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Rental price and rental duration under retail competition

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Abstract

Consider a retailer that rents products to customers for a pre-specified rental duration. By considering the dynamics of uncertain rental demand and return processes, we first present a base model that is intended to analyze the impact of rental duration on the stocking level, the rental price, and the retailer's profit. Due to the complexity of the base model, we develop an approximation scheme to obtain tractable results. Also, we apply the base model to analyze a situation in which a retailer enters a revenue sharing agreement with a distributor. Moreover, we expand our base model to address the issue of competition in rental duration and rental price. The analysis of our competitive model in a duopolistic environment suggests that the market equilibrium depends on the market potential and the rental duration sensitivity. Furthermore, we establish conditions under which one firm will charge a lower rental price while the other firm will offer a longer rental duration in equilibrium.

Keywords: Rental industry; Inventory models; Retail competition

1. Introduction

To provide convenient and economical solutions, the rental industry is growing at an impressive pace. One can rent almost anything that ranges from products such as mobile phones, furniture, books, videos, cars, computers, etc., to services such as apartments, hotel rooms, lockers, safe deposit boxes, uniform and linen cleaning/delivery, etc. Generally speaking, the rental industry can be divided into two groups. To provide convenience, the first group of rental companies allows the customers to specify the rental duration according to their needs. The first group of rental companies includes car rental companies, hotels, furniture rental companies, etc. To better manage asset utilization, return processes and customer services, the rental duration is pre-specified by the rental companies (not the customers) in the second group. The second group of rental companies includes locker rentals, safe deposit box rentals, video rentals, book rentals, laptop computer rentals at the UCLA library, etc. The first group is clearly preferred by the customers, but the second group of rental companies exists for the following reasons:

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1. *Limited supply.* When the number of rental units is in short supply, many renters may impose a pre-specified rental duration. For instance, the UCLA library rents laptop computers to customers for a pre-specified 4-hour rental duration. Next, most Caltrain stations offer bicycle lockers so as to give train commuters the flexibility of riding bicycles to or from the train stations. Due to limited supply of these bike lockers, commuters can rent these lockers for a six-month rental duration specified by Caltrain. The reader is referred to Caltrain (www.caltrain.com) for details. Other examples include safe-deposit boxes, mail boxes, student lockers, and even burial plots.[1]
2. *Regulatory requirement.* There are instances in which the rental duration is pre-specified according to certain regulatory requirements. For example, United States federal law requires high-risk facilities such as X-ray laboratories and nuclear plants to record and report the radioactive exposure of workers over a pre-specified period. To provide a cost effective solution to these high risk facilities, Global Dosimetry Solutions (GDS) (www.dosimetry.com) rents the wearable thermoluminescent badges to these high risk facilities so as to record radioactive exposure of their employees over a pre-specified period dictated by the US government. In order to measure and report the radioactive exposure of each badge worn by each employee, these badges must be returned to GDS for the measurement and refurbishment operations according to a pre-specified rental duration. The reader is referred to Bayiz and Tang (2004) for details.
3. *Asset utilization.* When the unit cost is high relative to the rental price, it is advantageous for a rental company to pre-specify the rental duration so that the rental company can increase the utilization of their rental units by having better control of the return process. For example, the video rental company Blockbuster (www.blockbuster.com) imposes a pre-specified rental duration. Other examples include book rental stores.

In this paper, we shall focus our attention on the second group of rental companies and we shall analyze the issues of rental price and rental duration for rental companies with a pre-specified rental duration. Specifically, we would like to examine the following questions:

1. Given the rental duration, what is the optimal number of rental units that the rental store should stock? What is the optimal rental price the rental store should charge?
2. When facing competition, how should the rental stores compete? Should they compete on rental duration by offering a longer rental duration with a higher rental price? Should they compete on rental price by offering a lower rental price with a shorter rental duration?

The answer to these questions could provide some insights for the manager to evaluate the trade-off between rental duration and rental price. In this paper, we first present a base model for analyzing the impact of rental duration on the stocking level, the rental price, and the retailer's profit. Our base model is a stylized model that is based on several assumptions. For example, we assume that the underlying rental demand in each period is an i.i.d random variable that is equal to the sum of a constant term and a random error term. The constant term is a linear function of the rental price plus an inverse function of the rental duration, while the random error term is a time independent random variable. This particular form of demand function would be more suitable for stable non-seasonal rental items such as mailboxes, safe deposit boxes, dosimetry badges, classical movies or classical books. Also, we assume that unmet demand is backordered in the sense that each customer is willing to rent the rental unit in a later period when the unit is not available at the store currently. Furthermore, the rental duration is treated as a given parameter in the base model. Even with these assumptions, the exact analysis of the base model is complex. To obtain tractable results, we develop an approximation scheme for the base model. Our approximate analysis shows that the optimal stocking level and the optimal rental price increase as the rental duration increases. This result is intuitive and it is consistent with common practice. We also extend our base model to analyze a revenue sharing scheme that is intended to entice the retailer to stock more rental units. Under the revenue sharing scheme, the retailer can purchase the item at a reduced price from the distributor, but the retailer has to share the rental revenue with the dis-

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[1] Due to space limitation, all public burial plots in Hong Kong and Singapore have a 15-year duration pre-specified by the government. The remains will be exhumed and cremated before the expiration of the fifteen-year term so that these burial plots can be recycled.

tributor in return. Under certain conditions, our analysis suggests that the optimal rental price is reduced when the retailer engages in a revenue sharing contract with the distributor.

As the competition in the rental industry becomes fierce, retailers would develop different strategies to compete. In this paper, we expand our base model to analyze the case in which two retailers compete on rental price and rental duration. We show that, in equilibrium, a retailer that offers a longer rental duration will always charge a higher rental price. This result is consistent with common practice. Moreover, we show that the market equilibrium depends on the market potential, price sensitivity and rental duration sensitivity. Finally, we present conditions under which, in equilibrium, one firm would compete on rental price by offering a lower rental price with a shorter rental duration while the other firm would compete on rental duration by offering a longer rental duration with a higher rental price.

This paper is organized as follows. In Section 2, we review some existing work that is related to our paper. Section 3 presents the base model and the associated analysis. We extend our base model to deal with the issue of revenue sharing in Section 4. In Section 5, we expand our base model to analyze the case in which the retailers compete on both rental duration and rental price. Also, we establish conditions under which a unique equilibrium calls for one firm to compete on rental price while the other competes on rental duration. This paper is concluded in Section 6.

2. Literature review

The research work that relates to our paper can be divided into four streams. The first stream deals with the interaction between stocking level and pricing. The interface between ordering and pricing decisions has received significant attention recently. Dada and Petruzzi (1999) analyze the optimal stocking level and optimal pricing for the single-period newsboy problem. For multiple period dynamic pricing models, the reader is referred to a comprehensive review provided by Elmaghraby and Keskinocak (2003).

The second stream of related research focuses on the analysis of the rental industry. Bayiz and Tang (2004) develop an integrated planning system for a dosimetry service company that is based on deterministic demand and return processes. This planning system is designed to help the firm to manage inventory in an effective manner by taking the return process into consideration. Other work that relates to the rental industry tends to concentrate on the analysis of the revenue sharing scheme. Dana and Spier (2001) show that the revenue sharing scheme is valuable in vertically separated industries in which demand is either uncertain or variable. They also show that revenue sharing enables the supply chain to achieve the first best outcome by softening retail price competition without distorting the retailer's stocking decisions. To explore the notion of revenue sharing contracts further, Cachon and Lariviere (2005) provide an analytical comparison between revenue sharing contracts and other supply chain contracts such as buy-back contracts, price-discount contracts, quantity-flexibility contracts, sales-rebate contracts, etc. They show that revenue sharing contracts are equivalent to buy back contracts in the newsvendor case and are equivalent to price discount contracts in the price-setting newsvendor case. Utilizing the panel data collected at 6137 video rental stores in the US between 1998 and 2000, Mortimer (2004) compares the stocking levels, rental prices, etc., across different stores for the same title as well as across different titles within the same store. In addition, she conducts regression analysis to examine the effect of revenue sharing scheme on the retailer's profit. Her analysis shows that the revenue sharing scheme has a small positive effect on the retailer's profit for popular titles, and a small negative effect for less popular titles.

The third stream of related research focuses on models in which firms compete on price and one other dimension. So (2000) develops a queueing model to analyze a situation in which firms compete on price and completion time in the service industry. In the case when retailers offer price discounts to customers who commit their orders in advance, Dana (1998) develops an economic model to analyze how firms can utilize the notion of an Advance Booking Discount (ABD) program to create price discrimination in competitive markets. More recently, McCardle et al. (2004) present a newsvendor type model to examine the notion of advance booking discount programs when retailers compete on price. In the context of competition within the rental industry, Dana (2001) presents a strategic model of competition in price and availability that enables us to gain a better understanding about the relationship between price and product availability. He shows that, in equilibrium, firms use higher rental prices to 'signal' higher product availability.

Finally, our paper is closely related to the literature that analyzes inventory systems with returns. Due to complexity of the analysis, most papers in this literature assume that the return process is independent of the demand process (c.f., Fleischmann et al. (1997)). There are a few exceptions. Kiesmüller and van der Lann (2001) and the references therein consider a single product model with fixed leadtimes and a finite horizon. The demand in each period is independent and follows a time-variant Poisson process. By considering the case in which the return process depends explicitly on the demand process, Kiesmüller and van der Lann (2001) present numerical comparisons between models based on demand-independent return processes and those based on demand-dependent return processes. Based on their numerical experiments, they conclude that neglecting the dependency between the return process and the demand process can lead to bad performance with respect to the total average relevant cost.

Our paper differs from the aforementioned streams in the following manner. First, our model is a multi-period model that captures the dynamics of uncertain rental demand and return processes. Second, our model is intended to analyze the issue of rental duration and rental price. Third, our model examines the case when the retailers compete on rental price and rental duration instead of product availability. While the objective of the model developed by Kiesmüller and van der Lann (2001) is to illustrate the importance of capturing the dependency between the return process and the demand process, the goal of our model is to analyze the issue of rental price and rental duration under competition. Our model differs from the model considered by Kiesmüller and van der Lann (2001) in the following ways: (i) we consider the rental price decision; (ii) we allow the customer to return the product anytime within the rental duration; (iii) we consider the case in which the retailers compete on rental price and rental duration; and (iv) we explicitly model the customer preference in terms of rental price and rental duration through the demand function.

3. The base model

To simplify the exposition of our model, we first consider a base model arising from a monopolistic environment. We shall extend this base model to the duopolistic case in a later section. Consider a retailer that purchases a single product from a distributor at c per unit,[2] rents the product to customers by charging a rental price p per unit for a pre-specified rental duration r periods, where r is a positive integer. In the base model, the rental duration r is treated as a given parameter. We assume that the rental demand D_t in period t possesses the following functional form:

$$D_t = \mu(p, r) + \epsilon,$$

where $\mu(p, r)$ denotes the expected demand in each period and ϵ represents demand uncertainty. We assume that ϵ is normally distributed with mean 0 and standard deviation σ and that ϵ is time independent. Notice that the rental demand is stationary, and hence, it is more suitable for modeling non-seasonal items such as mailboxes, lockers, dosimetry badges, classical videos, etc.

For any rental price p and rental duration r, we assume that the expected demand $\mu(p, r)$ possesses the following functional form:

$$\mu(p, r) = \alpha - \beta p - \frac{\gamma}{r}, \tag{1}$$

where α, β, $\gamma > 0$. Notice that α represents the market potential, β represents the rental price sensitivity, and γ represents the rental duration sensitivity. Observe that $\mu(p, r)$ is decreasing in rental price p but increasing in rental duration r. Hence, $\mu(p, r)$ captures the tradeoff between the rental price p and the rental duration r. Also, notice that $\mu(p, r)$ is increasing and concave in rental duration r, which captures the diminishing marginal value of rental duration. For example, in the video rental industry, longer rental duration offers additional flexibility to customers in terms of the time to watch the video and the time to return the video. However, the value of this additional flexibility diminishes as the rental duration increases.

In this paper, we consider the case in which the retailer needs to determine the total number of rental units at the beginning of the planning horizon, which we shall refer to as the initial stocking level I_0. Once I_0 is

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[2] In the event that the retailer rents mailboxes, safe deposit boxes, lockers, etc., the unit cost c corresponds to the construction cost.

determined, the retailer can rent these I_0 units according to customer demands and customer returns over a planning horizon of T periods. Once I_0 is specified in period 0, the retailer is not allowed to adjust the total number of rental units upward by acquiring additional units or downward by disposing certain units in subsequent periods. This assumption is reasonable when the fixed cost associated with subsequent acquisitions or disposals is high or when the disposal value is very low.[3] For any given rental duration r, the retailer needs to determine the initial stocking level I_0 and the rental price p. To ensure boundedness, we shall focus on the expected average profit per period over the planning horizon T in this paper.

Let S_t be the ‘actual’ number of rental units checked out from the retailer in period t. These S_t units are due by the end of period $t+r$. Out of these S_t units rented in period t, let k_iS_t denote the cumulative total number of units returned from period $t+1$ to period $t+i$. Therefore, $(k_i - k_{i-1})S_t$ corresponds to the number of units returned in period $t+i$. Hence, $(k_i - k_{i-1})S_t$ corresponds to early returns, on-time returns, and late returns when $i<r$, $i=r$, and $i>r$, respectively. To simplify our analysis, we assume all customers will return the product within the rental duration, i.e., $i \leqslant r$.[4] Thus, $0 = k_0 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_r = 1$. We also assume that the k_i’s are known and can be estimated from the historical return data. Since $(k_i - k_{i-1})S_t$ corresponds to the number of units returned in period $t+i$, we can express the inventory level at the end of period t (after accounting for the units returned during period t), denoted by I_t, as

$$I_t = \begin{cases} I_{t-1} + \sum\limits_{i=1}^{t-1}(k_i - k_{i-1})S_{t-i} - D_t, & t = 1,2,\ldots,r, \\ I_{t-1} + \sum\limits_{i=1}^{r}(k_i - k_{i-1})S_{t-i} - D_t, & t = r+1,\ldots,T. \end{cases} \tag{2}$$

In this paper, we shall consider the case in which all unmet demands are backordered. A backorder implies that the customer is willing to wait for the rental item when it is unavailable at the store. Define $[x]^+ = \max\{x,0\}$ and $[x]^- = \max\{-x,0\}$. Hence, $D_t + [I_{t-1}]^-$ corresponds to the ‘effective demand’ in period t and $[I_{t-1}]^+ + \sum_{i=1}^{r}(k_i - k_{i-1})S_{t-i}$ corresponds to the ‘effective supply’ at the beginning of period t, where $t = r+1,\ldots,T$. Therefore, the actual number of unit rented in period t, S_t, can be expressed as

$$S_t = \begin{cases} \mathrm{MIN}\left\{[I_{t-1}]^+ + \sum_{i=1}^{t-1}(k_i - k_{i-1})S_{t-i}, [I_{t-1}]^- + D_t\right\} & t = 1,2,\ldots,r, \\ \mathrm{MIN}\{[I_{t-1}]^+ + \sum_{i=1}^{r}(k_i - k_{i-1})S_{t-i}, [I_{t-1}]^- + D_t\} & t = r+1,\ldots,T. \end{cases} \tag{3}$$

Let h be the holding cost per unit per period. The holding cost captures the processing cost associated with the return and the storage processes. Let c be the unit cost associated with the unit purchased by the retailer at time 0. For each unit of unmet demand, let s be the shortage cost per unit that includes the penalty cost. The retailer’s expected average profit per period can be expressed as

$$\pi(I_0,p) = \frac{\sum_{t=1}^{T} pE(S_t) - cI_0 - \sum_{t=1}^{T}((hE([I_t]^+) + sE([I_t]^-))}{T}.$$

In this case, the retailer’s problem is to determine the rental price p and the initial stocking level I_0 so as to maximize the expected average profit per period over the planning horizon. Hence, the retailer’s problem can be formulated as the following program:

$$\text{(P1)} \quad pi' = \mathrm{MAX}_{p,I_0}\pi(I_0,p), \quad \text{subject to (2) and (3).}$$

Program (P1) is a complex problem and it does not yield closed form solutions for the following reasons. First, observe from (2) and (3) that I_t is not a convex function of I_0 for any given rental price p. Hence, the objective function of (P1) is not jointly concave in I_0 and p. Second, observe from (2) and (3) that I_t and S_t depend on the past history of I_i and S_i, where $i = t-1$, $t-2,\ldots$. This implies that we need to keep track of the rental

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[3] We offer two examples. First, the fixed cost associated with constructing additional safe deposit boxes at a bank is very high. Second, the fixed cost associated with ordering additional Thermoluminescent badges at the Global Dosimetry Solutions GDS is very high due to the inherent manufacturing process of these badges. Also, the disposal value of these badges is very low because these badges are based on firm-specific technology and cannot be used by other companies.

[4] However, our analysis can be extended to the case in which late returns are allowed.

history and the return history in order to determine S_t and I_t for each period t. Hence, the state space of the problem is quite large. Third, even when $T = 1$, problem (P1) can be reduced to the joint pricing and ordering newsvendor problem. As shown in Dada and Petruzzi (1999), there is no simple closed form solution for the joint pricing and ordering newsvendor problem. Therefore, problem (P1) does not yield closed form solutions even for the monopolistic case, which makes it difficult for us to analyze other managerial issues such as revenue sharing and retail competition. As a way to obtain closed form solutions so that we can examine these managerial issues analytically, we shall develop an approximation scheme in the following subsection.

3.1. An approximation scheme

We shall approximate problem (P1) by using two approximations. The first approximation is based on the fact that $D_t - S_t = [I_t]^- - [I_{t-1}]^-$. To see that, for any $t \geqslant r+1$, observe from (2) and (3) that

$$D_t - S_t = D_t + \mathrm{MAX}\left\{-[I_{t-1}]^+ - \sum_{i=1}^{r}(k_i - k_{i-1})S_{t-i}, -[I_{t-1}]^- - D_t\right\} = \mathrm{MAX}\{-[I_{t-1}]^- - I_t, -[I_{t-1}]^-\}$$
$$= [I_t]^- - [I_{t-1}]^-.$$

Similarly, we can show that $D_t - S_t = [I_t]^- - [I_{t-1}]^-$ for $t = 1, \ldots, r$. In this case, if the incremental change in the backorders (i.e., $[I_t]^- - [I_{t-1}]^-$) is sufficiently small, then $S_t \approx D_t$. This observation motivates us to approximate S_t by D_t. The first approximation would enable us to eliminate constraint (3) and simplify (2) as

$$I_t = \begin{cases} I_0 - \left(D_t + \sum_{i=1}^{t-1}(1-k_i)D_{t-i}\right) & t = 1, 2, \ldots, r, \\ I_0 - \left(D_t + \sum_{i=1}^{r}(1-k_i)D_{t-i}\right) & t = r+1, \ldots, T. \end{cases}$$

The second approximation is intended to consolidate the above expression for I_t into a single equation. To do so, let us introduce the terms D_{0-i}, where $D_{0-i} = \mu(p,r) + \epsilon$ for $i = 1, 2, \ldots, r$. Given D_{0-i}, we can approximate the above expression for I_t by

$$I_t = I_0 - \left(D_t + \sum_{i=1}^{r}(1-k_i)D_{t-i}\right) \quad t = 1, 2, \ldots, T. \tag{4}$$

The second approximation is reasonable when $T \gg r$. These two approximations enable us to approximate problem (P1) by the following program:

$$\text{(P2)} \quad \pi^* = \mathrm{MAX}_{p,I_0} \frac{\sum_{t=1}^{T} pE(D_t) - cI_0 - \sum_{t=1}^{T}(hE([I_t]^+) + sE([I_t]^-))}{T}, \quad \text{subject to (4)}.$$

We now analyze the solution to problem (P2).

3.2. Analysis

Since $E(D_t) = \mu(p,r)$, we can rewrite problem (P2) as

$$\text{(P3)} \quad \pi^* = \mathrm{MAX}_p\{p\mu(p,r) - \{\mathrm{MIN}_{I_0} C(I_0,p)\}\}, \quad \text{subject to (4)},$$

where

$$C(I_0,p) = \frac{cI_0 + \sum_{t=1}^{T}(hE([I_t]^+) + sE(I_t]^-))}{T}. \tag{5}$$

In this case, we can determine the optimal decisions; i.e., I_0^* and p^*, by solving an inner problem and an outer problem. For the inner problem, we determine $I_0^*(p)$ that minimizes $C(I_0,p)$ for any given rental price p. Then we solve the outer problem by determining p^* that maximizes the average profit per period; i.e., $p\mu(p,r) - C(I_0^*(p),p)$.

For any given rental price p, we now solve the inner problem by determining $I_0^*(p)$ that minimizes $C(I_0,p)$. For notational convenience, let $\hat{D}_r = D_t + \sum_{i=1}^{r}(1-k_i)D_{t-i}$. In this case, we can simplify the inventory balance equation (4) as

$$I_t = I_0 - \hat{D}_r \quad t = 1, 2, \ldots, T. \tag{6}$$

Let $F_r(.)$ denote the distribution of $\hat{D}_r$. Note that F_r is normally distributed with a mean $\hat{\mu}_r = E(\hat{D}_r) = \mu n_1(r)$ and a standard deviation $\hat{\sigma}_r = \sqrt{(\mathrm{Var}(\hat{D}_r))} = \sigma n_2(r)$, where $n_1(r) = 1 + \sum_{i=1}^{r}(1-k_i)$ and $n_2(r) = \sqrt{1 + \sum_{i=1}^{r}(1-k_i)^2}$.

In order to examine the impact of the rental duration r on the rental price and the retailer's profit, we need to impose certain characteristics on the cumulative return rate k_i. Let k_i^r be the 'cumulative' return rate when the rental duration is r periods. In this case, it is reasonable to assume that more customers will return the products within i periods when the rental duration is shorter; i.e., $k_i^r \geqslant k_i^{r+1}$. With this assumption, we can show that:

Lemma 1. *When k_i^r is decreasing in r, $n_1(r)$ and $n_2(r)$ are increasing in r.*

All proofs are given in the Appendix. Throughout this paper, we shall assume that $n_1(r)$ and $n_2(r)$ are increasing in r.

To find the optimal initial stock level I_0^* that minimizes average cost per period $C(I_0)$, we first substitute I_t from (6) into $C(I_0,p)$ given in (5). Then we differentiate $C(I_0,p)$ with respect to I_0 to obtain the following first order condition:

$$\frac{c}{T} - s + (h+s)F_r(I_0^*) = 0. \tag{7}$$

Hence, the optimal initial stocking level for any given p can be expressed as

$$I_0^*(p) = F_r^{-1}\left(\frac{s - c/T}{h+s}\right) = \mu n_1(r) + z^* \sigma n_2(r), \tag{8}$$

where $z^* = \Phi^{-1}(\frac{s-c/T}{h+s})$ and $\Phi(\cdot)$ is the standard normal distribution function.[5] Substitute the optimal inventory I_0^* into the total cost $C(I_0)$ in (5), getting

$$C(I_0^*(p)) = \frac{c\mu(p,r)n_1(r)}{T} + (h+s)\phi(z^*)\sigma n_2(r), \tag{9}$$

where $\phi(\cdot)$ is the standard normal probability density function. From (1), we have $\mu = \mu(p,r) = \alpha - \beta p - \gamma/r$. Combining this with Lemma 1, it can be easily shown that the optimal initial stocking level for any given p, $I_0^*(p)$, is increasing in the rental period r, the demand uncertainty σ and the length of the planning horizon T. Also, the minimum cost per period for any given p, $C(I_0^*(p))$, is increasing in the demand uncertainty σ, decreasing in the rental price p, increasing in the rental duration r, and decreasing in T. These results are quite intuitive.

Given the optimal average cost per period $C(I_0^*(p))$ in (9), we now solve the outer problem by determining p^* that maximizes the average profit per period. The outer problem can be expressed as

$$\begin{aligned}\pi^* &= \mathrm{MAX}_p\{p\mu(p,r) - C(I_0^*(p))\} = \mathrm{MAX}_p \pi(p) \\ &= \mathrm{MAX}_p\left\{\left(p - \frac{cn_1(r)}{T}\right)\left(\alpha - \beta p - \frac{\gamma}{r}\right) - (h+s)\phi(z^*)\sigma n_2(r)\right\}.\end{aligned} \tag{10}$$

By considering the first order condition, we can determine the optimal rental price p^* that maximizes the retailer's profit, where the optimal rental price p^* and the optimal profit π^* have the following properties:

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[5] Throughout this paper, we shall assume that $\frac{s-c/T}{h+s} \geqslant 0.5$. This assumption will ensure $z^* > 0$, which implies that it is optimal for the retailer to order above the mean demand.

Proposition 2. *The optimal rental price and the optimal retailer's profit can be expressed as*

$$p^* = \frac{T\alpha_r + \beta c n_1(r)}{2T\beta}, \tag{11}$$

$$\pi^* = \frac{\left(\alpha_r - \frac{\beta c n_1(r)}{T}\right)^2}{4\beta} - (h+s)\phi(z^*)\sigma n_2(r), \tag{12}$$

where $\alpha_r = \alpha - \frac{\gamma}{r}$. *Moreover, the optimal rental price* p^* *is increasing in the unit cost* c, *decreasing in the length of the planning horizon* T, *and is increasing in the rental duration* r. *Furthermore, the optimal profit* π^* *is decreasing in the unit cost* c, *decreasing in the demand uncertainty* σ, *and increasing in* T. *When* α *is sufficiently large,* π^* *is concave in* T.

Proposition 2 has the following implications. First, notice that the optimal rental price p^* increases as the rental duration r increases. This result is intuitive because the retailer has to increase the rental price in order to compensate for the implicit cost incurred by longer rental duration. The behavior of p^* and π^* with respect to the T is a consequence of the stationary demand assumption. The optimal rental price p^* reduces as the retailer commits to a longer planning horizon T. This result is also intuitive because the implicit cost of the rental unit is lower as the retailer spreads the upfront unit cost c over a longer planning horizon. Moreover, as T increases, the retailer will gain more profit because the product cost is incurred at period 0 (i.e., one time cost) and the revenue obtained in each subsequent period outweighs the total holding and shortage cost. While these results are intuitive, it is unclear if these results will continue to hold under retail competition. We shall investigate this issue in a later section.

Consider a special case in which the customers will return in any of the r periods with equal probability; i.e., $k_i - k_{i-1} = \frac{1}{r}$ for $i = 1,\dots,r$ and for $r = 1,2,\dots$. We can establish the following corollary.

Corollary 3. *If* $k_i - k_{i-1} = \frac{1}{r}$ *for* $i = 1,\dots,r$, *then the optimal rental price* p^* *is increasing and concave in the rental duration* r.

3.3. Goodness of approximation

The analyses and the closed form expressions for the optimal initial stocking level I_0^* and the optimal rental price p^* presented in Section 3.2 are based on the approximation scheme presented in Section 3.1. Before we utilize our approximate model to analyze the issue of revenue sharing and retail competition, we need to examine the goodness of our approximation scheme. In preparation, let I_0' and p' be the actual optimal decisions and let $\pi' = \pi(I_0', p')$ be the optimal profit associated with the original problem (P1). Also, recall from Section 3.2 that I_0^* and p^* correspond to the optimal decisions associated with the approximate problem (P3). Let $\hat{\pi} = \pi(I_0^*, p^*)$ denote the retailer's actual profit obtained from implementing the approximate decisions I_0^* and p^*.

To measure the goodness of our approximation scheme, we develop three specific error gaps. The first error gap $\Delta\pi$ measures the percentage error in the optimal profit, where $\Delta\pi = \frac{\pi' - \hat{\pi}}{\pi'}$. The second error gap ΔI and the third error gap Δ_p correspond to the percentage errors in the optimal initial stocking level and the optimal rental price, respectively, where $\Delta_I = \frac{I_0' - I_0^*}{I_0'}$ and $\Delta p = \frac{p' - p^*}{p'}$.

Given these three error gaps, We first examine how Δ_π, Δ_I, and Δ_p change with respect to the parameters h, s and T. To construct our numerical experiments for this purpose, we fix the rental duration $r = 3$, $\alpha_r = \alpha - \frac{\gamma}{r} = 50$ and $\beta = 5$. Thus, the mean demand is given by $\mu(p, r = 3) = 50 - 5p$. In addition, we fix $c = 10$ and $\sigma = 8$. Thus, the range of prices relevant to the problem is $[0, 10]$. The values of the parameters h, s and T are chosen so as to ensure that the approximate service level $\left(\frac{s - c/T}{h+s}\right)$ is within a reasonable range of 80–95%.

We use Monte-Carlo simulation to generate i.i.d. normally distributed demands. Calculating $\hat{\pi} = \pi(I_0^*, p^*)$ numerically is straightforward once the demands have been generated and the values of various parameters and decision variables are available. However, obtaining the optimal profit π' requires explicit search over I_0 and p. To do so, we first search over integer values for I_0 for a fixed value of p. Then we search over p

to determine I'_0, p' and the corresponding optimal profit π'. The values of various parameters used in the numerical experiments and the performance measures are shown in the figures below.

Fig. 1 examines the impact of T on the error gaps. We set $s = 1$, $h = 0.1$, but we vary T from 100 to 250. Since c is fixed at 10, the customer service level $(\frac{s-c/T}{h+s})$ varies from 0.82 to 0.87 as we vary T from 100 to 250. Observe from Fig. 1 that the profit gap Δ_π is below 4% and that the inventory gap Δ_I is below 5%. Next, notice that the inventory error gap Δ_I is always positive. This implies that the optimal initial stocking level generated by our approximation scheme I_0^* is always lower than the true optimal initial stocking level I'_0. The underlying reason for this phenomenon is that the shipment S_t is approximated by the demand D_t in each period t while $S_t \leqslant D_t$ in reality. By over-estimating the shipment S_t in each period t, we over-estimate the returns in each period in our approximation scheme. As such, our analysis of the approximate problem (P3) will lead us to choose a lower initial stocking level I_0^* than that of the actual case as stated in problem (P1). Finally, observe that the error gaps Δ_π, Δ_I and Δ_p decrease as T increases. This result is driven by two factors. First, as T increases, the objective function of our approximate problem (P2) better approximates the objective function of the actual problem (P1). This is because $\frac{\sum_{t=1}^{T} E[D_t]}{T}$ better approximates $\frac{\sum_{t=1}^{T} E[S_t]}{T}$ as T increases. Second, for large T so that $T \gg r$, the effect of the approximation in the inventory balance equation diminishes.

We now examine the impact of h on the error gaps. We set $s = 5$, $T = 250$ and we vary h from 0 to 1.5 so as to ensure the customer service level is within a reasonable range; i.e., between 80% and 95%. First, observe from Fig. 2 that the profit gap Δ_π is below 4%. Second, notice that the inventory gap $\Delta_I > 0$, and that ΔI increases as h increases. The underlying reason is essentially the same as discussed earlier. Recall from our discussion about Fig. 1 that our approximate analysis tends to over-estimate the shipments and subsequent returns. This over-estimation will lead us to choose a lower initial stocking level I_0^* than that of the true optimal initial stocking level I'_0. This explains why the inventory gap $\Delta_I > 0$ as reported in Fig. 2. As the inventory gap Δ_I increases, the profit gap Δ_π will increase also. This explains why Δ_π increases as h increases. Thus, we can conclude that our approximation scheme performs better for lower values of h.

We now examine the impact of the shortage cost s on the error gaps. We set $h = 0.2$, $T = 250$, but we vary s from 1 to 4. Fig. 3 shows that the approximation scheme works well for different values of s. Specifically, the profit gap $\Delta_\pi < 3\%$. Also, consistent with Figs. 1 and 2, we see that the approximation results in $\Delta_I > 0$. However, the impact of s on Δ_π is not straightforward. Increasing s has two counteracting effects. Firstly, as seen from Fig. 3, the inventory gap ΔI decreases with increasing s. As the inventory gap Δ_I decreases, the profit gap Δ_π may decrease as s increases. However, since our approximate analysis tends to lead us to choose a lower initial stocking level I_0^* than the actual optimal initial stocking level I'_0, the backorder cost associated with the

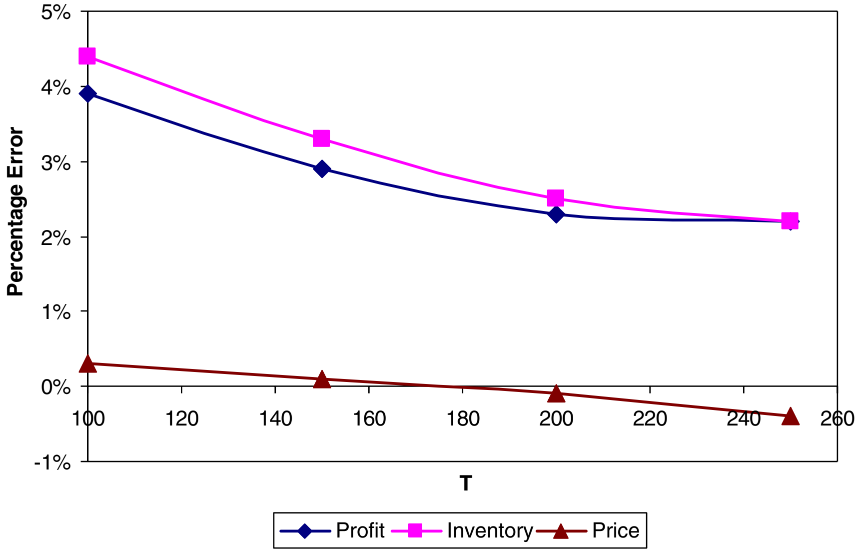

Fig. 1. Impact of time horizon ($c = 10$, $s = 1$, $h = 0.1$, $r = 3$).

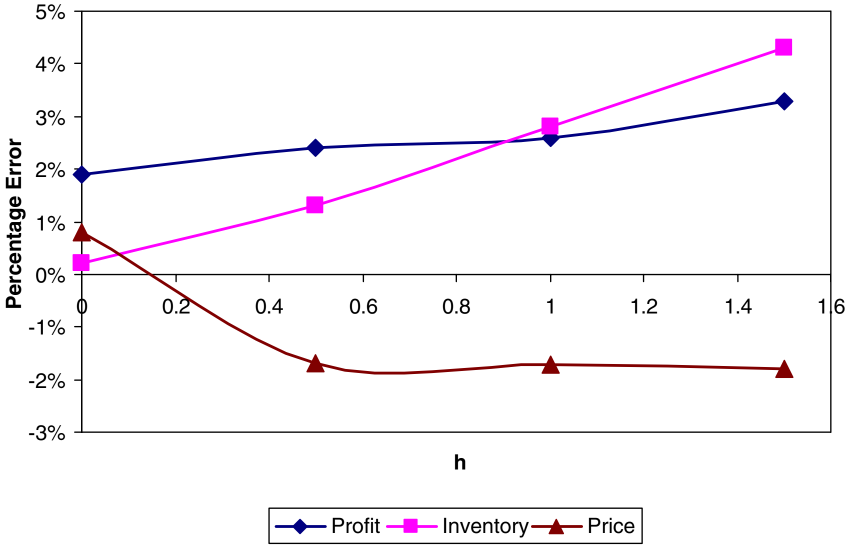

Fig. 2. Impact of holding cost ($c = 10$, $s = 5$, $T = 250$, $r = 3$).

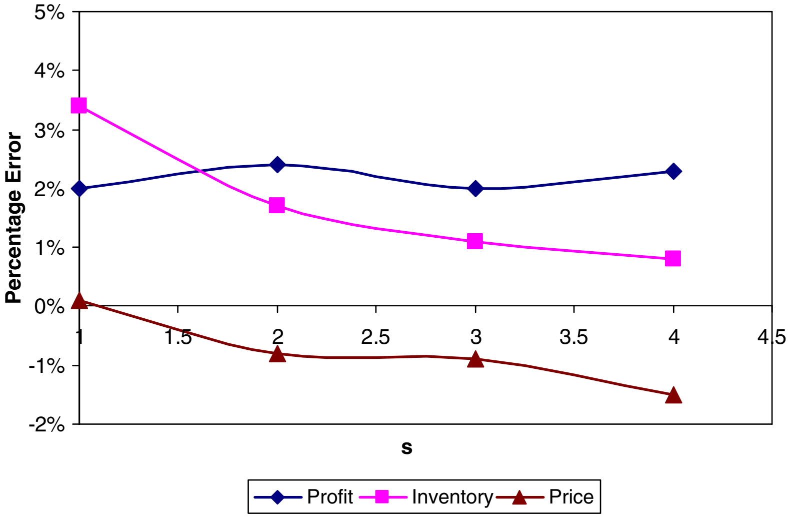

Fig. 3. Impact of backordering cost ($c = 10$, $h = 0.2$, $T = 250$, $r = 3$).

approximate solution I_0^* will increase as s increases even though the inventory gap Δ_I is decreasing in s. This could cause the profit gap Δ_π to increase as s increases. These two countering effects may explain why the profit gap can increase or decrease as s increases.

Next, let us examine the impact of the rental duration r on the error gaps. We set $h = 0.1$, $T = 250$, $s = 1$, but we vary r from 1 to 4. Fig. 4 shows that the approximation scheme performs well for these values of r. Essentially, all error gaps are small: the inventory gap $\Delta_I < 6\%$, the price gap $\Delta_p < 3\%$, and the profit gap $\Delta_\pi < 4\%$.

We have investigated the goodness of our approximation by varying the values of c and σ and our numerical results show that our approximation scheme performs well. We omit the details. In conclusion, our numerical experiments suggest that our approximation scheme as presented in Section 3.1 tends to work well for a wide range of parameters and the profit gap Δ_π is less than 4%. In addition, both error gaps Δ_I and Δ_p are

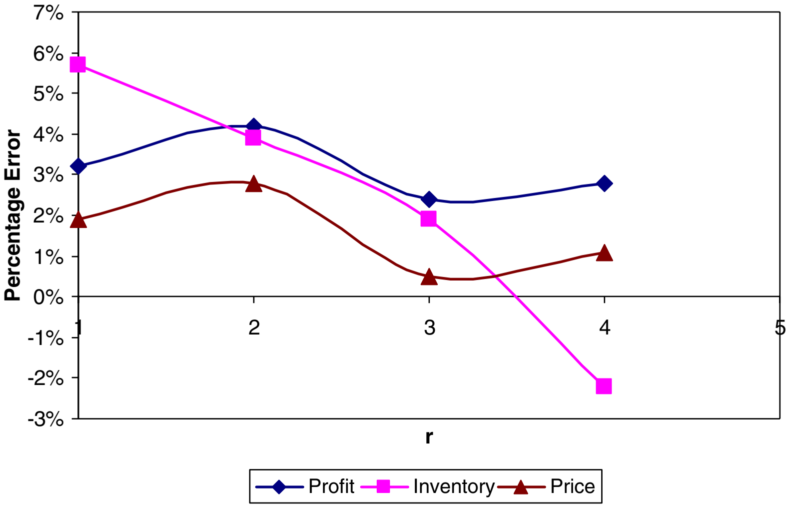

Fig. 4. Impact of rental duration ($c = 10$, $h = 0.1$, $s = 1$, $T = 250$).

reasonably small. Also, the profit gap Δ_π decreases as T increases, and increases as h increases. Since our approximation scheme as described in Section 3.1 tends to work well, we shall adopt this approximation scheme for the remainder of this paper.

4. Revenue sharing

We now utilize our approximate analysis of the base model presented in Section 3.2 to analyze a situation in which a retailer enters a revenue sharing contract with a distributor. Under the revenue sharing scheme as described in Cachon and Lariviere (2005), Dana and Spier (2001), and Mortimer (2004), the retailer pays a 'reduced' unit cost $\kappa_1 c$ with $0 < \kappa_1 < 1$ to the distributor. In return, the retailer has to share a portion of the rental revenue with the distributor. In effect, the retailer keeps only $\kappa_2 p$ for each rental unit, where $0 < \kappa_2 < 1$. However, since the customers are still paying the same rental price p as before, the mean demand $\mu(p,r)$ remains to be the same. By considering the effective unit cost $\kappa_1 c$ and the retailer's effective rental revenue $\kappa_2 p$, we can utilize the results in Section 3.2 to show that the optimal initial stocking level $\tilde{I}_0$, the minimum inventory cost $C(\tilde{I}_0)$ and the optimal average profit per period $\tilde{\pi}$ given in (8)–(10) can be rewritten as

$$\tilde{I}_0(p) = \mu(p,r)n_1(r) + \tilde{z}\sigma n_2(r),$$

$$C(\tilde{I}_0(p)) = \frac{\kappa_1 c\mu(p,r)n_1(r)}{T} + (h+s)\phi(\tilde{z})\sigma n_2(r),$$

$$\tilde{\pi} = \mathrm{MAX}_p \tilde{\pi}(p) = \mathrm{MAX}_p \left\{ \left(\kappa_2 p - \frac{\kappa_1 c n_1(r)}{T} \right)(\alpha - \beta p - \gamma/r) - (h+s)\phi(\tilde{z})\sigma n_2(r) \right\}, \tag{13}$$

where $\tilde{z} = \Phi^{-1}(\frac{s-\kappa_1 c/T}{h+s})$. Notice that $\tilde{z} > z^* > 0$ because of our assumption $\frac{s-c/T}{h+s} > 0.5$ and the fact that $0 < \kappa_1 < 1$. Hence, $\tilde{I}_0(p) > I_0^*(p)$. This result implies that for any given rental price, the retailer would increase the optimal stocking level under the revenue sharing scheme.

4.1. The benefits of revenue sharing

While our primary objective is to analyze the optimal decisions of the retailer under a revenue sharing contract, we begin by proving the existence of revenue sharing contracts which are mutually beneficial for both the distributor and the retailer. For this purpose, we present a highly stylized model of the distributor. Given the rental price p, the retailer would order $I_0^*(p)$ units in period 0, where $I_0^*(p)$ is given in (8). Therefore, the distributor's expected profit, denoted by $\pi_d(p)$, can be expressed as

$$\pi_d(p) = (c - m)I_0^*(p) = (c - m)[\mu n_1(r) + z^*\sigma n_2(r)], \tag{14}$$

where m corresponds to the unit cost incurred by the distributor. Next, under the revenue sharing scheme, the distributor has two streams of income. The first stream of income is generated from the retailer's initial order $\tilde{I}_0(p)$ that occurred in period 0. The second stream of income is derived from the revenue sharing scheme under which the distributor will receive an expected income of $(1 - \kappa_2)p\mu$ in each of the T periods. By considering the fact that the distributor charges $\kappa_1 c$ for each unit under the revenue sharing scheme, one can show that the distributor's expected profit under the revenue sharing scheme, denoted by $\tilde{\pi}_d(p)$, can be expressed as

$$\tilde{\pi}_d(p) = (\kappa_1 c - m)\tilde{I}_0(p) + (1 - \kappa_2)p\mu T = (\kappa_1 c - m)[\mu n_1(r) + \tilde{z}\sigma n_2(r)] + (1 - \kappa_2)p\mu T. \tag{15}$$

By comparing the expressions given in (14) and (15), we can establish the following lemma that specifies the conditions under which the distributor would obtain a higher expected profit under the revenue sharing scheme.

Lemma 4. *For any given rental price* p, $\tilde{\pi}_d(p) > \pi_d(p)$ *if and only if* $\kappa_2 < 1 - (1 - \kappa_1)\frac{cn_1(r)}{pT} - (1 - \kappa_1)\frac{cz^*\sigma n_2(r)}{p\mu T} + \frac{(\kappa_1 c - m)\sigma n_2(r)(\tilde{z} - z^*)}{p\mu T}$.

Turning our attention to the retailer, recall from (10) that the retailer's expected profit associated with the base case $\pi(p)$ for any given rental price p is given as

$$\pi(p) = \left(p - \frac{cn_1(r)}{T}\right)\mu - (h + s)\phi(z^*)\sigma n_2(r), \tag{16}$$

where $z^* = \Phi^{-1}(\frac{s - c/T}{h + s})$. Also, the retailer's expected profit under the revenue sharing scheme $\tilde{\pi}(p)$ is given in (13), where

$$\tilde{\pi}(p) = \left(\kappa_2 p - \frac{\kappa_1 c n_1(r)}{T}\right)\mu - (h + s)\phi(\tilde{z})\sigma n_2(r), \tag{17}$$

and $\tilde{z} = \Phi^{-1}(\frac{s - \kappa_1 c/T}{h + s})$. By comparing the expressions given in (16) and (17), we can establish the following lemma.

Lemma 5. *For any given rental price* p, $\tilde{\pi}(p) > \pi(p)$ *if and only if* $\kappa_2 > 1 - (1 - \kappa_1)\frac{cn_1(r)}{pT} - \frac{(h+s)\sigma n_2(r)[\phi(z^*) - \phi(\tilde{z})]}{p\mu}$.

Combining the results stated in Lemmas 4 and 5, we can prove the following proposition.

Proposition 6. *For any given rental price* p, $\tilde{\pi}(p) > \pi(p)$ *and* $\tilde{\pi}_d(p) > \pi_d(p)$ *if and only if* κ_2 *satisfies the following condition*: $1 - (1 - \kappa_1)\frac{cn_1(r)}{pT} - \frac{(h+s)\sigma n_2(r)[\phi(z^*) - \phi(\tilde{z})]}{p\mu} < \kappa_2 < 1 - (1 - \kappa_1)\frac{cn_1(r)}{pT} - (1 - \kappa_1)\frac{cz^*\sigma n_2(r)}{p\mu T} + \frac{(\kappa_1 c - m)\sigma n_2(r)(\tilde{z} - z^*)}{p\mu T}$.

Proposition 6 implies that there exists a revenue sharing scheme that would generate a 'win-win' situation for the retailer and the distributor. This result is consistent with the result presented in Cachon and Lariviere (2005).

4.2. Optimal rental price and optimal retail profit under revenue sharing

We now turn our attention to analyzing the optimal rental price for any given revenue sharing scheme (κ_1, κ_2). By differentiating the objective function in (13) with respect to p, we can determine the optimal rental price $\tilde{p}$ under the revenue sharing scheme and obtain the retailer's optimal profit $\tilde{\pi}$. Specifically, we have:

Proposition 7. *The optimal rental price* $\tilde{p}$ *and the retailer's optimal profit* $\tilde{\pi}$ *under the revenue sharing scheme are given as follows*:

$$\tilde{p} = \frac{T\alpha_r + \beta c n_1(r)\frac{\kappa_1}{\kappa_2}}{2T\beta}, \tag{18}$$

$$\tilde{\pi} = \frac{(\alpha_r\kappa_2 - \frac{\beta c n_1(r)\kappa_1}{T})^2}{4\beta\kappa_2} - (h+s)\phi\left(\Phi^{-1}\left(\frac{s-\kappa_1 c/T}{h+s}\right)\right)\sigma n_2(r), \tag{19}$$

where $\alpha_r = \alpha - \frac{\gamma}{r}$. Moreover, $\tilde{p}$ is increasing in the ratio $\frac{\kappa_1}{\kappa_2}$, and $\tilde{\pi}$ is convex in κ_1 and κ_2.

By comparing the optimal rental price p^* given in (11) and the optimal price $\tilde{p}$ under the revenue sharing scheme given in (18), it is easy to show that:

Corollary 8. *The optimal rental price* $\tilde{p}$ *under the revenue sharing scheme is lower than the optimal rental price* p^* *for the base case if and only if* $\frac{\kappa_1}{\kappa_2} < 1$.

To relate Corollary 8 with an actual application, let us consider the revenue sharing scheme in the video rental industry as described in Mortimer (2004). Under this scheme, the retailer pays a reduced unit cost \$8 (instead of \$65), but the retailer keeps approximately 50% of the rental revenue per rental unit. In this scenario, $\kappa_1 = \frac{8}{65} = 0.12$, $\kappa_2 = 0.5$, and $\frac{\kappa_1}{\kappa_2} = \frac{0.12}{0.5} = 0.24 < 1$. Therefore, Corollary 8 suggests that the optimal rental price under the revenue sharing scheme $\tilde{p}$ is lower than the optimal rental price p^* for the base case. This result is consistent with the empirical finding presented in Mortimer (2004). Specifically, Mortimer (2004) shows that, for any given movie title, stores that operate under revenue sharing contracts with their distributors tend to charge lower rental prices.

Similarly, by comparing the retailer's optimal profit $\tilde{\pi}$ under the revenue sharing scheme and the retailer's optimal profit π^* for the base case given in (12), we can show that:

Corollary 9. *Suppose* $\frac{\kappa_1}{\kappa_2} < 1$. *Then the retailer's optimal profit* $\tilde{\pi}$ *under the revenue sharing scheme is higher than the retailer's optimal profit for the base case* π^* *if* $\frac{\kappa_1}{\kappa_2}$ *is sufficiently small and* α *is sufficiently large.*

Corollary 9 suggests that, when $\frac{\kappa_1}{\kappa_2}$ is sufficiently small, the revenue sharing scheme will enable the retailer to obtain a higher profit only when the market potential α is sufficiently large. To relate this result with an actual application, we shall consider the revenue sharing scheme in the video rental industry as described in Mortimer (2004). Corollary 9 corroborates nicely with the empirical result generated from the regression analyses conducted by Mortimer (2004). Specifically, her empirical analysis suggests that revenue sharing contracts have a small positive effect on retailer profit for popular titles (high market potential). When $\frac{\kappa_1}{\kappa_2}$ is sufficiently small, Corollaries 8 and 9 suggest that the customers will enjoy a lower rental price and the retailer will enjoy a higher profit under the revenue sharing scheme.[6]

5. Retail competition

We now expand our base model to analyze a situation in which two retailers, say, firm A and firm B, compete on rental price and rental duration. Consider the case when both firms pay the same unit cost c; however, firm j charges a rental price p_j per unit for a rental duration r_j, where $j = A, B$. We assume that the rental demand for firm j in period t, denoted by D_{jt}, takes on the following form:

$$D_{jt} = \mu_j(p_A, p_B; r_A, r_B) + \epsilon \quad \text{for } j = A, B, \tag{20}$$

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[6] In order for the distributor to offer such revenue sharing scheme, the distributor needs to choose the parameters κ_1 and κ_2 carefully so that the distributor will enjoy a higher profit as well. Let $\tilde{\pi}_d(\kappa_1,\kappa_2)$ be the distributor's average profit per period under the revenue sharing scheme. Notice that $\kappa_1 = 1$ and $\kappa_2 = 1$ corresponds to the case with no revenue sharing. In this case, the distributor need to select the optimal parameters so as to maximize his profit while meeting the incentive compatibility requirement imposed by the retailer, say, $\tilde{\pi}(\kappa_1,\kappa_2) \geqslant K$, where $\tilde{\pi}(\kappa_1,\kappa_2)$ corresponds to the retailer's optimal profit given in (19). In this case, the distributor's problem can be formulated as: $\mathrm{MAX}_{0\leqslant\kappa_1,\kappa_2\leqslant 1}\tilde{\pi}_d(\kappa_1,\kappa_2)$, subject to $\tilde{\pi}(\kappa_1,\kappa_2) \geqslant K$. The analysis of the distributor's problem is beyond the scope of this paper.

where ϵ is normally distributed with mean 0 and standard deviation σ, and ϵ is time independent. To model the competition in rental price and rental duration, we assume that the mean demand possesses the following functional form:

$$\mu_j(p_A, p_B; r_A, r_B) = \left(\alpha - \beta p_j - \frac{\gamma}{r_j}\right) - \delta\left\{\beta(p_j - p_i) + \gamma\left(\frac{1}{r_j} - \frac{1}{r_i}\right)\right\} \quad \text{for } j = A, B, \tag{21}$$

where $\delta \in [0, 1]$.[7] Notice that the first term captures the underlying demand for firm j and the second term captures the loss (or gain) in demand due to the store switching behavior of some customers. For example, firm j will suffer a loss in demand when the rental price at firm i is lower or when the rental duration of firm i is longer.[8] The linear switching functional form expressed in the second term of (21) has been utilized in marketing research when considering brand switching behavior or store switching behavior. The reader is referred to Raju et al. (1995), McCardle et al. (2004) and Balasubramanina and Bhardwaj (2004) for details.

For notational convenience, let:

$$\begin{aligned} &\beta_1 = \beta(1+\delta), \quad \beta_2 = \beta\delta, \quad \gamma_1 = \gamma(1+\delta), \quad \gamma_2 = \gamma\delta, \\ &\alpha_{jr} = \alpha - \frac{\gamma_1}{r_j} + \frac{\gamma_2}{r_i}, \quad \text{for } j = A, B, \ i \neq j. \end{aligned} \tag{22}$$

In this case, the mean demand $\mu_j(p_A, p_B; r_A, r_B)$ can be simplified as

$$\mu_j(p_A, p_B; r_A, r_B) = \alpha_{jr} - \beta_1 p_j + \beta_2 p_i, \quad \text{for } j = A, B. \tag{23}$$

To simplify our exposition, we shall assume that the customers will follow the same return pattern regardless of the store they rented from. In other words, the customer's return pattern is not store specific. Since we index the firms by using i and j in this section, let k_u be the cumulative return rate so that $k_u D_{jt}$ corresponds to the cumulative total number of units returned to firm j between periods $t+1$ and $t+u$. Analogous to $n_1(r)$ and $n_2(r)$ in Section 3.2, let:

$$n_1(r_j) = 1 + \sum_{u=1}^{r_j}(1 - k_u), \quad \text{for } j = A, B, \quad \text{and} \tag{24}$$

$$n_2(r_j) = \sqrt{1 + \sum_{u=1}^{r_j}(1 - k_u)^2}, \quad \text{for } j = A, B. \tag{25}$$

By using the same approach as presented in Section 3.2 for the base case, we can derive the optimal initial stocking level and the optimal inventory cost for each firm and show that the optimal initial stocking level I_{j0}^* and the optimal inventory cost $C(I_{j0}^*)$ for firm j are increasing in r_j and decreasing in r_i for any given rental price p_i and p_j. We omit the details. Similar to the base case, we can utilize the expression for the optimal inventory cost $C(I_{j0}^*)$ for firm j to develop the expression for the profit function of firm j, getting:

$$\pi_j^* = \text{MAX}_{p_j}\pi_j, = \text{MAX}_{p_j}\left\{\left(p_j - \frac{cn_1(r_j)}{T}\right)(\alpha_{jr} - \beta_1 p_j + \beta_2 p_i) - (h+s)\phi(z^*)\sigma n_2(r_j)\right\}, \quad \text{for } j = A, B, \tag{26}$$

where $z^* = \Phi^{-1}(\frac{s-c/T}{h+s})$. Notice that z^* is independent of the firm, and hence, we can conclude that both firms have the same customer service level.

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[7] When $p_A = p_B = p$ and $r_A = r_B = r$, the mean rental demand of each store reduces to the mean demand for the base case; i.e., $\mu_A = \mu_B = \alpha - \beta p - \frac{\gamma}{r}$. Clearly, we could have reduced the mean rental demand of each store by half so that the sum of the mean rental demand of both stores will be equal to the mean demand for the base case. However, reducing the mean rental demand by half would require us to carry a constant 0.5 throughout, which would make the exposition more complex without adding managerial insights. For this reason, we scale the total demand of both retailers so that it suffices to use the above functional form.

[8] Note that when $\delta = 0$, the mean rental demand of each store reduces to the mean demand for the base case.

5.1. Equilibrium price and profit for any given rental duration

For any given pair of rental duration (r_A, r_B) and for any given rental price of firm i, we can determine the best response price for firm j. Specifically, for any given value of p_i, we can differentiate π_j given in (26) with respect to p_j. By considering the first order condition, the best response price for firm j, denoted by $p_j^b(p_i)$, is given as

$$p_j^b(p_i) = \frac{T\alpha_{jr} + T\beta_2 p_i + \beta_1 c n_1(r_j)}{2T\beta_1} \quad \text{for } j, i = A, B \;\; j \neq i.$$

Since $\alpha_{jr} = \alpha - \frac{\gamma_1}{r_j} + \frac{\gamma_2}{r_i}$, it is easy to show that the best response price $p_j^b(p_i)$ is increasing in r_j, decreasing in r_i, increasing in p_i, and decreasing in T.

For any given rental duration (r_A, r_B), the equilibrium prices of both firms, denoted by p_A^e and p_B^e, will satisfy the following condition simultaneously:

$$p_j^e = \frac{T\alpha_{jr} + T\beta_2 p_i^e + \beta_1 c n_1(r_j)}{2T\beta_1} \quad \text{for } j, i = A, B, \;\; j \neq i.$$

By solving the above equations for the variables p_A^e and p_B^e, the equilibrium price for any given pair of rental duration (r_A, r_B) can be expressed as

$$p_j^e = \frac{T(2\beta_1\alpha_{jr} + \beta_2\alpha_{ir}) + \beta_1(2\beta_1 c n_1(r_j) + \beta_2 c n_1(r_i))}{T(4\beta_1^2 - \beta_2^2)} \quad \text{for } j, i = A, B, \;\; j \neq i. \tag{27}$$

Notice that when both firms offer the same rental duration; i.e., $r_j = r_i$, the equilibrium rental prices for both firms are the same; i.e., $p_j^e = p_i^e$. By comparing the equilibrium prices of firm A and firm B, we can show that:

Corollary 10. *$p_B^e > p_A^e$ if and only if $r_B > r_A$.*

Corollary 10 implies that the firm that offers a longer rental duration will also charge a higher rental price in equilibrium.

By substituting the equilibrium price p_j^e into μ_j given in (23), we can express the mean demand for firm j at the equilibrium price, denoted by μ_j^e, as follows:

$$\mu_j^e = \frac{\beta_1}{T(4\beta_1^2 - \beta_2^2)}[T(2\beta_1\alpha_{jr} + \beta_2\alpha_{ir}) - ((2\beta_1^2 - \beta_2^2)c n_1(r_j) - \beta_1\beta_2 c n_1(r_i))] \quad \text{for } j = A, B. \tag{28}$$

Throughout this paper, we shall assume that the mean demand $\mu_j^e > 0$ in equilibrium. Similarly, by substituting the equilibrium price p_j^e into the profit function π_j given in (26), we can show that the profit attained at the equilibrium price, denoted by π_j^e, can be expressed as

$$\pi_j^e = \frac{\beta_1}{T^2(4\beta_1^2 - \beta_2^2)^2}[T(2\beta_1\alpha_{jr} + \beta_2\alpha_{ir}) - ((2\beta_1^2 - \beta_2^2)c n_1(r_j) - \beta_1\beta_2 c n_1(r_i))]^2 - (h+s)\phi(z^*)\sigma n_2(r_j). \tag{29}$$

When there is no competition or no store switching behavior; i.e., $\delta = 0$, it is easy to check that p_j^e and π_j^e reduce to the optimal price p^* given in (11) and the optimal profit π^* given in (12), respectively.

5.2. Equilibrium

When both firms compete on rental price and rental duration, the analysis is complex when each firm has many rental durations to choose from. To simplify the exposition of our analysis, we shall consider the case in which each firm can rent the product either for 1 period or 2 periods; i.e, $r_j = 1$ or $r_j = 2$ for $j = A, B$. In this case, we need to consider 4 scenarios, namely, $(r_A, r_B) = \{(1,1),(1,2),(2,1),(2,2)\}$. For each scenario (r_A, r_B), we can substitute the corresponding value of r_A and r_B into (27) and (29) to determine the within-scenario equilibrium price $p_j^e(r_A, r_B)$ and the within-scenario equilibrium profit $\pi_j^e(r_A, r_B)$. Consequently, we can construct the normal form of the competition metagame that lists each retailer's strategies (in terms of rental duration r_j) and the payoff associated with each scenario (r_A, r_B). By examining the payoff function for each of the four scenarios, we can obtain the equilibrium in terms of rental duration.

5.2.1. Within-scenario equilibrium price

We now analyze the within-scenario equilibrium price $p_j^e(r_A, r_B)$ associated with each of the four scenarios $(r_A, r_B) = \{(1,1),(1,2),(2,1),(2,2)\}$. It follows from (27) that, when $(r_A, r_B) = (1,1)$, we have:

$$p_A^e(1,1) = p_B^e(1,1) = \frac{T(\alpha - \gamma) + (1+\delta)\beta c n_1(1)}{T\beta(2+\delta)}. \tag{30}$$

When $(r_A, r_B) = (1,2)$, we have

$$p_A^e(1,2) = \frac{T(\alpha - \gamma[\frac{2+3.5\delta+0.5\delta^2}{2+3\delta}]) + (1+\delta)\beta c[\frac{2(1+\delta)n_1(1)+\delta n_1(2)}{2+3\delta}]}{T\beta(2+\delta)}, \quad \text{and} \tag{31}$$

$$p_B^e(1,2) = \frac{T(\alpha - \gamma[\frac{1+\delta-0.5\delta^2}{2+3\delta}]) + (1+\delta)\beta c[\frac{2(1+\delta)n_1(2)+\delta n_1(1)}{2+3\delta}]}{T\beta(2+\delta)}. \tag{32}$$

When $(r_A, r_B) = (2,1)$, it is easy to show that $p_A^e(2,1) = p_B^e(1,2)$ and $p_B^e(2,1) = p_A^e(1,2)$, due to symmetry. Finally, when $(r_A, r_B) = (2,2)$, we have

$$p_A^e(2,2) = p_B^e(2,2) = \frac{T(\alpha - 0.5\gamma) + (1+\delta)\beta c n_1(2)}{T\beta(2+\delta)}. \tag{33}$$

Similar to the numerical analysis presented in Section 3.3 for the monopoly case, we also checked for the goodness of approximation for each of the four scenarios $(r_A, r_B) = \{(1,1),(1,2),(2,1),(2,2)\}$. We embedded the simulation program for the monopoly case in an iterative procedure to search for the exact equilibrium prices. The exact equilibrium prices (Table IA) matched closely with the approximate analytical expressions given in Eqs. (30)–(32) (Table IB).

Table IA
Prices for Firm A and Firm B using exact numerical solution ($\sigma = 5$, s = 4, h = 0.1, c = 10, T = 20, $\delta = 0.5$)

	$r_B = 1$	$r_B = 2$
$r_A = 1$	4.4, 4.4	4.3, 5.1
$r_A = 2$	5.1, 4.3	5.0, 5.0

Table IB
Prices for Firm A and Firm B using approximate analytical solution ($\sigma = 5$, s = 4, h = 0.1, c = 10, T = 20, $\delta = 0.5$)

	$r_B = 1$	$r_B = 2$
$r_A = 1$	4.3, 4.3	4.2, 4.8
$r_A = 2$	4.8, 4.2	4.7, 4.7

Table IIA
Profits for Firm A and Firm B using exact numerical solution ($\sigma = 5$, s = 4, h = 0.1, c = 10, T = 20, $\delta = 0.5$)

	$r_B = 1$	$r_B = 2$
$r_A = 1$	2232, 2232	2175, $\underline{2641}$
$r_A = 2$	$\underline{2641}$, 2175	$\underline{2562}$, $\underline{2562}$

Best responses are marked by $\underline{\quad}$.

Table IIB
Profits for Firm A and Firm B using approximate analytical solution ($\sigma = 5$, s = 4, h = 0.1, c = 10, T = 20, $\delta = 0.5$)

	$r_B = 1$	$r_B = 2$
$r_A = 1$	2034, 2034	1967, 2285
$r_A = 2$	$\underline{2285}$, 1967	$\underline{2543}$, $\underline{2543}$

Best responses are marked by ___.

For each of the four scenarios, we used the exact equilibrium prices to calculate the exact equilibrium profit for both firms (Table IIA). We compared the exact profits to the profits π_A^e and π_B^e given in (29) that are based on the approximate analytical solutions in (30)–(32) (Table IIB). We found that the equilibrium in the meta-game predicted by the exact and the approximate methods is $(r_A,r_B) = (2,2)$. This indicates that our analytical approximation works well even in the competitive case.

In the remainder of the paper, we shall use our analytical approximation to analyze the competitive case. Let us define a term called τ, where τ is derived from the analysis when we compare the within-scenario equilibrium prices $p_j^e(r_A, r_B)$. Let:

$$\tau = \frac{\beta c(n_1(2) - n_1(1))}{0.5T}. \tag{34}$$

By examining the expression for τ, it is more likely to have the rental duration sensitivity $\gamma > \tau$ when the planning horizon T is sufficiently long. For this reason, we shall restrict our attention to the case when $\gamma > \tau$ for the remainder of this paper. The analysis associated with the case when $\gamma < \tau$ is essentially the same. We omit the details.

By comparing the within-scenario equilibrium prices $p_j^e(r_A, r_B)$, we can establish the following Proposition:

Proposition 11. *The within-scenario equilibrium price $p_j^e(r_A, r_B)$ has the following properties*:

1. $p_A^e(1,1) > p_A^e(1,2)$ *if and only if* $\gamma > \tau$.
2. $p_A^e(1,2) < p_A^e(2,2)$.
3. $p_A^e(2,1) > p_A^e(1,1)$.
4. $p_A^e(2,1) > p_A^e(2,2)$ *if and only if* $\gamma > \tau$.
5. $p_B^e(1,1) < p_B^e(1,2)$.
6. $p_B^e(1,2) > p_B^e(2,2)$ *if and only if* $\gamma > \tau$.
7. $p_B^e(2,1) < p_B^e(1,1)$ *if and only if* $\gamma > \tau$.
8. $p_B^e(2,1) < p_B^e(2,2)$.

By examining items 2, 3, 5, and 8 in Proposition 11, we can show that the within-scenario equilibrium price of firm j is increasing in r_j for any fixed rental duration of firm i; i.e., $p_A^e(1,r_B) < p_A^e(2,r_B)$ and $p_B^e(r_A,1) < p_B^e(r_A,2)$. This observation implies that a firm will increase his rental price when offering a longer rental duration. Next, by observing items 1, 4, 6 and 7 as stated in Proposition 11, we notice that, for any fixed rental duration of firm j, the within-scenario equilibrium price of firm j is decreasing in r_i when $\gamma > \tau$; i.e., $p_A^e(r_A,1) > p_A^e(r_A,2)$ and $p_B^e(1,r_B) > p_B^e(2,r_B)$. This observation implies that a retailer will reduce his rental price when his competitor offers a longer rental duration.

5.2.2. Within-scenario equilibrium profit

We now analyze the within-scenario equilibrium price $\pi_j^e(r_A, r_B)$ associated with each of the four scenarios $(r_A,r_B) = \{(1,1),(1,2),(2,1),(2,2)\}$. It follows from (29) that, when $(r_A,r_B) = (1,1)$, we have

$$\pi_A^e(1,1) = \pi_B^e(1,1) = \frac{1+\delta}{T^2\beta(2+\delta)^2}[T(\alpha-\gamma) - c\beta n_1(1)]^2 - (h+s)\phi\sigma n_2(1). \tag{35}$$

When $(r_A, r_B) = (1,2)$, we have

$$\pi_A^e(1,2) = \frac{1+\delta}{T^2\beta(2+\delta)^2}\left[T\left(\alpha-\gamma\left[\frac{2+3.5\delta+0.5\delta^2}{2+3\delta}\right]\right)-c\beta\left[\frac{(2+4\delta+\delta^2)n_1(1)-\delta(1+\delta)n_1(2)}{(2+3\delta)}\right]\right]^2 - (h+s)\phi\sigma n_2(1), \quad \text{and} \tag{36}$$

$$\pi_B^e(1,2) = \frac{1+\delta}{T^2\beta(2+\delta)^2}\left[T\left(\alpha-\gamma\left[\frac{(1+\delta-0.5\delta^2)}{(2+3\delta)}\right]\right)-c\beta\left[\frac{(2+4\delta+\delta^2)n_1(2)-\delta(1+\delta)n_1(1)}{(2+3\delta)}\right]\right]^2 - (h+s)\phi\sigma n_2(2). \tag{37}$$

When $(r_A, r_B) = (2,1)$, it is easy to show that $\pi_A^e(2,1) = \pi_B^e(1,2)$ and $\pi_B^e(2,1) = \pi_A^e(1,2)$ due to symmetry. Finally, when $(r_A, r_B) = (2,2)$, we have

$$\pi_A^e(2,2) = \pi_B^e(2,2) = \frac{1+\delta}{T^2\beta(2+\delta)^2}[T(\alpha-0.5\gamma)-c\beta n_1(2)]^2 - (h+s)\phi\sigma n_2(2). \tag{38}$$

5.2.3. Equilibrium

Let us define two terms X and Y, which will be shown to be critical points that signify changes in the rental duration to be chosen by each firm in equilibrium. Specifically, these two critical points are derived from the analysis when we compare the within-scenario equilibrium profit of each firm under different scenarios. Let:

$$X = 0.5\gamma\frac{3+4\delta-0.5\delta^2}{2+3\delta} + \beta c\frac{(2+2\delta-\delta^2)n_1(1)+(2+4\delta+\delta^2)n_1(2)}{2T(2+3\delta)} + \frac{(h+s)\phi\sigma(n_2(2)-n_2(1))}{[\gamma-\tau]\frac{(1+\delta)(2+4\delta+\delta^2)}{\beta(2+\delta)^2(2+3\delta)}},$$

$$Y = 0.5\gamma\frac{3+5\delta+0.5\delta^2}{2+3\delta} + \beta c\frac{(2+4\delta+\delta^2)n_1(1)+(2+2\delta-\delta^2)n_1(2)}{2T(2+3\delta)} + \frac{(h+s)\phi\sigma(n_2(2)-n_2(1))}{[\gamma-\tau]\frac{(1+\delta)(2+4\delta+\delta^2)}{\beta(2+\delta)^2(2+3\delta)}}. \tag{39}$$

Lemma 12. *Suppose* $\gamma > \tau$. *Then* $Y > X$.

The following Proposition compares the within-scenario equilibrium profits $\pi_j^e(r_A, r_B)$:

Proposition 13. *Suppose* $\gamma > \tau$. *Then the within-scenario equilibrium profit* $\pi_j^e(r_A, r_B)$ *has the following properties*:

1. $\pi_A^e(1,1) > \pi_A^e(1,2)$.
2. $\pi_A^e(1,2) < \pi_A^e(2,2)$ *if and only if* $\alpha > Y$.
3. $\pi_A^e(2,1) > \pi_A^e(1,1)$ *if and only if* $\alpha > X$.
4. $\pi_A^e(2,1) > \pi_A^e(2,2)$.
5. $\pi_B^e(1,1) < \pi_B^e(1,2)$ *if and only if* $\alpha > X$.
6. $\pi_B^e(1,2) > \pi_B^e(2,2)$.
7. $\pi_B^e(2,1) < \pi_B^e(1,1)$.
8. $\pi_B^e(2,1) < \pi_B^e(2,2)$ *if and only if* $\alpha > Y$.

By considering the payoff matrix associated with the four scenarios $(r_A, r_B) = \{(1,1),(1,2),(2,1),(2,2)\}$ and by utilizing the results provided in the above proposition, we can establish the following Proposition:

Proposition 14. *Suppose* $\gamma > \tau$. *Then*:

1. *Both firms will choose* 1*-period rental duration* (*i.e.*, $r_A = 1, r_B = 1$) *in equilibrium when* $\alpha < X$.
2. *One firm will compete on price while the other firm will compete on rental duration* (*i.e.*, $(r_A, r_B) = (1,2)$ *or* $(2,1)$) *in equilibrium when* $X < \alpha < Y$; *i.e., one firm will offer a lower rental price while the other firm will offer a longer rental duration.*
3. *Both firms will choose* 2*-period rental duration* (*i.e.*, $r_A = 2, r_B = 2$) *in equilibrium when* $\alpha > Y$.

Proposition 14 has the following implications. The market equilibrium depends on the rental duration sensitivity γ, the market potential α, and the critical points X and Y, where the critical points depend on the price sensitivity β and the switching coefficient δ. Therefore, we can conclude that the market equilibrium depends heavily on the market condition. Specifically, if the market potential is low (i.e., when $\alpha < X$), Proposition 14 suggests that both firms should offer shorter rental durations so as to reduce their inventory costs in equilibrium. On the other hand, if the market potential is high (i.e., when $\alpha > Y$), then both firms should offer longer rental durations so as to increase the rental demand in equilibrium. Finally, if the market potential is in the middle range (i.e., when $X < \alpha < Y$), then one firm should compete on rental price by offering a lower price with a shorter rental duration while the other firm should compete on rental duration by offering a longer rental duration with a higher rental price in equilibrium. The same analysis will go through for the case when $\gamma < \tau$. We omit the details.

The results presented in Proposition 14 also hold for the monopolistic case. To elaborate, let us consider the case when $\delta = 0$. When $\delta = 0$, it is easy to check from (21) that the mean average demand μ_j would reduce to the mean demand for the monopolistic case μ given in (1). Also, it is easy to check from (39) that X and Y can be simplified as:

$$X = Y = 0.5\gamma\frac{3}{2} + \beta c\frac{n_1(1) + n_1(2)}{2T} + \frac{(h+s)\phi\sigma(n_2(2) - n_2(1))}{[\gamma - \tau]\frac{1}{4\beta}}.$$

Since $X = Y$, we can apply Proposition 14 to show that, in a monopolistic situation, it is optimal for the retailer j to set $r_j = 1$ when $\alpha < X$, and set $r_j = 2$ when $\alpha > X$.

6. Concluding remarks

To examine the impact of rental duration on the optimal initial stocking level, the optimal rental price and the retailer's optimal profit, we have constructed a base model that captures the dynamics of uncertain rental demand and return processes. In addition, we have developed an approximation scheme that enabled us to obtain tractable results. Our base model has the following limitations. Firstly, we consider the case in which the underlying rental demand in each period is an i.i.d. random variable. This assumption is appropriate for stable and non-seasonal rental items. However, for seasonal rental items such as new releases of videos, the rental process is likely to be non-stationary. Secondly, our base model does not allow replenishment or disposal of stocks in subsequent periods. In certain situations, the retailer may wish to dispose some stocks especially when the rental demand decreases over time. Thirdly, our demand model assumes that all unmet demands are backordered. We address this issue by imposing a shortage cost. In certain situations, certain customers may not be willing to wait for the products in future periods. While the base model is a stylized model, we believe that our base model can be used as a building block for analyzing more in-depth questions. Some potential in-depth questions could include: How should a rental store react when one of the competitors offers longer rental duration? lower rental price? quantity discount (e.g., rent two different products for the price of one)? When renting multiple products, how should a rental store determine the optimal product assortment? How should a rental store account for the fact that the products are substitutable? How should a rental store behave strategically when there are many competitors?

There is a potentially new research issue arising from the on-line video rental store Netflix. Netflix provides an interesting value proposition to their customers for $20 per month. Specifically, each Netflix customer can provide a wish list of movies he would like to watch, and Netflix will send three DVDs by mail along with prepaid return postage envelopes that would allow customers to return the DVDs free of charge. Also, Netflix does not impose any rental duration so that their customers can return the DVDs anytime. In the context of our model, Netflix's business model charges a flat fee per month for a longer rental duration. Notice that Netflix's business model enables them to 'pool' their inventory across different movie titles because Netflix is required to ship any three DVDs on the customer's wish list. By pooling the inventory across different titles, Netflix can reduce inventory cost significantly. We plan to develop a model to analyze Netflix's business model in the near future.

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Appendix

Proof of Lemma 1. For any rental duration r, $k_i^r \geqslant k_i^{r+1}$ for $i = 1, \ldots, r$. Therefore, we have $(1 - k_i^{r+1}) \geqslant (1 - k_i^r)$ for $i = 1, \ldots, r$. Hence, it is easy to show that $n_1(r+1) \geqslant n_1(r)$ and $n_2(r+1) \geqslant n_2(r)$. □

Proof of Proposition 2. Differentiate the profit function π given in (10) with respect to p and then rearrange the terms, we obtain (11). By substituting p^* into π, we obtain the optimal profit function π^* given in (12). Clearly, p^* is increasing in c and decreasing in T. Also, it follows from Lemma 1 that $n_1(r)$ is increasing in r, we can conclude that p^* is increasing in r. □

Next, observe that the optimal profit π^* is decreasing in c and σ. To show that π^* is increasing in T, let us differentiate π^* with respect to T, getting

$$\frac{\mathrm{d}\pi^*}{\mathrm{d}T} = \frac{\alpha_r - \frac{\beta c n_1(r)}{T}}{2\beta} \cdot \frac{\beta c n_1(r)}{T^2} + (h+s)\sigma n_2(r)\left[\left(\frac{c}{T^2(h+s)}\right)\Phi^{-1}\left(\frac{s - c/T}{h+s}\right)\right].$$

Hence, we can conclude that $\frac{\mathrm{d}\pi^*}{\mathrm{d}T} > 0$. By considering the second derivative of π^*, one can show that π^* is concave in T when α is sufficiently large. We omit the details.

Proof of Corollary 3. When $k_i - k_{i-1} = \frac{1}{r}$, it is easy to see that $k_i = \frac{i}{r}$. In this case, $n_1(r) = 1 + \sum_{i=1}^{r}(1 - k_i) = 1 + \frac{1}{r}\sum_{i=1}^{r}(r - i) = \frac{(r+1)}{2}$. Combining the fact that $\alpha_r = \alpha - \frac{\gamma}{r}$ is concave in r and the fact that $n_1(r)$ is linear in r, we can conclude that the optimal rental price p^* given in (11) is concave in r. □

Proof of Lemma 4. By examining the expressions given in (16) and (17), we can show that:

$$\tilde{\pi}(p) - \pi(p) = (1 - \kappa_1)\frac{c\mu n_1(r)}{T} - (1 - \kappa_2)p\mu + (h+s)\sigma n_2(r)[\phi(z^*) - \phi(\tilde{z})].$$

Hence, $\tilde{\pi}(p) - \pi(p) > 0$ if and only if $(1 - \kappa_1)\frac{c\mu n_1(r)}{T} - (1 - \kappa_2)p\mu + (h+s)\sigma n_2(r)[\phi(z^*) - \phi(\tilde{z})] > 0$. We can prove the statement by rearranging the terms. This completes the proof. □

Proof of Lemma 5. By examining the expressions given in (14) and (15), we can show that

$$\tilde{\pi}_d(p) - \pi_d(p) = -(1 - \kappa_1)c(\mu n_1(r) + z^*\sigma n_2(r)) + (\kappa_1 c - m)\sigma n_2(r)[\tilde{z} - z^*] + (1 - \kappa_2)p\mu T.$$

Hence, $\tilde{\pi}_d(p) - \pi_d(p) > 0$ if and only if $-(1 - \kappa_1)c(\mu n_1(r) + z^*\sigma n_2(r)) + (\kappa_1 c - m)\sigma n_2(r)[\tilde{z} - z^*] + (1 - \kappa_2)p\mu T > 0$. We can prove the statement by rearranging the terms. This completes the proof. □

Proof of Proposition 6. By combining the conditions stated in Lemmas 4 and 5, we can obtain the condition. To prove that there exists an $1 > \kappa_2 > 0$ that satisfies the inequalities stated in the Proposition, we need to show that: $-(1 - \kappa_1)\frac{cz^*\sigma n_2(r)}{p\mu T} + \frac{(\kappa_1 c - m)\sigma n_2(r)(\tilde{z} - z^*)}{p\mu T} > -\frac{(h+s)\sigma n_2(r)[\phi(z^*) - \phi(\tilde{z})]}{p\mu}$. This inequality will hold if the function $U(\kappa_1) > 0$ for $0 < \kappa_1 < 1$, where

$$U(\kappa_1) = -(1 - \kappa_1)\frac{cz^*}{T} + \frac{(\kappa_1 c - m)(\tilde{z} - z^*)}{T} + (h+s)[\phi(z^*) - \phi(\tilde{z})].$$

Let us consider the case when $\kappa_1 = 1$. When $\kappa_1 = 1$, it is easy to show that $\tilde{z} = z^*$ and that $U(1) = 0$. To show that the function $U(\kappa_1) > 0$ for $0 < \kappa_1 < 1$, it is sufficient to show that the function $U(\kappa_1)$ is decreasing in κ_1. By differentiating the function $U(\kappa_1)$ with respect to κ_1, one can show that

$$\begin{aligned}\frac{dU(\kappa_1)}{d\kappa_1} &= \frac{c}{T}\tilde{z} + \frac{\kappa_1 c - m}{T}\frac{d\tilde{z}}{d\kappa_1} - (h+s)\frac{d\phi(\tilde{z})}{d\kappa_1} = \frac{c}{T}\tilde{z} + \frac{\kappa_1 c - m}{T}\left(-\frac{c}{T(h+s)}\right)\tilde{z} - (h+s)\frac{c}{T(h+s)}\tilde{z}\\ &= -\frac{\kappa_1 c^2 - mc}{T^2(h+s)}\tilde{z} < 0. \end{aligned}\tag{40}$$

This completes the proof. □

Proof of Proposition 7. Differentiate the profit function π given in (13) with respect to p and then rearrange the terms, we obtain $\tilde{p}$ given in (18). By substituting $\tilde{p}$ into $\tilde{\pi}$ given in (13), we can obtain the optimal profit function $\tilde{\pi}$ as given in (19). Clearly, $\tilde{p}$ is increasing in the ratio $\frac{\kappa_1}{\kappa_2}$. For any given κ_1, it is clear that the first term of the optimal profit function $\tilde{\pi}$ given in (19) is convex in κ_2 and the second term is independent of κ_2. Hence, $\tilde{\pi}$ is convex in κ_2. Next, for any given κ_2, one can see that the first term of the optimal profit function $\tilde{\pi}$ given in (19) is convex in κ_1. By examining the second term, one can see that $\tilde{\pi}$ is convex in κ_1 if the term $\phi\left(\Phi^{-1}\left(\frac{s-\frac{\kappa_1 c}{T}}{h+s}\right)\right)$ is concave in κ_1. This is evident when we differentiate the term $\phi\left(\Phi^{-1}\left(\frac{s-\frac{\kappa_1 c}{T}}{h+s}\right)\right)$ with respect to κ_1 twice, getting

$$\frac{\mathrm{d}^2\phi\left(\Phi^{-1}\left(\frac{s-\frac{\kappa_1 c}{T}}{h+s}\right)\right)}{d\kappa_1^2} = -\frac{\frac{c^2}{T^2(h+s)^2}}{\phi\left(\Phi^{-1}\left(\frac{s-\frac{\kappa_1 c}{T}}{h+s}\right)\right)} < 0.$$

Hence, we can conclude that $\tilde{\pi}$ is convex in κ_1.

Proof of Corollary 8. The result is immediate when we compare $\tilde{p}$ given in (18) and p^* given in (11). □

Proof of Corollary 9. In preparation, let us observe that the function $\phi\left(\Phi^{-1}\left(\frac{s-\frac{\kappa_1 c}{T}}{h+s}\right)\right)$ is increasing in κ_1. This is evident when we differentiate this term with respect to κ_1, getting

$$\frac{\mathrm{d}\phi\left(\Phi^{-1}\left(\frac{s-\frac{\kappa_1 c}{T}}{h+s}\right)\right)}{\mathrm{d}\kappa_1} = \frac{c}{T(h+s)}\Phi^{-1}\left(\frac{s-\frac{\kappa_1 c}{T}}{h+s}\right) > 0.$$

Since $\kappa_1 < 1$ and since $\phi\left(\Phi^{-1}\left(\frac{s-\frac{\kappa_1 c}{T}}{h+s}\right)\right)$ is increasing in κ_1, we can conclude that $\phi(\tilde{z}) < \phi(z^*)$.

Let us now compare $\tilde{\pi}$ with π^* given in (19) and (12), respectively. By rearranging the terms, one can show that

$$\tilde{\pi} - \pi^* = \frac{1}{4\beta}\cdot\left(2\alpha_r c'(1-\kappa_1) - (1-\kappa_2)\alpha_r^2 - (c')^2\left(1 - \frac{\kappa_1^2}{\kappa_2}\right)\right) + (h+s)\sigma n_2(r)(\phi(z^*) - \phi(\tilde{z})),\tag{41}$$

where $c' = \frac{\beta c n_1(r)}{T}$.

Since $\phi(\tilde{z}) < \phi(z^*)$, we can conclude $\tilde{\pi} > \pi^*$ when $\left(2\alpha_r c'(1-\kappa_1) - (1-\kappa_2)\alpha_r^2 - (c')^2(1-\frac{\kappa_1^2}{\kappa_2})\right) > 0$. In this case, it is easy to see that $\left(2\alpha_r c'(1-\kappa_1) - (1-\kappa_2)\alpha_r^2 - (c')^2(1-\frac{\kappa_1^2}{\kappa_2})\right) > 0$ when κ_1 is small, κ_2 is large, and α is sufficiently large. This completes the proof. □

Proof of Corollary 10. It follows from (27) that:

$$p_B^e - p_A^e = \frac{T[(2\beta_1 - \beta_2)(\alpha_{Br} - \alpha_{Ar})] + \beta_1 c[(2\beta_1 - \beta_2)(n_1(r_B) - n_1(r_A))]}{T(4\beta_1^2 - \beta_2^2)}.$$

When $r_B > r_A$, we have $\alpha_{Br} = \alpha - \frac{\gamma}{r_B} > \alpha - \frac{\gamma}{r_A} = \alpha_{Ar}$, and we have $n_1(r_B) > n_1(r_A)$ because $n_1(r)$ is increasing in r. Combining these observations with the fact that $2\beta_1 = 2\beta(1+\delta) > \beta\delta = \beta_2$, we can conclude that $p_B^e - p_A^e > 0$. This completes the proof. □

Proof of Corollary 11. Observe from (22) that $\beta_1\beta_2 = \delta(1+\delta)\beta^2$, $2\beta_1^2 - \beta_2^2 = \beta^2(2 + 4\delta + \delta^2)$, and $4\beta_1^2 - \beta_2^2 = \beta^2(2+\delta)(2+3\delta)$. In this case, one can check from (30) and (32) that

$$\begin{aligned} p_A^e(1,1) - p_A^e(1,2) &= \frac{1}{T\beta(2+\delta)}\left[T\gamma\left[\frac{0.5\delta(1+\delta)}{2+3\delta}\right] - (1+\delta)\beta c\left[\frac{\delta}{2+3\delta}\right][n_1(2)-n_1(1)]\right] \\ &= \frac{0.5\delta(1+\delta)}{\beta(2+\delta)(2+3\delta)}[\gamma-\tau]. \end{aligned} \tag{42}$$

Hence, $p_A^e(1,1) > p_A^e(1,2)$ if and only if $\gamma > \tau$. Next, it follows from (32) and (33), we can show that

$$p_A^e(2,2) - p_A^e(1,2) = \frac{1}{T\beta(2+\delta)}\left[T\gamma\left[\frac{1+2\delta+0.5\delta^2}{2+3\delta}\right] + (1+\delta)\beta c\left[\frac{2+2\delta}{2+3\delta}\right][n_1(2)-n_1(1)]\right].$$

Since $n_1(r)$ is increasing in r, $n_1(2) - n_1(1) > 0$. Hence, we can conclude that $p_A^e(2,2) > p_A^e(1,2)$. By following the same approach, we can prove all other inequalities. We omit the details. □

Proof of Lemma 12. By comparing Y and X given in (39) and by rearranging the terms, we can show that $Y - X = \frac{0.5\delta(1+\delta)}{(2+3\delta)}[\gamma-\tau]$. Therefore, we can conclude that $Y > X$ if and only if $\gamma > \tau$. □

Proof of Proposition 13. It follows from (35) and (36) that:

$$\begin{aligned} \pi_A^e(1,1) - \pi_A^e(1,2) = {} & \frac{1+\delta}{T^2\beta(2+\delta)^2} \\ & * \left[[T(\alpha-\gamma) - c\beta n_1(1)] + \left[T\left(\alpha - \gamma\left[\frac{2+3.5\delta+0.5\delta^2}{2+3\delta}\right]\right) - c\beta\left[\frac{(2+4\delta+\delta^2)n_1(1) - \delta(1+\delta)n_1(2)}{(2+3\delta)}\right]\right]\right] \\ & * \left[[T(\alpha-\gamma) - c\beta n_1(1)] - \left[T\left(\alpha - \gamma\left[\frac{2+3.5\delta+0.5\delta^2}{2+3\delta}\right]\right) - c\beta\left[\frac{(2+4\delta+\delta^2)n_1(1) - \delta(1+\delta)n_1(2)}{(2+3\delta)}\right]\right]\right]. \end{aligned}$$

Notice from (28) that the mean demand $\mu_A^e > 0$ when $(r_A, r_B) = (1,1)$ and when $(r_A, r_B) = (1,2)$. This implies that $[T(\alpha-\gamma) - c\beta n_1(1)] > 0$ and that $[T(\alpha - \gamma[\frac{2+3.5\delta+0.5\delta^2}{2+3\delta}]) - c\beta[\frac{(2+4\delta+\delta^2)n_1(1)-\delta(1+\delta)n_1(2)}{(2+3\delta)}]] > 0$. Therefore, $\pi_A^e(1,1) - \pi_A^e(1,2) > 0$ if and only if the term $[T(\alpha-\gamma) - c\beta n_1(1)] - [T(\alpha - \gamma[\frac{2+3.5\delta+0.5\delta^2}{2+3\delta}])$ $-c\beta[\frac{(2+4\delta+\delta^2)n_1(1)-\delta(1+\delta)n_1(2)}{(2+3\delta)}]] > 0$. As it turns out, this term can be simplified as $\frac{0.5T\delta(1+\delta)}{(2+3\delta)}[\gamma-\tau]$. Since $\gamma > \tau$, we can conclude that $\pi_A^e(1,1) > \pi_A^e(1,2)$. This proves the first statement. □

By examining $\pi_A^e(2,2)$ and $\pi_A^e(1,2)$ given in (38) and (36), we can show that:

$$\begin{aligned} \pi_A^e(2,2) - \pi_A^e(1,2) = {} & \frac{(1+\delta)(2+4\delta+\delta^2)}{\beta(2+\delta)^2(2+3\delta)} * [\gamma-\tau] \\ & * \left[\alpha - 0.5\gamma\frac{3+5\delta+0.5\delta^2}{2+3\delta} - \beta c\frac{(2+4\delta+\delta^2)n_1(1) + (2+2\delta-\delta^2)n_1(2)}{2T(2+3\delta)}\right] - (h+s)\phi\sigma(n_2(2) - n_2(1)). \end{aligned}$$

By rearranging the terms, it is easy to check that $\pi_A^e(2,2) > \pi_A^e(1,2)$ when $\alpha > Y$, where Y is given in (39).

By using the same approach, we can prove all other statements in Proposition 13. We omit the details. □

Proof of Proposition 14. When $\alpha < X$, it is easy to apply the results stated in Proposition 13 to show that $(r_A, r_B) = (1,1)$ is the unique equilibrium for the competition game. To see this, notice that firm A will be worse off if he offers a longer rental duration because $\pi_A^e(2,1) < \pi_A^e(1,1)$ when $\alpha < X$. Similarly, firm B will be worse off if he offers a longer rental duration because $\pi_B^e(1,2) < \pi_B^e(1,1)$ when $\alpha < X$. We can use the same argument to show that $(r_A, r_B) = (2,2)$ is the unique equilibrium when $\alpha > Y$.

It remains to show that one firm competes on price while the other competes on rental duration when $X < \alpha < Y$. Specifically, when $X < \alpha < Y$, we have two equilibria, namely, $(r_A, r_B) = \{(1,2),(2,1)\}$. Notice that $\pi_A^e(2,2) < \pi_A^e(1,2)$ and $\pi_B^e(1,1) < \pi_B^e(1,2)$ when $X < \alpha < Y$. Therefore, we can conclude that $(r_A, r_B) = (1,2)$ will be an equilibrium in this case because neither firm A nor firm B has the incentive to change their rental duration. For this equilibrium $(r_A, r_B) = (1,2)$, we now show that firm A competes on price and firm B competes on rental duration by showing that $p_A^e(1,2) < p_B^e(1,2)$. To see that, observe from Proposition 8 that

$p_A^e(1,2) < p_A^e(2,2) = p_B^e(2,2) < p_B^e(1,2)$. Therefore, we can conclude that firm A competes on a lower rental price while firm B competes on a longer rental duration when $X < \alpha < Y$.

By symmetry, we can show that $(r_A, r_B) = (2,1)$ will be the second equilibrium when $X < \alpha < Y$. By using the exact same approach, we can show that firm B competes on rental price while firm A competes on rental duration in this case. We omit the details. □

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