010$	1.5566	15.3913	1.8550	4.8083	3.4708
$\sigma_Y = 0.10$	1.5632	15.5383	1.8883	4.2891	3.1282
$\sigma_u = 0.02$	1.5881	16.0875	2.0151	3.1374	2.3696
$\rho_{Yu} = 0.2$	1.5726	15.7470	1.9361	3.7426	2.7680
$\rho_{Y\widehat{A}} = 0.2$	1.5874	16.0725	2.0116	7.0553	4.8384
Baseline parameters: $r = 0.15, \gamma = 0.75, \theta = 0.5, \mu = 0.13, \sigma = 0.45$,					
$\mu_Y = 0.005, \sigma_Y = 0.20, \sigma_u = 0.06, \mu_p = 0.015$,and $\rho_{Y\widehat{A}} = \rho_{u\widehat{A}} = \rho_{Yu} = 0$.					
These values imply $m = -0.0184$ and $s = 0.2691$.					
The calculations of Q before and after upgrade use equation (31),					
assuming $\delta = 0.1$, and $a = \overline{a}$ (before adjustment) and $a = 1$ (after adjustment).					
Parameters are expressed in annual terms where appropriate.					

Table 1: Table Caption

(shown in the third column) is only 1.8883 years. The mean time between upgrades is so much larger than the median time between upgrades because the frontier level of technology, $\widehat{A}_t$, can decline and take a very long excursion before rising enough to trigger an upgrade to technology.

The final two columns of Table 1 report the value of Q immediately before and after upgrades. As the frontier technology $\widehat{A}_t$ increases toward its trigger, $\overline{a} \times A_t$, the value of Q increases as the prospect of a technology upgrade draws near. When Q reaches 4.2891 (shown in the fourth column, assuming that the current value of the depreciation rate, δ_t, is 0.1), the firm upgrades its technology and takes a gulp of physical capital. This jump in the firm's total capital stock causes Q to jump downward to 3.1282 (shown in the fifth column).

The second row of Table 1 shows the effect of reducing the fixed cost of upgrading, θ, to 0.25 from its baseline value of 0.5. Not surprisingly, reducing the cost of upgrading reduces $\overline{a}$, the threshold for upgrading, which reduces both the mean and median times between upgrades. The reduction in the cost of upgrading raises the value of the option to upgrade and thus increases the value of Q immediately before and after upgrades, as shown in the final two columns.

The third row of the table shows the effect of reducing the instantaneous standard deviation, σ, of the geometric Brownian motion for $\widehat{A}_t$ in equation (14) to 0.30 from its value of 0.45 in the baseline. The reduction in the standard deviation reduces the value of $\overline{a}$ slightly. Interestingly, the reduction in σ causes the mean and median times between successive upgrades to move in opposite directions. The mean time between upgrades falls by about two thirds because the reduction in σ reduces the importance of long excursions of $\widehat{A}_t$ before $\widehat{A}_t$ rises enough to trigger an upgrade. However, the reduction in σ increases the median time between successive upgrades to 2.4025 years from 1.8883 years in the baseline. The reduction in σ reduces the value of the option to upgrade, which lowers the value of Q both immediately before and immediately after upgrades.

The fourth and fifth rows show the effects of changing the drift, μ_Y, and instantaneous standard deviation, σ_Y, of the geometric Brownian motion for the demand parameter Y_t. This demand parameter operates through X_t, which summarizes the non-technology factors affecting operating profits. The fourth row shows the effect of increasing μ_Y to 0.010 from its value of 0.005 in the baseline. The increase in μ_Y increases m, which is the drift in X_t. As shown in the fourth row, the threshold $\overline{a}$ falls slightly, thereby causing small decreases in the mean and median times between successive upgrades. The increase in m reduces the discount factor $r - m$, thereby increasing the value of Q immediately before and immediately after upgrades. The fifth row of the table shows the effect of reducing σ_Y to 0.10 from its value of 0.20 in the baseline. All five entries in this row are identical to the corresponding entries in the baseline. This invariance with respect to σ_Y of $\overline{a}$, mean and median times between consecutive upgrades, and Q immediately before and after upgrades is an analytic feature of the special case in which $\rho_{Yu} = \rho_{X\widehat{A}} = 0$.[20]

The sixth row shows the effects of reducing the standard deviation of the user cost factor, σ_u, to 0.02 from its value of 0.06 in the baseline. In the model, σ_u operates

[20] When $\rho_{Yu} = 0$, the parameter m does not depend on σ_Y. When $\rho_{X\widehat{A}} = 0$, the parameter s (which depends on σ_Y) does not appear in the quadratic equation in equation (18). Therefore, when $\rho_{Yu} = \rho_{X\widehat{A}} = 0$, the roots of the quadratic equation in equation (18) are invariant to σ_Y. Since m and the root ϕ are invariant to σ_Y, equation (24) indicates that $\overline{a}$ is invariant to σ_Y. With an unchanged $\overline{a}$, and an unchanged process for $\widehat{A}_t$, the times between successive upgrades are unchanged. Also, with unchanged m and $\overline{a}$, equation (31) indicates that Q immediately before and after upgrades is unchanged.

through its effect on the parameters m and s of the geometric Brownian motion for X_t, which summarizes the non-technology factors affecting operating profits. Specifically, in the baseline case in which $\rho_{Y\widehat{A}} = \rho_{u\widehat{A}} = \rho_{Yu} = 0$, a decrease in σ_u decreases both the drift, m, and the instantaneous standard deviation, s. A reduction in the growth rate of X increases the effective discount rate, $r-m$, applied by the firm. As shown in the table, the reduction in σ_u increases the threshold $\overline{a}$, and increases both the mean and median times between successive upgrades. The reduction in σ_u, working through the reductions in m and s, substantially reduces the value of Q both immediately before and immediately after an upgrade.

The final two rows of the table allow for correlations among stochastic processes in the model. The seventh row introduces a positive correlation between the level of demand for the firm's product, measured by Y_t, and the user cost factor u_t. This positive correlation increases the threshold $\overline{a}$, which increases both the mean and median times between successive upgrades. From the viewpoint of the firm, an increase in demand, Y_t, is a favorable event but an increase in the user cost factor, u_t, is an unfavorable event. The positive correlation of a favorable event and an unfavorable event reduces the option value of the firm, thereby reducing the value of Q immediately before and immediately after upgrades.

The final row of the table reports the effects of a positive correlation between demand, Y_t, and the frontier technology, $\widehat{A}_t$. This correlation increases the threshold $\overline{a}$ and increases both the mean and median times between successive upgrades. Since increases in Y_t and $\widehat{A}_t$ are both favorable events, their positive correlation increases the option value of the firm, which increases the of Q immediately before and immediately after upgrades. Among the eight parameter configurations respresented in Table 1, the configuration in the final row represents the case in which growth options are the most important, as evidenced by the highest values of Q.

5.2 Investment Regressions

For each of the parameter configurations in Table 1, we generate an artifical set of panel data for 500 firms for a sample period of 80 quarters. We then run various investment regressions on the generated panel. We repeat this process 100 times. Table 2 reports the average values of the estimated regression coefficients on Tobin's Q and the cash flow-to-capital ratio, c^*, and their average standard errors (reported in parentheses) across the 100 replications. The first four columns report results for (simulated) quarterly data and the final four columns report results for (simulated) annual data. We will describe the construction of the quarterly data from the underlying intervals, which are 1/240 of a year in length. The creation of annual data is done in the same manner.

For each interval, gross investment is calculated as the sum of three terms: (1) the net increase in the physical capital stock multiplied by the purchase price of physical capital; (2) the amount of physical capital lost to depreciation multiplied by the price of physical capital; and (3) the upgrade expenditure θAX, whenever the firm upgrades its technology during the interval. Gross investment for quarter t, I_t, is calculated by summing these three terms over the 60 intervals in the quarter, and then multiplying by 4 to express investment at an annual rate. The investment-capital ratio in quarter t, ι_t, is calculated as I_t divided by the replacement cost of the total capital stock, $[1+(1-\gamma)\,\theta c]\,pK$ from equation (29), in the final interval of the previous quarter. For quarter t, the value of Tobin's Q, Q_t, is the value of Q in the final interval of the quarter. Cash flow in quarter t is calculated as the average value of cash flow (cash flow for each interval is expressed at annual rates) over the 60 intervals in the quarter. The normalized cash flow in quarter t, c_t^*, is cash flow during the entire quarter (expressed at annual rates) divided by the replacement cost of the total capital stock, $[1+(1-\gamma)\,\theta c]\,pK$, in the final interval of the quarter.

The first two columns, labeled "univariate", report the results of univariate regressions of the investment rate, ι_t, on Q_{t-1} and c_{t-1}^*, respectively. The first column of Table 2 reports the results of regressing ι_t on Q_{t-1} alone. In all cases, the estimated coefficient is positive, ranging from 0.093 to 0.333, and is at least three times the size of its standard error. The second column reports the results of regressing ι_t on c_{t-1}^* alone. The estimated coefficients range from 0.313 to 0.717 and are greater than 3 times their estimated standard errors. In all cases but one (the exception is the case in which $\sigma_u = 0.02$) the estimated coefficient on c_{t-1}^* is between 0.60 and 0.72 and is at least ten times the size of the estimated standard error. Recall from equation (37) that for continuous investment, the coefficient of ι on c^* is $\gamma + (1-\gamma)\,\theta\Gamma$, where $\Gamma \equiv r - m - \sigma_u^2 + \rho_{Xu} s \sigma_u$. In general, $(1-\gamma)\,\theta\Gamma$ is small, so this coefficient will be slightly larger than $\gamma = 0.75$. In the baseline case,[21] $\Gamma = 0.1540$, so with $\gamma = 0.75$ and $\theta = 0.5$, the coefficient on c^*, $\gamma + (1-\gamma)\,\theta\Gamma$, is $0.75 + (1-0.75)\,(0.5)\,(0.1540) = 0.7692$. Most of the estimated coefficients in the second column are close to, but less than,

[21] In the baseline, $r = 0.15$, $m = -0.0184$, $s = 0.2691$, $\sigma_u = 0.06$, and $\rho_{Xu} = -0.6690$, so $\Gamma = 0.1540$.

0.75. Recall that the expression for continuous investment in equation (37) holds instantaneously, but our regressions on run on time aggregated data, where investment in period t is regressed on cashflow in period $t-1$, which would tend to produce a lower estimated coefficient. To test this explanation, we again divided the year into 240 intervals and created "semi-monthly" data by aggregating data over 10 intervals into 24 half-month periods per year. Using the baseline parameter values for this semi-monthly data yields a coefficient on cash flow of 0.8111. Consistent with finding of a higher cashflow coefficient for the finer observation interval is that moving from quarterly observations to annual observations uniformly reduces (in most cases, by about one half) the estimated cash flow coefficients.

When Q and cash flow are simultaneously included in the investment regressions (reported in columns 3 and 4 of the results in Table 2), the coefficient on Q is virtually unchanged from the univariate regressions and the coefficient on c^* uniformly falls relative to the univariate regressions, in most cases by a substantial amount. In one case, in which the frontier technology is much less volatile ($\sigma = 0.3$) so growth options are less important, the cash flow coefficient even becomes negative. In all of the other cases, the coefficient on c^* ranges from 0.216 to 0.340 and is at least twice its estimated standard error. The coefficients on c^* in the multiple regressions are smaller than the coefficients of c^* in the corresponding univariate regressions because Q and c^* are correlated. However, this correlation does not lead to much difference between the coefficient on Q in univariate and multiple regressions. The reason for the asymmetry between the large effects on the coefficient on c^*, and the tiny effect on the coefficients on Q when moving from univariate to multiple regressions is that the variance of Q is substantially larger than the variance of c^* and substantially larger than the covariance of Q and c^*.[22] When multiple regressions appear in the empirical literature, the estimated coefficient on Q is typically very small and the estimated coefficient on cash flow is typically much larger. This pattern is found in the bottom

[22] Let σ_{QQ} be the variance of Q, $\sigma_{c^*c^*}$ be the variance of c^*, σ_{Qc^*} be the covariance of Q and c^*, $\sigma_{Q\iota}$ be the covariance of Q and ι, and $\sigma_{c^*\iota}$ be the covariance of c^* and ι. Then $b_Q \equiv \sigma_{Q\iota}/\sigma_{QQ}$ is the coefficient on Q in a univariate regression of ι on Q, and $b_{c^*} \equiv \sigma_{c^*\iota}/\sigma_{c^*c^*}$ is the coefficient on c^* in a univariate regression of ι on c^*. In a multiple regression of ι on Q and c^*, the coefficient on Q is β_Q and the coefficient on c^* is β_{c^*}. It can be shown that $\beta_Q = \left[b_Q - \left(\sigma_{Qc^*}/\sigma_{QQ}\right) b_{c^*}\right] / \left(1-\rho^2\right)$ and $\beta_{c^*} = \left[b_{c^*} - \left(\sigma_{Qc^*}/\sigma_{c^*c^*}\right) b_Q\right] / \left(1-\rho^2\right)$, where $\rho^2 \equiv \sigma^2_{Qc^*} / \left(\sigma_{QQ}\sigma_{c^*c^*}\right)$ is the square of the covariance of Q and c^*. Since σ_{QQ}, the variance of Q, is much greater than $\sigma_{c^*c^*}$, the variance of the normalized cash flow c^*, β_{c^*} is substantially smaller than b_{c^*} while β_Q does not differ much from b_Q.

row of Table 2 ($\rho_{Y\widehat{A}} = 0.2$) where growth options are important, as evidenced by the high values of Q in Table 1.

We examine the impact of time aggregation in the final four columns of Table 2, which report the results of regressions run on data aggregated to annual frequency. Time aggregation has very little effect on the estimated coefficients on Q. For both univariate and multiple regressions, the coefficient on Q is smaller for annual data than for quarterly data (expressed at annual rates), but only very slightly smaller. The major impact of time aggregation is on the coefficient on cash flow. For univariate regressions, the coefficient on c^* for annual data is about one half the size of the corresponding coefficient for quarterly data, in all cases except one; for these cases, the estimated coefficient is at least six times the size of its estimated standard error. For the exceptional case ($\sigma_u = 0.02$), the coefficient on cash flow is negative for annual data. For multiple regressions, the coefficient on c^* again falls as we move from quarterly data to annual data; in the the exceptional case, $\sigma = 0.3$ and the coefficient on c^* becomes negative in the annual regressions. As we noted for quarterly data, in multiple regressions on annual data, c^* has a larger coefficient than does Q in the final row of the table, in which growth options are most important. Again, this finding is consistent with the findings in the empirical literature.

Table 2: Estimated Coefficients on Tobin's Q and Cash Flow								
deviation from baseline:	Quarterly				Annual			
	univariate		multiple		univariate		multiple	
	Q	c^*	Q	c^*	Q	c^*	Q	c^*
none	0.184	0.648	0.178	0.256	0.172	0.332	0.169	0.169
	(0.005)	(0.052)	(0.005)	(0.053)	(0.006)	(0.046)	(0.006)	(0.044)
$\theta = 0.25$	0.148	0.603	0.142	0.276	0.145	0.310	0.142	0.172
	(0.005)	(0.049)	(0.005)	(0.050)	(0.005)	(0.044)	(0.005)	(0.043)
$\sigma = 0.30$	0.333	0.717	0.363	-0.602	0.262	0.353	0.271	-0.184
	(0.007)	(0.060)	(0.008)	(0.065)	(0.008)	(0.053)	(0.008)	(0.053)
$\mu_Y = 0.01$	0.154	0.647	0.149	0.283	0.145	0.329	0.142	0.181
	(0.004)	(0.052)	(0.004)	(0.053)	(0.005)	(0.046)	(0.005)	(0.045)
$\sigma_Y = 0.10$	0.184	0.685	0.178	0.269	0.172	0.393	0.169	0.201
	(0.004)	(0.047)	(0.005)	(0.047)	(0.005)	(0.044)	(0.005)	(0.042)
$\sigma_u = 0.02$	0.316	0.313	0.316	0.216	0.279	-0.153	0.280	0.047
	(0.007)	(0.092)	(0.007)	(0.090)	(0.008)	(0.064)	(0.008)	(0.061)
$\rho_{Yu} = 0.2$	0.230	0.672	0.224	0.227	0.212	0.359	0.209	0.164
	(0.006)	(0.049)	(0.006)	(0.050)	(0.007)	(0.045)	(0.007)	(0.043)
$\rho_{Y\widehat{A}} = 0.2$	0.093	0.631	0.090	0.340	0.088	0.301	0.087	0.213
	(0.003)	(0.053)	(0.003)	(0.053)	(0.003)	(0.047)	(0.003)	(0.046)
Baseline parameters: $r = 0.15, \gamma = 0.75, \theta = 0.5, \mu = 0.13, \sigma = 0.45,$								
$\mu_Y = 0.005, \sigma_Y = 0.20, \sigma_u = 0.06, \mu_p = 0.015,$and $\rho_{Y\widehat{A}} = \rho_{u\widehat{A}} = \rho_{Yu} = 0.$								
These values imply $m = -0.0184$ and $s = 0.2691$.								

6 Variance Bounds

Equity prices empirically exhibit "excess volatility" relative to the dividends on which they are a claim. This observation was formalized by Leroy and Porter (1981) and most provocatively by Shiller (1981), though assuming that equity prices and dividends were trend stationary. West (1988) showed that equities were indeed more volatile than justified by a dividend-discount model even allowing for non-stationarity. The model examined in this paper could, in principle, address this puzzle, since growth options generate variation in the value of the firm that is unrelated to the firm's current profitability. This variation might induce "excess volatility" in the firm's valuation compared to its underlying cash flows.

Two issues must be addressed in evaluating this potential explanation of excess volatility. First, our model produces excess volatility in the firm's value during the intervals of continuous investment between consecutive upgrades, but the opposite occurs at the time of a technological upgrade. Recall from equation (28) that the value of the firm is proportional to its current cash flow, but the proportionality factor, $\frac{\gamma}{u_t} + \frac{1-\gamma}{r-m} H\left(\frac{a_t}{\bar{a}}\right)$, varies with the user cost factor, u_t, and the state of the technological frontier, a_t.[23] These additional sources of variation may contribute to apparent excess volatility. The variance of firm value, V, depends on the variance of cash flow, C, as well as the variances of the user cost factor, u, and relative technology, a, and importantly, the covariances among these processes. While the variances of u and a increase the volatility of V compared to C, the covariances can, depending on their sign, either reinforce this effect or have an opposing effect. Even when the underlying stochastic processes are mutually independent, there are two sources of correlation that can affect the volatility of V. First, the user cost factor, u, is negatively correlated with cash flow, $C = AX/(1-\gamma)$ (even though it is positively correlated with cash flow *per unit of capital*) because the composite variable X depends inversely on the user cost factor, which induces a negative correlation between X and u.[24] Since C is negatively correlated with u, it is positively correlated with $1/u$, which according to equation (28), tends to increase the volatility of V. Working in the opposite

[23] The literature on excess volatility has argued that variation in discount rates is not sufficient to explain the magnitude of the excess volatility in equity valuations compared to dividends. These arguments could apply to variation in r and u_t in the current model, but do not apply to variation in a_t.

[24] As stated in footnote 4, $s\rho_{Xu} = \rho_{Yu}\sigma_Y - \frac{\gamma}{1-\gamma}\sigma_u$. Therefore, if $\rho_{Yu} = \rho_{Y\widehat{A}} = \rho_{\widehat{A}u} = 0$, the correlation ρ_{Xu} is negative.

direction is the comovement of C and A at the time of an upgrade. At any instant at which the firm upgrades its technology, the user cost factor remains unchanged, but cash flow jumps *upward* with the discrete increase in the installed technology, A, and in the physical capital stock, K, while a jumps *downward* from $\overline{a}$ to one. Thus, after aggregating over regimes of continuous investment and upgrades, it is not clear that the volatility of the firm's value will exceed the volatility of its cash flow. Greater volatility of firm value relative to its cash flow should be observed during continuous investment regimes (if the underlying stochastic processes are mutually independent), but could be reversed by the negative covariance of cash flow and the relative technology at upgrade times.

The second important issue to be confronted when assessing variance bounds in this model is that the model generates neither stock prices nor dividends, which are usually the empirically measured variables in the excess volatility literature. The model is set in perfect markets, so neither capital structure nor dividends are determined (since neither affects the value of the firm). This issue cannot be explicitly addressed without leaving the perfect markets paradigm, which is beyond the scope of the paper (and also outside the spirit of the current excercise–to examine the implications of growth options without other market imperfections). In order to examine volatility bounds in our model, we assume that the firm has no debt, and hence the value of the firm, V, is equal to its equity value. Our calculations thus provide a floor on the equity variance, since leverage would only increase the variance of the value of equity. If dividends are smoother than cash flows, then the variance of cash flows that we calculate provides an upper bound for the dividend variance.[25] In this case, the ratio of the variance of V to the variance of C (in log differences) that we calculate is a lower bound on the variance ratio for stock prices versus dividends.

Since C and V are nonstationary, we follow West (1988) and take differences to induce stationarity. In West's model, arithmetic differences were assumed sufficient to induce stationarity, while in our structure (with geometric Brownian motion), log differences are required. Table 3 reports the standard deviation of the log change in cash flow, $\Delta \ln C$, the standard deviation of the log change in value, $\Delta \ln V$, and the variance ratio, $\frac{var(\Delta \ln V)}{var(\Delta \ln C)}$. The volatilities of quarterly changes are reported in the

[25] If dividends are literally a smoother version of cash flows (and both must integrate to the same value), as in Lintner (1956), then the variance of cash flow should exceed the variance of dividends. Recent work, such as Brav, et al (2003), tends to confirm that dividends are smoothed relative to cash flows.

first three columns of results, and the volatilities of annual changes are reported in the final three columns of results. Each cell in Table 3 contains two entries. The top entry reports the relevant statistic from the model with optimally chosen technology upgrades; the bottom entry, which appears in parentheses, reports the corresponding statistic for a more conventional model of productivity shocks in which the level of productivity A_t follows an exogenous stochastic process. We model this exogenous stochastic process as a geometric Brownian motion. In fact, we simply set A_t equal to the exogenous stochastic variable $\widehat{A}_t$ at all times, and ignore any upgrade decisions or upgrade costs. In this case, cash flow is simply

$$C_t = \frac{\widehat{A}_t X_t}{1-\gamma}, \text{ if } A_t \equiv \widehat{A}_t \tag{38}$$

from equation (12). The value of the firm is[26]

$$V_t = \left(\frac{\gamma}{u_t} + \frac{1-\gamma}{r - m - \mu - \rho_{X\widehat{A}} s\sigma} \right) C_t, \text{ if } A_t \equiv \widehat{A}_t. \tag{39}$$

For all of the parameter configurations in Table 3, for both quarterly and annual data, and for both the model with optimally chosen technology upgrades and the conventional model with exogenous technology, A_t, the variance ratio, $\frac{var(\Delta \ln V)}{var(\Delta \ln C)}$, exceeds one. The largest values of the variance ratio occur for the model with optimally chosen technology upgrades for the parameter configuration at the bottom of the table, where growth options are the most important. The increase in the variance ratio in this case, relative to the baseline case of the model in the first row, results entirely from an increase in the volatility of the value of the firm as growth options become more important. Indeed, moving from the baseline case to case at the bottom of the table, the volatility of cash flows increases slightly, and so would, contribute to a decrease in the variance ratio.

Comparing quarterly and annual volatilities, notice that if, for example, $\Delta \ln V$ were i.i.d. over time, the annual standard deviation of annual $\Delta \ln V$ would be double the standard deviation of quarterly $\Delta \ln V$. As it turns out, for all eight of the

[26]When A_t is exogenous and always equal to $\widehat{A}_t$, the equality of the required return and the expected return in equation (16) can be written as $r\Psi = \widehat{A}X + mX\Psi_X + \frac{1}{2}s^2X^2\Psi_{XX} + \mu\widehat{A}\Psi_{\widehat{A}} + \frac{1}{2}\sigma^2\widehat{A}^2\Psi_{\widehat{A}\widehat{A}} + \rho_{X\widehat{A}}s\sigma X\widehat{A}\Psi_{X\widehat{A}}$. This partial differential equation is satisfied by $\Psi = \frac{\widehat{A}X}{r-m-\mu-\rho_{X\widehat{A}}s\sigma}$, or equivalently, $\Psi = \frac{(1-\gamma)C}{r-m-\mu-\rho_{X\widehat{A}}s\sigma}$. Since the value of the firm is $pK + \Psi$, we have $V = pK + \frac{(1-\gamma)C}{r-m-\mu-\rho_{X\widehat{A}}s\sigma}$, which implies $V = \left[\frac{pK}{C} + \frac{1-\gamma}{r-m-\mu-\rho_{X\widehat{A}}s\sigma}\right]C$. Finally, use the fact that $\frac{pK}{C} = \frac{\gamma}{u}$ from equation (13) to obtain $V = \left(\frac{\gamma}{u} + \frac{1-\gamma}{r-m-\mu-\rho_{X\widehat{A}}s\sigma}\right)C$.

parameter configurations of the model with optimal upgrades, the annual standard deviations of $\Delta \ln V$ are about double (specifically, from 2.01 to 2.06 times as large as) the quarterly standard deviations of $\Delta \ln V$. Similarly, for these cases, the annual standard deviation of $\Delta \ln C$ are also about double (specifically, from 2.02 to 2.09 times as large as) the quarterly standard deviations. For the more conventional model with exogenously evolving productivity, the bottom entries in each cell reveal a very similar pattern. For all 8 parameter configurations, annual standard deviations of $\Delta \ln V$ and $\Delta \ln C$ are also about double (more precisely, 1.99 times) the size of the corresponding quarterly standard deviations.

The values of the variance ratio for the model with optimally chosen upgrades range from 1.60 to 2.25. To see why the variance ratio is greater than one, first consider the conventional case with exogenous productivity evolving according to a geometric Brownian motion. In this case, if the user cost factor, u_t, were constant over time, then equation (39) reveals immediately that the value of the firm, V, would be strictly proportional to contemporaneous cash flow, C. With V proportional to C, $\Delta \ln V$ would be identically equal to $\Delta \ln C$, and the variance ratio would be exactly equal to one. However, allowing for variation in the user cost factor in the conventional model breaks the proportionality between value, V, and cash flow, C. As explained earlier in this section, if the underlying stochastic processes are mutually independent, then cash flow and the user cost factor will be negatively correlated so cash flow and $\frac{1}{u_t}$ are positively correlated. This positive correlation, along with variation in u_t, will increase the variance of V relative to C and will cause the variance ratio to exceed one. For all eight parameter configurations in Table 3, for both quarterly and annual growth rates, the variance ratios, shown in parentheses, are slightly greater than 1.5 (all of the values are between 1.50 and 1.58). The contribution of the endogenous optimal choice of technology to the variance ratio is the extent to which the variance ratio of the top entry in each cell exceeds the bottom entry in each cell.[27] For the baseline case, endogenous optimal technology

[27] Throughout Table 3, the standard deviations of $\Delta \ln C$ and $\Delta \ln V$ are higher for the conventional model of productivity growth than for the model with optimally chosen upgrades. One might think that in the conventional model in which A_t follows a geometric Brownian motion, the standard devation of $\Delta \ln C$ would be lower than in the model of optimal technology upgrades in which A_t jumps upward by about 50% when technology is upgraded. However, if A_t follows a geometric Brownian motion, it can fall as well as rise; in fact, it can take long excursions below its previous peaks. In the model with optimal technology adoptions, the downside variability in A_t is eliminated because the firm would never choose to incur a cost to reduce its level of technology. If the downside

choice increases the variance ratio in the baseline case by 24% for quarterly growth rates and by 20% for annual growth rates. For the parameter configuration at the bottom of the table, endogenous optimal technology choice increases the variance ratio by 47% for quarterly growth rates and by 40% for annual growth rates. This increase in variance ratios arising from the endogenous optimal choice of technology accounts for some of the excess volatility of stock prices relative to dividends.

Table 3: Volatility of Growth of Firm Value and Cash Flow

deviation from baseline:	Quarterly $sd(\Delta \ln C)$	Quarterly $sd(\Delta \ln V)$	Quarterly $\frac{var(\Delta \ln V)}{var(\Delta \ln C)}$	Annual $sd(\Delta \ln C)$	Annual $sd(\Delta \ln V)$	Annual $\frac{var(\Delta \ln V)}{var(\Delta \ln C)}$
none	0.1195 (0.2141)	0.1652 (0.2661)	1.9105 (1.5447)	0.2461 (0.4269)	0.3348 (0.5297)	1.8510 (1.5393)
$\theta = 0.25$	0.1176 (0.2141)	0.1659 (0.2661)	1.9893 (1.5447)	0.2442 (0.4269)	0.3359 (0.5297)	1.8918 (1.5393)
$\sigma = 0.30$	0.1336 (0.1646)	0.1701 (0.2064)	1.6216 (1.5732)	0.2695 (0.3281)	0.3470 (0.4111)	1.6574 (1.5694)
$\mu_Y = 0.01$	0.1194 (0.2141)	0.1667 (0.2658)	1.9491 (1.5408)	0.2460 (0.4269)	0.3373 (0.5290)	1.8805 (1.5354)
$\sigma_Y = 0.10$	0.0961 (0.2021)	0.1404 (0.2515)	2.1359 (1.5493)	0.2007 (0.4030)	0.2859 (0.5006)	2.0291 (1.5430)
$\sigma_u = 0.02$	0.0975 (0.2027)	0.1252 (0.2491)	1.6497 (1.5107)	0.2033 (0.4039)	0.2573 (0.4957)	1.6027 (1.5066)
$\rho_{Yu} = 0.2$	0.1091 (0.2085)	0.1498 (0.2589)	1.8867 (1.5423)	0.2258 (0.4156)	0.3050 (0.5153)	1.8247 (1.5374)
$\rho_{Y\widehat{A}} = 0.2$	0.1207 (0.2277)	0.1808 (0.2813)	2.2456 (1.5259)	0.2498 (0.4541)	0.3641 (0.5599)	2.1246 (1.5203)

Numbers in parenthesis are for case in which $a_t = 1$ for all t.

Baseline parameters: $r = 0.15, \gamma = 0.75, \theta = 0.5, \mu = 0.13, \sigma = 0.45$,

$\mu_Y = 0.005, \sigma_Y = 0.20, \sigma_u = 0.06, \mu_p = 0.015$,and $\rho_{Y\widehat{A}} = \rho_{u\widehat{A}} = \rho_{Yu} = 0$.

These values imply $m = -0.0184$ and $s = 0.2691$.

variability in A_t is eliminated in the conventional model by specficying that $A_t = max_{s \leq t}\left\{\widehat{A}_t\right\}$, then in the baseline case, the standard deviation of $\Delta \ln C$ is 0.1251 for quarterly growth rates and 0.2700 for annual growth rates, which are much closer to the corresponding values to the case with optimal technology adoption than to the case with exogenous technology.

7 Comments and Conclusions

The value of a firm, measured as the expected present value of payouts to claimholders, summarizes a variety of information about the current and expected future cash flows of the firm. Tobin's Q is an empirical measure, based on the value of the firm, that is designed to capture a firm's incentive to invest in capital. However empirical regressions of investment on Tobin's Q and cash flow often find only a weak effect of Q but find an important role for cash flow in explaining investment. Moreover, there is strong evidence of excess volatility of equity values relative to their underlying dividends. We show that growth options can account for these phenomena.

In our model, growth options arise because the firm's level of productivity is a choice variable. The firm can choose to upgrade its technology to the frontier level of technology, whenever it choose to pay the cost of upgrading. This opportunity to upgrade to the frontier is reflected in the firm's value. Fluctuations in the frontier technology will thus induce volatility in the firm's value that are unrelated to the currently installed technology or to contemporaneous cash flows, thereby helping to account for excess volatility. During the intervals of time between consecutive technology upgrades, investment in physical capital is driven by the same factors that drive cash flow, so investment will be postively correlated with cash flow, but investment is uncorrelated with Tobin's Q during these intervals. The correlation between investment and Tobin's Q arises from the forward-looking nature of the value of the firm. As the frontier technology gets sufficiently far ahead of the technology currently in use, an upgrade in technology, with its associated investment expenditures, becomes imminent, and the value of the firm increases, which increases Tobin's Q. That is, a high value of Tobin's Q indicates a high likelihood of imminent capital expenditures associated with an upgrade. In discretely sampled data, this relationship will appear as a positive correlation between investment and Tobin's Q.

Our simulations show that the model can generate empirically realistic investment regressions when the growth option component of the firm is fairly important. Specifically, when growth options are important, investment regressions on Q and cash flow yield small positive coefficients on Q and larger positive coefficients on cash flow. This finding is noteworthy because empirical findings of a large cash flow coefficient are often interpreted as evidence of financing constraints. However, capital markets in our model are perfect, so there are no financing constraints. The model also generates excess volatility of firm value relative to cash flow, especially when growth options are an important component of the firm's value.

An avenue for further work is to allow for factor adjustment costs. In the current model, both capital and labor are costlessly adjustable. As a result, the investment rate is very volatile, which is consistent with plant-level, but not firm-level data (see Doms and Dunne (1998)). This could again be addressed by explicitly incorporating adjustment costs for capital. Another approach to matching the firm-level data on investment would be to model the behavior of plants, and then to aggregate the behavior of plants into firms. This aggregation would reduce some of the investment spikes associated with technology upgrades at individual plants. Aggregating further to economy-wide valuation, earnings, and dividends would allow us to investigate further the excess volatility results of Leroy and Porter (1981), Shiller (1981) and West (1988).

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