INTERNATIONAL TRANSACTIONS IN OPERATIONAL RESEARCH

Intl. Trans. in Op. Res. 12 (2005) 595–629

Optimal procurement, disposal and pricing policies for managing rental goods

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Received 29 September 2005; received in revised form 3 October 2005; accepted 4 October 2005

Abstract

Consider a video rental retailer who procures DVDs or video cassettes from a distributor and rents them to the customers. To meet the time-varying rental demand, the retailer needs to develop cost-effective procurement and disposal policies. In this paper, we first present a base model in which the underlying rental demand is decreasing over time, backorders are not allowed and the disposal price is exogenous. For this base model, we show that the optimal procurement quantity is equal to the sum of effective demands (rental demand net of returns) over an integral number of periods, and the optimal disposal policy can be determined by solving a simple dynamic program with polynomial complexity. We then analyze the case of endogenous disposal prices and derive optimal disposal policies by solving a quadratic optimization problem with tree constraints. We also extend the base model to allow for backorders and to cases where the retailer has multiple procurement opportunities and a contractual period where disposals are not allowed. We show that the qualitative nature of the procurement policy is preserved in these cases and the optimal procurement and selling policies can be determined using similar dynamic programming algorithms.

Keywords: Procurement policy; disposal policy; pricing policy

1. Introduction

The video rental industry has grown significantly recently, especially with the introduction of the DVD format. Netflix reported in their 2003 annual report that the DVD rental market in the US accounted for $5 billion in 2003 and is estimated to grow to $10 billion by 2008. The reader is referred to Netflix's Annual report available at www.netflix.com. A video rental retailer such as Blockbuster can enter either a revenue-sharing contract or a linear pricing contract with a distributor such as Warner Brothers. In the former, the retailer can purchase the item at a reduced price, but in turn has to share the rental revenue with the distributor. In the latter, the retailer pays the usual price, which is much higher than that paid under revenue sharing, but gets to keep all the

Published by Blackwell Publishing Ltd, 9600 Garsington Road, Oxford, OX4 2DQ, UK and 350 Main St, Malden, MA 02148, USA.

rental revenue. The reader is referred to www.rentrakonline.com for a detailed description of the revenue-sharing contracts. A retailer can enter into different contracts for different titles, and different retailers can sign different contracts for the same title.

Based on weekly rental data provided on www.imdb.com, we notice that the rental demand for each title decreases over time as the popularity of the movie declines and as the number of persons who have already viewed the movie increases. As rental demand drops below a certain level, the video rental retailer would dispose of some inventory procured earlier under the linear pricing contract and move the title from the ''new releases'' section to the 'catalog' section. This need to dispose of inventory is accentuated by the limited shelf space and the need to make space for newer titles. For example, video rental stores such as Blockbuster, Hollywood and 20/20 regularly dispose of used DVDs in the secondary market. When disposing excess inventories, the retailer faces important decisions like timing of disposal, quantity to be disposed of and the selling price. Moreover, given the decreasing nature of the rental demand, the retailer also needs to decide the initial quantity to be ordered from the distributor so that he can satisfy the higher initial rental demand without incurring very high holding and procurement costs.

In this paper, we aim to derive the optimal stocking and disposal policy for a video rental retailer when the underlying rental demand is deterministic and decreasing over time. In practice, the demand might not be known to the retailer with certainty, but can be predicted fairly accurately by using box office collections and DVD sales (cf. Lehman and Weinberg, 2000). To simplify the exposition of our model, we introduce the notion of effective demand, which is the base rental demand net of returns. We first develop a base model, where we assume that the disposal price is provided exogenously to the retailer, that all the rental demand has to be met exactly in each period and that there is a fixed cost per period associated with the disposal. We model this problem as an extension of the dynamic lot sizing model by incorporating returns and analyze it using the methodology of concave production networks. We find that the optimal initial order quantity is equal to the sum of effective demands over an integral number of periods. The optimal selling policy can be derived by solving a dynamic program of complexity $O(T^3)$, where T is the time horizon of the problem. In a special case where returns possess a certain structure (i.e., more returns occur in later periods), the optimal disposal quantities can be obtained by solving a simpler dynamic program of complexity $O(T^2)$. The base methodology is later extended to account for cases where (i) the retailer might have multiple procurement opportunities; (ii) the retailer might be bound contractually by the distributor not to sell in the secondary market for a certain period of time; and (iii) the customers are patient and it might be possible to backorder unmet demand. We find that the nature of the optimal policy is preserved in all of these cases but the actual computation and quantitative results vary in each.

In this paper, we also present a model where the retailer jointly decides the optimal initial order quantity, disposal quantity and disposal price over time. We find that when the demand for disposed-off quantity is linear in disposal price, the decision problem of the retailer can be modeled as a quadratic optimization problem. Dualization of this problem results in a quadratic optimization problem with tree constraints that can be solved in polynomial time. In the case where the relevant parameters of the problem have a special structure, we show that a myopic policy is optimal. The optimal disposal policy is to sell nothing in periods with positive effective demand and to sell the minimum of the unconstrained optimum quantity and the absolute value of effective demand in periods with negative effective demand. However, in the absence of the special structure, the myopic policy is not optimal and we outline the method for finding the optimal selling quantities and initial order quantity in such a case.

The rest of the paper is organized as follows: in Section 2, we review previous research related to our work. Section 3 presents and analyzes the base model. Section 4 analyzes the case of endogenous pricing. We extend our base model to incorporate multiple purchases, contractual period of no sale and backordering in Section 5. Section 6 presents concluding remarks and future research directions.

2. Literature review

In this section, we review the literature that is relevant to our paper and outline our contribution. A considerable amount of research has been carried out to analyze various contracting arrangements in the video rental industry. Dana and Spier (2001) show the value of revenue-sharing arrangements in a vertically separated industry where the demand in the downstream sector is variable or uncertain. Revenue sharing enables the supply chain to achieve the first best outcome by softening retail price competition without distorting the retailer's stocking decisions. Cachon and Lariviere (2005) analyze the revenue-sharing contracts and compare them with other type of contracts like buy-back contracts, price discount contracts and quantity-flexibility contracts. Mortimer (2004) conducted an empirical analysis to examine the effect of the revenue-sharing scheme on the retailer's profit and concluded that it has a small positive effect for popular titles and a small negative effect for less popular titles. She also compared stocking levels, rental prices, etc. across different stores for the same title and across different titles within the same store. In a recent paper, Ferguson et al. (2005) studied contracts that coordinate the supply chain in the case of false failure returns, products returned by the customers with no functional or cosmetic defect.

Relatively less attention has been devoted to the nature of the competition among retailers in the rental business and even less to the operational decisions of a single retailer. Dana (2001) presents a strategic model of competition among retailers in price and availability. He shows that in equilibrium, retailers use higher rental prices to "signal" higher availability. Tang and Deo (2004) analyze the decisions of a single retailer facing uncertain rental demand and return process. They derive the optimal order quantity, price and profit of the retailer and study the impact of rental duration on each of these decisions. They also analyze the competition in the case of duopoly and establish conditions under which one retailer competes on price and the other competes on the rental duration. Our paper is different from Tang and Deo (2004), in which the rental demand follows a stationary independent normal distribution in each period. In that paper, the disposal decision of the retailer is not considered.

Another stream of literature closely related to our paper is the one addressing the question of optimal disposal policies. Disposal decisions have received attention in the capacity management literature. Rajagopalan et al. (1998) model capacity acquisition, disposal and replacement decisions of a firm in an environment with uncertain technological breakthroughs. They present an efficient regeneration point-based dynamic programming algorithm to solve the problem. The motivation for capacity disposals in their problem could be technological progress or decreasing demand. In the context of inventory management, Heyman (1977) is one of the earliest works to address the issue of optimal disposal policies in the case of returns. He uses diffusion process approximation to model the system inventory in the presence of returns. In this literature, the approach closest to ours can be found in Beltran and Krass (2002). They use the classical dynamic lot sizing model (cf. Wagner and Whitin, 1958) to analyze the optimal procurement and disposal decisions under time-varying deterministic demand with returns. In terms of methodology, our approach extends the traditional dynamic lot sizing models in the case of returns with backordering, where returns can be used to satisfy demand in later periods.

Our paper differs substantially from Beltran and Krass (2002) in many ways. Specifically, we model the decreasing nature of the rental demand, which is natural in the video rental industry. We also account for the rental duration and model the dynamics of the return pattern explicitly in our model. Beltran and Krass (2002) assume that the rental duration equals one period in their model, while we allow for any rental duration. In addition, we also consider the case of endogenous pricing and backordering of demand that has not been considered in Beltran and Krass (2002). Thus, we believe that our model provides important insights into various decisions faced by a retailer who manages a video rental store and is a first step in developing a decision support system to operationalize these decisions. While we focus specifically on the video rental industry as an application, our model can also be applied to other rental products and recycled products. A potential application of the multiple purchase case can be found in Bayiz and Tang (2004).

3. The base model: exogenous pricing with no backordering

Consider a retailer who purchases a product from a distributor at c_0 per unit, rents the product to customers by charging a rental price ρ per unit and then rents the product for a pre-specified rental duration of r periods, where r is a positive integer. To simplify our exposition, we shall assume that the rental price ρ is constant; however, our model can be easily extended to the case in which the rental price is exogenous and time-varying. We also assume that the rental demand $D_t \geqslant 0$ in period t is known with certainty. We do not assume any specific functional form for the demand but only assume that it is decreasing in time; i.e., $D_t \geqslant D_{t+1}, \forall t = 1, 2, 3, \ldots, T-1$. In any period t, D_t units are checked out from the retailer that are due back by the beginning of period $t+r+1$. Out of these D_t units that are rented in period t, let k_i be the return rate so that k_iD_t corresponds to the total number of units returned in period $t+i$. We assume that these return parameters k_i are known to the retailer and $\sum_{i=1}^{r+1} k_i = 1$. In each period, returns from the previous periods are followed by realization of the rental demand followed by the disposal of units in the secondary market, if any.

For any rental duration r, we consider the case in which the retailer has to decide the initial stocking level I_0 in period zero. We assume that the retailer can acquire the stock in period zero only, i.e., no replenishments are allowed in subsequent periods. In the base model, backordering of unmet demand is not allowed; however, we shall extend our base model to the case in which backorders are allowed in a later section. When backorders are not allowed, demand must be met in each period. Hence, the inventory at the end of each period is non-negative, i.e., $I_t \geqslant 0, \ \forall t = 1, 2, 3, \ldots, T$. We also assume $I_T = 0$ as the boundary condition, where T is the time horizon under consideration.

Let h_t denote the inventory holding cost per unit per period. The inventory holding cost includes the physical cost of holding and maintaining inventory as well as the cost of working capital blocked. As the demand is decreasing over time and as the units are acquired only once in period zero, the retailer would want to dispose of inventory in the later periods in the secondary market, albeit at a discount. Let $Y_t \geqslant 0$ represent the number of units sold in period t at a price p_t. We assume that the retailer incurs a fixed cost F_t per period to undertake this selling activity. This fixed cost could include the marketing and promotional expenses and the employee time associated with the selling campaign. In the base model, the retailer is allowed to dispose of inventory in any period starting from $t = 1$. In reality, there could be contractual obligations with the distributor barring the retailer from early sale in the secondary market. Such contracts are common in the video rental industry and are considered later in this paper. The reader is referred to www.rentrakonline.com for further details on contractual agreements in the video rental industry. We also assume that the disposals in the secondary market do not cannibalize the rental revenue.

We assume that the disposal price is decreasing in time and that the unit cost of procurement is greater than or equal to all of these prices; i.e., $c_0 \geqslant p_1 \geqslant p_2 \geqslant \cdots \geqslant p_T$. This avoids trivial cases where the retailer makes non-negative profits purely through purchase and sale of titles. This is also reasonable as the popularity of a movie title decreases over time and hence the price that it can command in the secondary market also drops. Also, it is regularly observed in the industry that retailers dispose of DVDs at a loss as compared with the initial purchase price.

We shall determine the optimal initial order quantity I_0 and the selling quantities in every period Y_t. We first formulate the decision problem as a mathematical program and then derive some interesting structural properties that would help us to obtain the optimal policies using dynamic programming.

3.1. Problem formulation and optimal solution structure

We write the payoff to the retailer in terms of the total costs incurred over T periods, which is given by

$$\pi(I_0; I_t; Y_t) = c_0 I_0 + \sum_{t=1}^{T} h_t I_t - \sum_{t=1}^{T} p_t Y_t + \sum_{t=1}^{T} F_t \delta(Y_t) - \sum_{t=1}^{T} \rho D_t,$$

where

$$\delta(Y_t) = \begin{cases} 1, & Y_t > 0, \\ 0, & \text{otherwise}. \end{cases}$$

The first term denotes the cost of initial procurement in period zero, the second term is the total holding cost, the third term is the total disposal revenue, the fourth term represents the fixed cost incurred in periods when units are disposed and the last term represents the total rental revenue in T periods. To determine the optimal initial procurement quantity I_0 and the optimal disposal quantity Y_t in period t that minimizes the total cost $\pi(I_0; I_t; Y_t)$, the retailer has to solve the following problem:

$$\Pi = \min_{I_0, I_t, Y_t} \{\pi(I_0; I_t; Y_t)\}$$

$$\text{subject to}: \quad I_t, Y_t, I_0 \geqslant 0, \text{ where}$$

$$I_t = \begin{cases} I_{t-1} - Y_t - D_t + \sum_{i=1}^{r+1} k_i D_{t-i}, & t \geqslant r+2, \\ I_{t-1} - Y_t - D_t + \sum_{i=1}^{t-1} k_i D_{t-i}, & t \leqslant r+1. \end{cases} \tag{1}$$

To simplify the inventory balance equations (1) and to provide some additional insights into the problem, let us define the following terms:

$$\begin{aligned} \hat{D}_1 &= D_1, \\ \hat{D}_2 &= D_2 - k_1 D_1, \\ \hat{D}_3 &= D_3 - k_1 D_2 - k_2 D_1, \\ &\vdots \\ \hat{D}_t &= \begin{cases} D_t - \sum_{i=1}^{t-1}, k_i D_{t-i} & t \leqslant r+1, \\ D_t - \sum_{i=1}^{r+1} k_i D_{t-i}, & t \geqslant r+2. \end{cases} \end{aligned}$$

As we assume that, in any period, returns occur before the new rentals in that period, $\hat{D}_t$ can be interpreted as the "effective demand" in period t, net of all returns originating from demand in earlier periods. As all demand must be met in each period, $\sum_{t=1}^{T} \rho D_t$ is a constant. Thus, by considering the relevant terms and using the notation of effective demand to rewrite the inventory balance equations in (1), the retailer's problem can be reformulated as

$$\Pi = \min_{I_0, I_t, Y_t} \left\{ c_0 I_0 + \sum_{t=1}^{T} h_t I_t - \sum_{t=1}^{T} p_t Y_t + \sum_{t=1}^{T} F_t \delta(Y_t) \right\}$$

$$\begin{aligned} &\text{subject to}: \quad I_t, Y_t, I_0 \geqslant 0, \text{ where} \\ &\qquad I_t = I_{t-1} - Y_t - \hat{D}_t, \quad \forall t \geqslant 1. \end{aligned} \tag{2}$$

The retailer's problem is a mixed integer linear program (MILP) that can be solved using the usual methods available to solve MILPs. Another option is to recast the retailer's problem as a production network flow problem that can be solved easily by using dynamic programming methodology (cf., Denardo, 1982). Specifically, we first characterize the optimal solution by exploiting the underlying structure of the production network. We then utilize these special properties of the optimal solution to determine the optimal solution by solving a simple dynamic program.

We now recast the retailer's problem as a network flow problem. The corresponding network is depicted in Fig. 1. Notice that the nodes of the network represent time periods and flows along the arcs represent inventories, purchase and disposal quantities. In this network, a loop is a cycle such that the flow of each arc in the cycle is non-zero.

$$\hat{D}_t = \begin{cases} D_t - \sum_{i=1}^{t-1} k_i D_{t-i}; & t \le r+1 \\ D_t - \sum_{i=1}^{r+1} k_i D_{t-i}; & t > r+1 \end{cases}$$

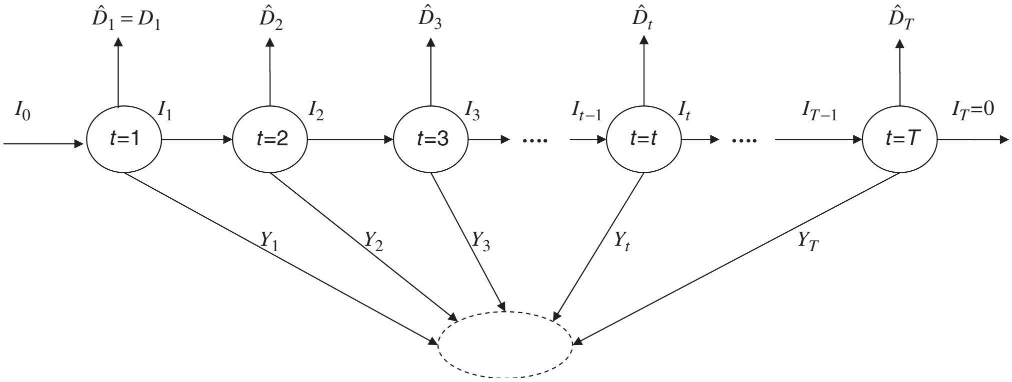

Fig. 1. A network flow diagram capturing inventories, effective demands and disposal quantities.

By examining the network structure in Fig. 1, we establish the following Proposition.

Proposition 1. *There exists an optimal solution that is loopless.*

Proof. All proofs are provided in Appendix A.

Proposition 1 enables us to reduce the state space for the subsequent dynamic programming formulation because we can search for the optimal solution by focusing on the set of feasible flows that satisfy the loopless property. However, in order to use this property effectively, we first need to derive some useful properties of effective demand. We know that

$$\hat{D}_t = D_t - \sum_{i=1}^{r+1} k_i D_{t-i}, \quad t \geqslant r+2.$$

Also, since k_i is the return rate as defined above, $\sum_{i=1}^{r+1} k_i = 1$. Substituting this into the above equation, we obtain

$$\begin{aligned} \hat{D}_t &= D_t \left(\sum_{i=1}^{r+1} k_i \right) - \sum_{i=1}^{r+1} k_i D_{t-i} \\ &= \sum_{i=1}^{r+1} (D_t - D_{t-i}) k_i \\ &< 0, \quad \text{since } D_t \leqslant D_{t-i} \quad \forall i. \end{aligned}$$

This observation is restated in Lemma 2.

Lemma 2. $\hat{D}_t \leqslant 0 \; \forall t \geqslant r+2$, *i.e., the effective demand in each period during or after* $t = r+2$ *is non-positive.*

Even though the underlying demand D_t is decreasing in t, the effective demand $\hat{D}_t$ may not be decreasing in t. In fact, it is easy to verify that the effective demand can change signs from positive to negative and back in each of the earlier periods $2 \leqslant t \leqslant r+1$ depending on the relative values of k_i and D_t. Specifically, negative effective demand implies that the return from earlier rentals is greater than the rental demand during that period. We know that effective demand is negative for all periods after $r+2$. But it is also possible that the effective demand might change sign and become negative for the last time before $r+2$. We define this period as τ, where

$$\tau = \text{argmax}\{t : \hat{D}_t > 0\}. \tag{3}$$

Based on the definition of τ, we have $\hat{D}_t \leqslant 0, \; \forall t > \tau$. Given τ, the next two lemmas explore the implication of this property on the decision variables of the problem I_0 and Y_t and the state variables I_t.

Lemma 3. (i) *There exists a loopless solution only if* $s \leqslant \tau$ *where* $s = \text{argmin}\ \{t : I_t^* = 0\}$.

(ii) *It is optimal not to sell before and during period s, i.e.,* $Y_t^* = 0$ *for* $1 \leqslant t \leqslant s$.

(iii) *The optimal initial purchase quantity is given by* $I_0^* = \sum_{t=1}^{s} \hat{D}_t$.

Lemma 3 is key to designing an efficient dynamic programming algorithm. Result (i) implies that a necessary condition for an optimal loopless solution is that the inventory in one of the periods s where $s \leqslant \tau < r+2$ should be zero. This property is similar to the regenerative point property in dynamic lot sizing models and we show that it can be extended to the model with returns. Result (ii) of the lemma shows that the time horizon is effectively divided into two parts and it is optimal not to dispose in the first part, i.e., until period s. Result (iii) shows that procurement policy has a simple form and equals the sum of effective demands over an integral period of time. While we know that the actual quantity can be calculated only after finding s where $s \leqslant \tau$, we already know that the upper bound on the initial procurement quantity is given by $\hat{I}_0 = \sum_{t=1}^{\tau} \hat{D}_t$.

Since we also know that $I_T^* = 0$, Lemma (i) guarantees that there exist two time periods $t = i, j; s \leqslant i, j \leqslant T$ such that $I_i^* = 0, I_j^* = 0$. Hence, we characterize each solution by picking two time periods and setting the inventory in each of these periods to zero. The next step is to find the optimal disposal decisions Y_t^* between $t = i$ and $t = j$ and then to calculate eventually the total cost of operation between these two periods.

Lemma 4. *Consider the case where* $I_i^* = 0, I_j^* = 0$ *and* $I_t^* > 0, s \leqslant i, i < t < j$. *We have*:

(i) $\hat{D}_{i+1} < 0$.

(ii) *If* $\hat{D}_j < 0$, *then* $Y_j^* = -\sum_{t=i+1}^{j} \hat{D}_t$ *and* $Y_t^* = 0;\; i+1 \leqslant t \leqslant j-1$.

(iii) *If* $\hat{D}_j > 0$, *then there is at most one k, such that* $Y_k^* > 0$, *where* $i+1 \leqslant k \leqslant j$.

Result (i) states that in the optimal solution, the inventory at the end of any period $i \geqslant s$ can be zero only if the effective demand in the next period is going to be negative. This is because otherwise it would be impossible to satisfy a positive effective demand in the next period without any inventory on hand. Results (ii) and (iii) imply that the optimal solution would involve selling only in one period between i and j in order to avoid the formation of a loop. The optimal period of selling depends on the sign of the effective demand in period j. Hence, selling can occur only in period j when the effective demand in period j is negative, because if that is not the case, we would end up with positive inventory at the end of period j. On the other hand, if the effective demand in period j is positive, then selling can occur in any period between i and j. In the optimal solution, this period would be chosen such that it minimizes the cost of operation between i and j. Notice that result (iii) suggests that it is possible to have no selling between periods i and j if the negative and positive effective demands exactly balance each other.

3.2. Dynamic programming formulation

Having obtained the necessary structural properties of the optimal solution, we now propose a dynamic programming algorithm to solve the retailer's problem optimally. Let:

- $f(1,s) \equiv$ minimum cost of operation from period 1 through s,
- $V(s) \equiv$ minimum cost of operation from period s to period T and
- $c(i,j) \equiv$ minimum cost of operation from period $i+1$ through period j, where $I_i^* = I_j^* = 0$.

Then, the results in Lemma 3 imply that

$$\Pi = \min_{s=1,2,\ldots,\tau} \{f(1,s) + V(s)\}. \tag{4}$$

Lemmas 3 and 4 enable us to decompose effectively the original problem into two subproblems. In the first subproblem, we use Lemma 3 to calculate the function $f(1,s)$. Specifically, results (ii) and (iii) in Lemma 3 suggest that $f(1,s) = c_0 \sum_{t=1}^{s} \hat{D}_t + \sum_{t=1}^{s-1} h_t I_t$, where $I_t = \sum_{i=t+1}^{s} \hat{D}_i$. Thus, for a given value of s, $f(1,s)$ is very easy to calculate. In the second subproblem, we use Lemma 4 to formulate a dynamic program to calculate the function $V(s)$. Note that $I_s^* = 0$ in the optimal solution and $I_T^* = 0$. Hence, there exists at least one case in which $I_i^* = 0$ and $I_j^* = 0$. Then for any given pair (i,j) that has $I_i^* = 0$ and $I_j^* = 0$, we can formulate the second subproblem as the following dynamic program: $V(i) = \min_{j>i}\{c(i,j) + V(j)\}$. For a given i, this is exactly the shortest path problem that can be solved in polynomial time if we know the value of $c(i,j)$. Next, we set out to find the cost $c(i,j)$. Again, the result in Lemma 4 can be used for this purpose.

Case I. $\hat{D}_{i+1} < 0, \hat{D}_j < 0$.
We know from Lemma 4 that $Y_j^* = -\sum_{t=i+1}^{j} \hat{D}_t$ and $Y_t^* = 0; i+1 \leqslant t \leqslant j-1$. Then we have

$$c(i,j) = F_j - p_j^* Y_j^* + \sum_{t=i+1}^{j} h_t I_t^*,$$

where

$$I_t^* = -\sum_{k=i+1}^{t} \hat{D}_k, \quad i+1 \leqslant t \leqslant j-1. \tag{5}$$

Case II. $\hat{D}_{i+1} < 0, \hat{D}_j > 0$.

Again from Lemma 4 we know that there exists at most one $k; i+1 \leqslant k \leqslant j$ such that $Y_k^* = -\sum_{t=i+1}^{j} \hat{D}_t$. As $c(i,j)$ denotes the minimum cost of operation between periods i and j, we have

$$c(i,j) = \min_k \left\{ F_k - p_k Y_k^* + \sum_{t=i+1}^{j} h_t I_t^* \right\},$$

where

$$I_t^* = \begin{cases} -\sum\limits_{l=i+1}^{t} \hat{D}_l, & i+1 \leqslant t \leqslant k-1, \\ -\sum\limits_{l=i+1}^{t} \hat{D}_l - Y_k^*, & k \leqslant t \leqslant j-i. \end{cases} \tag{6}$$

However, if $I_t^* < 0$ for any t in (5) or (6), we have an infeasible solution and hence $c(i,j) = \infty$. To summarize:

$$c(i,j) = \begin{cases} F_j + p_j \sum\limits_{t=i+1}^{j} \hat{D}_t + \sum\limits_{t=i+1}^{j-1} h_t \left(-\sum\limits_{l=i+1}^{t} \hat{D}_l \right), & \hat{D}_{i+1} < 0, \hat{D}_j < 0, \\ \min\limits_k \left\{ F_k + p_k \sum\limits_{t=i+1}^{j} \hat{D}_t + \sum\limits_{t=i+1}^{j-1} h_t \left(-\sum\limits_{l=i+1}^{t} \hat{D}_l \right) + \sum\limits_{t=k}^{j-1} h_t \sum\limits_{l=i+1}^{j} \hat{D}_l \right\}, & \hat{D}_{i+1} < 0, \hat{D}_j > 0, \\ \infty; & \text{otherwise.} \end{cases} \tag{7}$$

We now analyze the complexity of our dynamic programming algorithm. We begin by observing that the inventory in period zero is also zero. Next, we expand the definition of $c(i,j)$ to include $f(1,s)$ as follows:

$$c(i,j) = f(1,s), \quad i = 0, \quad j = s.$$

Then noting the similarity in the structure of Π from (4) and $V(i)$, we can reformulate the retailer's problem in (4) as

$$\Pi = V(0).$$

Calculation of $V(i)$ is a shortest path problem and takes $O(T^2)$ steps if the values of $c(i,j)$ are known. However, in this case, calculation of $c(i,j)$ takes $O(T)$ steps as it involves a search over period k. Thus, it is easy to see that the complexity of the problem is $O(T^3)$.

3.3. Special case

We now consider a special version of the above problem where the return parameters k_i are in ascending order, i.e., $k_t \geqslant k_{t-1}$. When $k_t \geqslant k_{t-1}$, $\forall t$, more customers will return the items in later periods before the due date stipulated by the retailer. Thus, this condition is reasonable and likely to be commonly found in practice. It also simplifies the solution procedure considerably and allows us to obtain a simpler ordering policy.

Lemma 5. *If $k_t \geqslant k_{t-1}$, then $\hat{D}_t$ is decreasing in t for $t \leqslant r+1$.*

Thus the ascending order of k_i results in a decreasing trend of the effective demand for all $t \leqslant r+1$. By observing the fact that $\hat{D}_1 > 0$, $\hat{D}_t$ is non-increasing in t from Lemma 5 and $\hat{D}_t < 0$ for $t \geqslant r+2$ from Lemma 2, there must exist a time period $\tau = \text{argmax}\,\{t: \hat{D}_t > 0\}$, where $\tau \leqslant r+1$. In this case we have:

Lemma 6. *Let $\tau = argmax\{t : \hat{D}_t > 0\}$. Then the optimal procurement quantity I_0^* and the disposal quantity Y_t^* possess the following properties:*

(i) $I_\tau^* = 0$;
(ii) $Y_t^* = 0\ \forall t \leqslant \tau$, *i.e., it is not optimal to sell before period* τ; *and*
(iii) $I_0^* = \sum_{t=1}^{\tau} \hat{D}_t$.

Lemma 6 implies that the entire time horizon is divided into two regions: one from period 1 through τ with positive effective demands and the other from period $\tau+1$ through T with negative effective demands. In this case, part (ii) of Lemma 6 suggests that it is optimal not to dispose of any inventory when the effective demand is positive. In addition, the ordering policy is much easier to compute as compared with the general case. Once all the demands D_t are known, we just need to calculate the effective demands $\hat{D}_t$ in all periods and determine τ. Then we can sum all the positive effective demands to obtain the initial ordering quantity I_0^*. In this special case, as $\hat{D}_t$ is decreasing, we do not have a case where $I_i^* = I_j^* = 0$ and $\hat{D}_{i+1} < 0, \hat{D}_j > 0$. Thus, from (7) we can see that calculation of $c(i,j)$ does not require a minimization over k like the general case. Thus the complexity of the dynamic programming algorithm for this special case is $O(T^2)$.

4. Endogenous pricing with no backordering

In the previous section we considered the case in which the retailer decides on the optimal procurement and disposal policy when the disposal prices are given exogenously and do not affect the quantity being disposed of. While this is not completely unreasonable, we expect that the demand for purchase of rental goods is likely to be dependent on the disposal price. For example, in the home video market, different titles are disposed by retailers at different prices. Also, the authors have found that different retailers can have different prices for the same titles. This indicates that disposal price is also one of the decision variables available to the retailer and hence we explicity model disposal price dependence of demand in the secondary market.

To simplify the analysis we assume that there is no fixed cost per period associated with disposals. We assume that the quantity that can be disposed of in any period is a linear function of the disposal price $Y_t = a_t - b_t p_t$. Here a_t can be interpreted as the market potential and b_t as the price sensitivity. However, our model and analysis are more general and can accommodate any demand function such that price is a decreasing concave function of the selling quantity. Carrying forward all the notation from the previous section, the retailer's decision problem is modified from (2) as follows:

$$\Pi = \min_{I_0, Y_t, p_t, I_t} \left\{ c_0 I_0 + \sum_{t=1}^{T} h_t I_t - \sum_{t=1}^{T} p_t Y_t \right\}$$

$$\text{subject to:} \quad p_t, I_0, I_t, Y_t \geqslant 0, \text{ where}$$
$$I_t = I_0 - \sum_{k=1}^{t} Y_k - \sum_{k=1}^{t} \hat{D}_k \quad \forall t \geqslant 1, \text{ and} \tag{8}$$
$$Y_t = a_t - b_t p_t \quad \forall t \geqslant 1.$$

Substituting for I_t in the objective function in (8) and using the disposal demand function $Y_t = a_t - b_t p_t$, we obtain

$$\Pi_1 = \min_{I_0, Y_t} \left\{ \left(c_0 + \sum_{t=1}^{T} h_t \right) I_0 - \sum_{t=1}^{T} \left\{ h_t \left(\sum_{k=1}^{t} Y_k \right) - \frac{a_t}{b_t} Y_t + \frac{1}{b_t} Y_t^2 \right\} \right\} - \sum_{t=1}^{T} \left(h_t \sum_{k=1}^{t} \hat{D}_k \right)$$
$$\text{subject to: } I_0, Y_t \geqslant 0, \tag{9}$$

where

$$\sum_{k=1}^{t} Y_k \leqslant I_0 - \sum_{k=1}^{t} \hat{D}_k, \quad \forall t \geqslant 1. \tag{10}$$

Note again that the last term in (9) is constant and does not figure in the optimization problem. In addition, we introduce some simplification in the notation as follows:

$$\begin{aligned}
&\overline{h_i} \equiv \sum_{t=i}^{T} h_t, && i \geqslant 1,\\
&\alpha_t \equiv \tfrac{1}{b_t} > 0, && t \geqslant 1,\\
&\beta_t \equiv \left(\overline{h}_t + \tfrac{a_t}{b_t} \right) > 0, && t \geqslant 1,\\
&u_1 \equiv I_0 - \hat{D}_1 &&\\
&u_k \equiv - \hat{D}_k, && k \geqslant 2.
\end{aligned}$$

As $\hat{D}_t$ corresponds to the effective demand in period t, we can interpret u_t as the "effective supply" in period t. In the first period, u_1 is equal to the initial order quantity less the demand in the first period. In all subsequent periods, u_t is equal to the negative of the effective demand. Notice that $\sum_{t=1}^{T} h_t \left(\sum_{k=1}^{t} Y_k \right) = \sum_{t=1}^{T} \overline{h_t} Y_t$. Utilizing these expressions, we can simplify (9) as:

$$\Pi_2 = \min_{I_0, Y_t} \left\{ (c_0 + \overline{h}_1) I_0 - \sum_{t=1}^{T} \beta_t Y_t + \sum_{t=1}^{T} \alpha_t Y_t^2 \right\}$$

$$\text{subject to:} \quad \sum_{k=1}^{t} Y_k \leqslant \sum_{k=1}^{t} u_k, \quad \forall t \geqslant 1,$$
$$I_0, Y_t \geqslant 0, \quad \forall t \geqslant 1.$$

We solve this problem by solving an inner and an outer optimization problem. In particular, we first fix I_0 and solve the inner minimization problem to obtain the optimal Y_t^* and then plug these back to solve the outer optimization problem and obtain optimal I_0^*. For a fixed I_0, the inner optimization problem is given by

$$\begin{aligned} \Pi_3 = \min_{Y_t} & \left\{ \sum_{t=1}^{T} \alpha_t Y_t^2 - \sum_{t=1}^{T} \beta_t Y_t \right\} \\ \text{subject to:} \quad & \sum_{k=1}^{t} Y_k \leqslant \sum_{k=1}^{t} u_k, \quad \forall t \geqslant 1, \\ & Y_t \geqslant 0 \quad \forall t \geqslant 1. \end{aligned} \tag{11}$$

Problem (11) is a convex (quadratic) optimization problem with linear inequalities and hence can be solved to optimality using conventional methods of convex programming. We demonstrate two methods that can be used to solve this problem optimally. The first method attempts to solve the primal problem directly. While this method provides the necessary qualitative insights, it becomes unwieldy for large problems and the computational effort increases exponentially with T. Hence in the second approach, we use duality. We derive the dual problem, which is a convex (quadratic) program with tree constraints and can be solved in polynomial time.

4.1. Primal approach

The primal method can be used to solve the problem in (11) for short planning horizon T. We first demonstrate the method for $T = 2$, primarily to understand the nature of the optimal selling policy and then generalize. We start by rewriting problem (11) for $T = 2$ as

$$\begin{aligned} & \min_{Y_1, Y_2} \{ \alpha_1 Y_1^2 - \beta_1 Y_1 + \alpha_2 Y_2^2 - \beta_2 Y_2 \} \\ \text{subject to:} \quad & Y_1 \leqslant u_1, \\ & Y_1 + Y_2 \leqslant u_1 + u_2, \\ & Y_1, Y_2 \geqslant 0. \end{aligned}$$

To guarantee feasibility, we consider the cases in which $u_1, u_2 \geqslant 0$. We first fix Y_1 and solve for Y_2 and then substitute it back to obtain the optimal Y_1. So the inner optimization problem is

$$\begin{aligned} & \min_{Y_2} \ \alpha_2 Y_2^2 - \beta_2 Y_2 \\ \text{subject to:} \quad & Y_2 \leqslant u_1 + u_2 - Y_1, \\ & Y_2 \geqslant 0. \end{aligned}$$

The optimal solution is given by

$$Y_2^* = \begin{cases} \dfrac{\beta_2}{2\alpha_2} & \text{if } \dfrac{\beta_2}{2\alpha_2} \leqslant u_1 + u_2 - Y_1, \\ u_1 + u_2 - Y_1 & \text{if } \dfrac{\beta_2}{2\alpha_2} > u_1 + u_2 - Y_1. \end{cases}$$

Now in order to solve the outer minimization problem, we have to consider the following two cases.

Case I. $Y_1 \leqslant u_1 + u_2 - \dfrac{\beta_2}{2\alpha_2}$.

The outer optimization problem is given by

$$\min_{Y_1} \ \alpha_1 Y_1^2 + \beta_1 Y_1$$

$$\text{subject to: } Y_1 \leqslant u_1,$$

$$Y_1 \leqslant u_1 + u_2 - \frac{\beta_2}{2\alpha_2},$$

$$Y_1 \geqslant 0.$$

We now need to consider two subcases:

Case Ia. $u_2 \leqslant \dfrac{\beta_2}{2\alpha_2}$.

In this case, the second constraint $Y_1 \leqslant u_1 + u_2 - \beta_2/2\alpha_2$ is binding and the optimal solution is given by

$$Y_1^* = \begin{cases} \dfrac{\beta_1}{2\alpha_1} & \text{if } \dfrac{\beta_1}{2\alpha_1} \leqslant u_1 + u_2 - \dfrac{\beta_2}{2\alpha_2}, \\ u_1 + u_2 - \dfrac{\beta_2}{2\alpha_2} & \text{if } \dfrac{\beta_1}{2\alpha_1} > u_1 + u_2 - \dfrac{\beta_2}{2\alpha_2}. \end{cases}$$

Case Ib. $u_2 > \dfrac{\beta_2}{2\alpha_2}$.

In this case the first constraint $Y_1 \leqslant u_1$ is binding and the optimal solution is given by

$$Y_1^* = \begin{cases} \dfrac{\beta_1}{2\alpha_1} & \text{if } \dfrac{\beta_1}{2\alpha_1} \leqslant u_1, \\ u_1 & \text{if } \dfrac{\beta_1}{2\alpha_1} > u_1. \end{cases}$$

Case II. $Y_1 > u_1 + u_2 - \dfrac{\beta_2}{2\alpha_2}$

Then, the optimization problem is given by

$$\min_{Y_1} \ \alpha_1 Y_1^2 + \beta_1 Y_1$$

$$\text{subject to: } Y_1 \leqslant u_1,$$

$$Y_1 > u_1 + u_2 - \tfrac{\beta_2}{2\alpha_2}.$$

Clearly, for feasibility we should have $u_2 < \beta_2/2\alpha_2$. Then the optimal solution is given by

$$Y_1^* = \begin{cases} \dfrac{\beta_1}{2\alpha_1} & \text{if } u_1 > \dfrac{\beta_1}{2\alpha_1} \text{ and } u_1 + u_2 < \dfrac{\beta_1}{2\alpha_1} + \dfrac{\beta_2}{2\alpha_2}, \\ u_1 & \text{if } u_1 < \dfrac{\beta_1}{2\alpha_1}. \end{cases}$$

Combining all the above calculations, we summarize to obtain the following cases:

1.	$u_1 < \frac{\beta_1}{2\alpha_1}; u_2 < \frac{\beta_2}{2\alpha_2}$		$Y_1^* = u_1$	$Y_2^* = u_2$
2.	$u_1 > \frac{\beta_1}{2\alpha_1}; u_2 > \frac{\beta_2}{2\alpha_2}$		$Y_1^* = \frac{\beta_1}{2\alpha_1}$	$Y_2^* = \frac{\beta_2}{2\alpha_2}$
3.	$u_1 < \frac{\beta_1}{2\alpha_1}; u_2 > \frac{\beta_2}{2\alpha_2}$	$u_1 + u_2 < \frac{\beta_1}{2\alpha_1} + \frac{\beta_2}{2\alpha_2}$	$Y_1^* = u_1$	$Y_2^* = \frac{\beta_2}{2\alpha_2}$
4.	$u_1 < \frac{\beta_1}{2\alpha_1}; u_2 > \frac{\beta_2}{2\alpha_2}$	$u_1 + u_2 > \frac{\beta_1}{2\alpha_1} + \frac{\beta_2}{2\alpha_2}$	$Y_1^* = u_1$	$Y_2^* = \frac{\beta_2}{2\alpha_2}$
5.	$u_1 > \frac{\beta_1}{2\alpha_1}; u_2 < \frac{\beta_2}{2\alpha_2}$	$u_1 + u_2 > \frac{\beta_1}{2\alpha_1} + \frac{\beta_2}{2\alpha_2}$	$Y_1^* = \frac{\beta_1}{2\alpha_1}$	$Y_2^* = \frac{\beta_2}{2\alpha_2}$
6.	$u_1 > \frac{\beta_1}{2\alpha_1}; u_2 < \frac{\beta_2}{2\alpha_2}$	$u_1 + u_2 < \frac{\beta_1}{2\alpha_1} + \frac{\beta_2}{2\alpha_2}$	$Y_1^* = u_1 + u_2 - \frac{\beta_2}{2\alpha_2}$	$Y_2^* = \frac{\beta_2}{2\alpha_2}$
			$Y_1^* = \frac{\beta_1}{2\alpha_1}$	$Y_2^* = u_1 + u_2 - \frac{\beta_1}{2\alpha_1}$

The interpretation for the primal optimal solution Y_t^* is as follows: the first two cases are straightforward. In the first case when the effective supplies u_1 and u_2 are both very small, the optimal policy is to dispose of u_1 and u_2 so that constraints are binding in both periods. On the other hand, in the second case when the effective supplies in both periods are very large, the optimal policy is to sell at the unconstrained optimum. Now consider the third and the fourth case. When u_1 is small and u_2 is large, the sum is irrelevant to the decision making. As u_1 is small, it is optimal to sell up to this bound in the first period and as u_2 is large, it is optimal to sell at the unconstrained optimum in the second period. On the other hand, when u_2 is small and u_1 is large, the sum is relevant to decision making. In the fifth case, u_1 is large enough to make the sum u_1+u_2 large enough so that it is optimal for the retailer to sell at the unconstrained optimum in both periods. But in the final case, u_1 is not large enough and hence u_1+u_2 is small. Then, the retailer has essentially to decide between choosing an unconstrained optimum in either the first period or the second period. Thus, there are two potential optimal solutions and the retailer needs to compare the total cost for the two options and choose the one with minimum cost, which depends on the relative values of the parameters. Thus, there are cases where a myopic policy might be optimal and other cases where the retailer has to take into account the intertemporal effect of his actions. This solves the inner optimization problem. Now we use the optimal values of Y_1^* and Y_2^* to calculate the optimal I_0^* in the outer optimization problem. We can see from (10) and (9) that

$$I_0^* = \max(Y_1^* + \hat{D}_1, Y_1^* + Y_2^* + \hat{D}_1 + \hat{D}_2).$$

Thus we have determined the retailer's optimal procurement and selling policy. Now, given the values of Y_1^* and Y_2^*, the optimal disposal prices are given by

$$p_t^* = \frac{a_t - Y_t^*}{b_t}, \quad t = 1, 2.$$

Now we outline the procedure for general T periods. We first begin with solving for Y_T given that $Y_1, Y_2, \ldots, Y_{T-1}$ are fixed. Y_T will be the minimum of two values: the unconstrained optimum and the upper bound, which in turn is a function of $Y_1, Y_2, \ldots, Y_{T-1}$. For two possible values of Y_T, we formulate two problems to solve for optimal Y_T as a function of $Y_1, Y_2, \ldots, Y_{T-2}$. We continue in this manner until we solve for the optimal Y_1. Next, we need to enumerate the optimal solutions for all the different ranges of the relevant parameter values like above. In the case of multiple options, one has to find the candidate that yields the lowest cost. Clearly, this would not be feasible for very large values of T and the computational effort would quickly explode. Hence, we use the dual approach to solve for problems of realistic size.

4.2. Dual approach

In this section, we use the Lagrangian method to write the dual problem and show that it can be solved using a well-known and efficient optimization procedure. Given that the original problem is a quadratic program with linear constraints, strong duality is guaranteed. Thus, we can obtain the optimal primal variable, which is the optimal selling policy in this case. We first introduce dual variables $\mu_t \geqslant 0$ for constraints $Y_t \geqslant 0$ and $\lambda_t \geqslant 0$ for $\sum_{k=1}^{t} Y_k \leqslant \sum_{k=1}^{t} u_k$, respectively. The Lagrangian for the optimization problem in (11) can be written as

$$L(Y_t, \mu_t, \lambda_t) = \sum_{t=1}^{T} \left\{ \alpha_t Y_t^2 - \beta_t Y_t - \mu_t Y_t + \lambda_t \sum_{k=1}^{t} (Y_k - u_k) \right\}. \tag{12}$$

Again redefining $\overline{\lambda}_k \equiv \sum_{t=k}^{T} \lambda_t$ we simplify (12) to obtain

$$L(Y_t, \mu_t, \overline{\lambda}_t) = \sum_{t=1}^{T} \left\{ \alpha_t Y_t^2 - (\beta_t + \mu_t - \overline{\lambda}_t) Y_t - \overline{\lambda}_t u_t \right\}. \tag{13}$$

Note that as $\lambda_t \geqslant 0$, we have $\overline{\lambda}_t \geqslant \overline{\lambda}_{t+1}$. Thus the dual problem can be written as

$$\max_{\overline{\lambda}_t \geqslant 0, \mu_t \geqslant 0} \left\{ \min_{Y_t} \left\{ \sum_{t=1}^{T} \{ \alpha_t Y_t^2 - (\beta_t + \mu_t - \overline{\lambda}_t) Y_t - \overline{\lambda}_t u_t \right\} \right\} \tag{14}$$

$$\text{subject to:} \quad \overline{\lambda}_t \geqslant \overline{\lambda}_{t+1}, \quad \forall t.$$

The inner unconstrained optimization over Y_t yields: $Y_t^* = (\beta_t + \mu_t - \overline{\lambda}_t)/2\alpha_t$. Notice that this corresponds to the unconstrained optimal solution $Y_t^* = \beta_t/2\alpha_t$ when $\overline{\lambda}_t = \mu_t = 0$. Substituting Y_t^* back into (14) yields

$$\min_{\mu_t \geqslant 0} \left\{ \min_{\overline{\lambda}_t \geqslant 0} \sum_{t=1}^{T} \left\{ \frac{(\beta_t + \mu_t - \overline{\lambda}_t)^2}{4\alpha_t} + \overline{\lambda}_t u_t \right\} \right\} \tag{15}$$

$$\text{subject to:} \quad \overline{\lambda}_t \geqslant \overline{\lambda}_{t+1}, \quad \forall t.$$

We solve this problem by first fixing μ_t and solving the inner optimization problem for optimal $\overline{\lambda}_t^*$ and then substituting them back to obtain the optimal μ_t^* in the outer optimization problem. The inner optimization problem for a fixed μ_t is a convex optimization problem with tree constraints ($\overline{\lambda}_t \geqslant \overline{\lambda}_{t+1}$; $t = 1, 2, \ldots, T-1$) and can be solved optimally in polynomial time by a reduction scheme presented in Tang (1990). The problem is solved first by ignoring the tree constraints. If all tree constraints are satisfied, then the optimal solution is obtained in the first iteration. If not, then the problem is resolved only for latest periods (largest t and $t+1$) in which the constraint is violated by substituting $\overline{\lambda}_t = \overline{\lambda}_{t+1}$, and this process is continued until all the tree constraints are satisfied. Thus, at each iteration, the size of the problem is reduced and hence the inner optimization problem is guaranteed to be solved in polynomial time. Having solved the inner optimization problem, solving the outer optimization problem is easy as the problem is separable in μ_t, and as the problems are quadratic. Hence, the original problem can also be solved in polynomial time. We outline the solution procedure of a convex program with tree constraints for our dual problem (15) in Appendix B.

5. Extensions to the base model

Having analyzed the endogenous price case, we revisit the base model where the disposal price is available exogenously to the retailer, and discuss a few extensions. In the first extension, we allow multiple purchase opportunities, i.e., the retailer can purchase from the distributors in any period in addition to period zero. In the second extension, we study the impact of contracts between the distributor and the retailer that prevents the retailer from disposing of inventory for a stipulated number of periods. Finally, we extend the model to incorporate the possibility of backordering of unmet demand.

5.1. Multiple purchase opportunities

We now extend the base model to the case in which procurement is allowed in each period. Let X_t denote the quantity purchased from the distributor in period t at a unit cost of c_t, where $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_T$. Let G_t denote the fixed cost associated with purchase in period t. In each period, we assume that the returns from the previous periods are followed by procurement, if any. Then the rental demand is realized followed by disposal in the secondary market, if any. We assume that both the unit cost of purchase (c_t) and unit disposal price (p_t) are available exogenously. Again, in order to avoid the possibility of arbitrage through purely selling and purchasing, we assume that $c_t \geqslant p_t$. In this case, the retailer's problem can be formulated as

$$\Pi = \min_{I_0, Y_t, Y_t, X_t} \left\{ \sum_{t=1}^{T} \{ c_t X_t + h_t I_t - p_t Y_t + F_t \delta(Y_t) + G_t \delta(X_t) \} \right\} \quad (16)$$

$$\text{subject to:} \quad I_t, Y_t, X_t \geqslant 0,$$

where

$$I_t = I_{t-1} - Y_t + X_t - \hat{D}_t,$$
$$\delta(z) = \begin{cases} 1 & \text{if } z > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

All the terms in the objective function are the same as in the base model (2) except the first and the last, which denote costs related to purchases (variable and fixed costs, respectively) in period t.

Just like the base case, we have left out the term denoting the rental revenue as it is constant and hence not relevant.

5.1.1. Structural properties of the optimal solution

As in the base case, we first derive the structural properties of the optimal solution, which would help us to design an efficient dynamic programming algorithm.

Lemma 7. *There exists an optimal solution that has* $X_t^* Y_t^* = 0, \ \forall t$.

Given the cost structure and the exogenous prices in the market and their relative values, Lemma 7 implies that it is never optimal for the retailer to both sell and purchase inventory in the same time period.

Proposition 8. *There exists an optimal solution that is loopless.*

Proposition 8 shows that the loopless property of the optimal solution for the base case in Proposition 1 is preserved when we extend our analysis to the case of multiple purchases and sales. A concomitant result is the existence of $s; 1 < s < T$ such that $I_s^* = 0$. We can prove this by contradiction. Suppose that $I_t^* > 0, \ \forall t < T$. Then as $I_T^* = 0$, the retailer has to dispose of Y_T in period T. In this case, the solution $(X_1^*, I_1^*, \ldots, I_{T-1}^*, Y_T^*)$ forms a loop, which contradicts Proposition 8. We omit the formal proof as it is exactly similar to the first case of Lemma 3. Given that there exists a period with zero inventory, we proceed with our analysis as before to quantify the cost of operation between two periods with zero inventory.

Corollary 9. *For any optimal solution that has* $I_i^* = 0$ *and* $I_j^* = 0$ *and* $I_t^* > 0; i < t < j$, *there exists at most one* k *such that* $Y_k^* > 0$ *or* $X_k^* > 0$, *where* $i < k \leqslant j$.

This result has been shown independently by Beltran and Krass (2002) in a different context. However, this result is different from the corresponding result for the conventional model with multiple purchases and no returns, cf. Denardo (1982). When there are no returns, the optimal solution has the property that $X_t^* I_t^* = 0, \ \forall t$. In other words, it is optimal to purchase only in periods when the inventory drops to zero. This is not true in our model because of the possibility of negative effective demand, i.e., returns from earlier periods are greater than the rental demand during that period. Even though the inventory at the end of a period might be zero, the effective demand in the next period might be negative, and hence, it is profitable to postpone the purchase to a later period and purchase at a lower cost. Given this nature of the optimal selling and purchasing policy, we calculate the cost of operation between two periods with zero inventory.

Corollary 10. *Suppose* $I_i^* = 0$ *and* $I_j^* = 0$ *and* $I_t > 0; i < t \leqslant j$. *Then, the optimal procurement decision* X_t^* *and optimal disposal decision* Y_t^* *possess the following properties*:

(i) *If* $\hat{D}_{i+1} > 0$, *then* $X_{i+1}^* = \sum_{t=i+1}^{j} \hat{D}_t > 0, X_t = Y_t = 0; i+1 < t \leqslant j$.
(ii) *If* $\hat{D}_{i+1} < 0$ *and* $\hat{D}_j < 0$, *then* $Y_j^* = -\sum_{t=i+1}^{j} \hat{D}_t > 0, X_t = Y_t = 0; i+1 \leqslant t < j$.

(iii) *If* $\hat{D}_{i+1} < 0$ *and* $\hat{D}_j > 0$, *then* $X_k^* + Y_k^* > 0$ *for at most one* k; $i+1 < k \leqslant j$ *and* $X_t = Y_t = 0$; $\forall t \neq k$ *where* $X_k^* = [\sum_{t=i+1}^{j} \hat{D}_t]^+$ *and* $Y_k^* = [-\sum_{t=i+1}^{j} \hat{D}_t]^+$.

Corollary 10 has implications different from those of Lemma 4 for the base case. Here, we can have a positive effective demand in the period beginning with zero inventory. In this case, as shown in result (i), the only feasible policy and hence the optimal policy is to purchase in this period. This is because between two periods with zero inventory, the number of periods with purchases and sales cannot be more than one in order to satisfy the loopless property. As such, the purchase quantity must be equal to the sum of the effective demands between two such periods. If the effective demand in the period i with zero inventory is negative, then there are two cases depending on the sign of the effective demand in period j. Clearly, if the effective demand in period j is negative, result (ii) suggests that it is optimal for the retailer to sell in period j in order to satisfy the zero inventory condition. But this means that he cannot sell or purchase in any other period between i and j and hence again the total sale quantity has to be equal to the sum of all effective demands. If, on the other hand, the effective demand in period j is positive, then the retailer can make a higher revenue by advancing the sale to an earlier period and still satisfying the zero inventory condition in period j. However, there is a possibility where it is optimal for the retailer not to sell or purchase in any period between i and j if the sum of all positive and negative effective demands is zero, i.e., if the demands are self-sufficient between periods i and j, as stated in result (iii).

In each of the cases studied in Corollary 10, we have assumed that the solution is feasible, i.e., the effective demands are such that it is feasible to sell or purchase as recommended. However, it is easy to verify for infeasibility. In other words, if $I_t^* < 0$ for some t between i and j, then the solution is infeasible and calculating X_k^* or Y_k^* is redundant. For such cases, we assign a cost of ∞. We now use the preceding result to come up with a dynamic programming formulation for the retailer's problem.

5.1.2. *Dynamic programming formulation*

We extend the dynamic programming formulation of the base model and use the same definitions here. Thus, the original problem is equivalent to

$$\Pi = \min_{s=1,2,\ldots,T-1} \{f(1,s) + V(s)\},$$

where

$$V(i) = \min_{j>i} \{c(i,j) + V(j)\}.$$

Although the formulation is very similar to the base case, it is important to highlight a few important differences. Here, s can be anything from zero to $T-1$, unlike the base case, where $s \leqslant \tau$. Also, the cost parameters $c(i,j)$ are very different in this case. The next step is to find $f(1,s)$ and $c(i,j)$, where

$$f(1,s) = c_1 X_1^* + \sum_{t=1}^{s} h_t I_t^* = c_1 \sum_{t=1}^{s} \hat{D}_t + \sum_{t=1}^{s} h_t \left(X_1^* - \sum_{i=1}^{t} \hat{D}_i \right).$$

The term $c(i,j)$ can be calculated using Corollary 10 in the case of a feasible solution. We can summarize as follows:

$$c(i,j) = \begin{cases} G_{i+1}\delta([\sum_{t=i+1}^{j} \hat{D}_t]^+) + c_{i+1}[\sum_{t=i+1}^{j} \hat{D}_t]^+ + \sum_{t=i+1}^{j} h_t I_t^*, & \hat{D}_{i+1} > 0, I_t^* > 0, \forall_t, \\ F_j - p_j[-\sum_{t=i+1}^{j} \hat{D}_t]^+ + \sum_{t=i+1}^{j} h_t I_t^*, & \hat{D}_{i+1} < 0, \hat{D}_j < 0, I_t^* > 0, \forall_t \\ \min_k \left\{ F_k\delta\left(\left[-\sum_{t=i+1}^{j} \hat{D}_t\right]^+\right) - p_k\left[-\sum_{t=i+1}^{j} \hat{D}_t\right]^+ + \right. & \hat{D}_{i+1} < 0, \hat{D}_j < 0, I_t^* > 0, \forall_t \\ \left. G_k\delta\left(\left[\sum_{t=i+1}^{j} \hat{D}_t\right]^+\right) + c_k\left[\sum_{t=i+1}^{j} \hat{D}_t\right]^+ + \sum_{t=i+1}^{k} h_t I_t^* \right\}, & \\ \infty, & \text{otherwise,} \end{cases}$$

where I_t^* can be found using the relevant material balance equations. We can show that the complexity of the dynamic programming algorithm is $O(T^3)$ by reformulating the problem as in the base case.

5.2. Contractual period of no sale

In the video rental industry, retailers commonly engage in contracts with distributors that prevent them from disposing of DVDs in the secondary market for a certain stipulated period. These contracts can serve two purposes: (i) guarantee that the sales of a retailer who is selling DVDs in the primary market would not be cannibalized; and (ii) prevent the rental retailer from speculating and making arbitrage profits by purely buying and selling DVDs instead of his primary business of rentals.

We modify our base model to incorporate this feature of the video rental industry. We introduce $\hat{t}$ as the period until which the retailer cannot dispose of the rental units in the secondary market. We allow for a single purchase opportunity in period zero as in the base case. Additionally, we assume that $\hat{t} > r+1$. This is reasonable in the context of the video rental industry, where the rental duration r is of the order of 5–6 days, while $\hat{t}$ is of the order of a few weeks or months. As usual, we first derive some useful structural properties of the optimal solution and then use them to write a dynamic programming algorithm to obtain the solution.

Proposition 11. *There exists an optimal solution that is loopless.*

The proof is similar to that of Proposition 1. Specifically, we assume that a solution with a loop is optimal and then construct variants with lower costs and hence contradict the original supposition. As the formal proof is exactly similar to Proposition 1, we omit it here for brevity. Also, like the base case, we can show an existence of period $s < \tau$ (where τ is as defined in (8)) such that $s = \text{argmin}\{t : I_t^* = 0\}$ and hence a concomitant result is that $X_1^* = \sum_{t=1}^{s} \hat{D}_t$. Now that we have shown the existence of time periods with zero inventory, we can proceed as in previous cases to characterize the optimal solution between two periods with zero inventory. We shall later use this characterization to calculate the optimal cost of operation between two such periods and hence the total cost of operation over the entire time horizon.

Lemma 12. *Suppose $I_i^* = I_j^* = 0, j > i$ and $I_t > 0, i < t < j$. Then, the optimal disposal policy possesses the following properties:*

(i) *If $j < \hat{t}$, then $\sum_{t=i+1}^{j} \hat{D}_t = 0$.*
(ii) *If $j > \hat{t}$, then $Y_j^* = -\sum_{t=i+1}^{j} \hat{D}_t > 0$, $Y_t^* = 0$ $i+1 \leqslant t < j$.*

If both the zero inventory periods i and j are before $\hat{t}$, then it must be true that the negative and the positive effective demands balance each other as it is not possible to sell in any of the periods before $\hat{t}$. If j is after $\hat{t}$, then j is also after $r+1$ and hence the effective demand in period j is negative. So the only feasible and hence the optimal solution is to sell in period j as we can sell only in one period between i and j to avoid loops and we also need to satisfy the zero inventory property.

5.2.1. Dynamic programming formulation

Next, we formulate the dynamic programming algorithm using the above structural properties. Similar to the base case, the original problem is equivalent to the following dynamic program:

$$\Pi = \min_{s=1,2,\ldots,\tau} \{f(1,s) + V(s)\},$$

where

$$V(i) = \min_{j>i}\{c(i,j) + V(j)\}.$$

Thus, the formulation is very similar to the base model, with minor changes in calculating the cost parameters as shown below:

$$f(1,s) = c_0 X_1^* + \sum_{t=1}^{s-1} h_t I_t^*,$$

where

$$I_t^* = X_1^* - \sum_{i=1}^{t} \hat{D}_i, \quad X_1^* = \sum_{t=1}^{s} \hat{D}_t.$$

The term $c(i,j)$ can be calculated using Lemma 12 in the case of a feasible solution. This is given by

$$c(i,j) = \begin{cases} \sum_{t=i+1}^{j} h_t I_t^*, & j < \hat{t}, I_t^* > 0, \forall t, \\ F_j - p_j \sum_{t=i+1}^{j} \hat{D}_t + \sum_{t=i+1}^{j} h_t I_t^*, & j > \hat{t}; I_t^* > 0, \forall t, \\ \infty, & \text{otherwise}, \end{cases}$$

where I_t^* is calculated using the relevant material balance equations. Note that the complexity of the dynamic program in this case is only $O(T^2)$ because the calculation of $c(i,j)$ does not include any search over T as can be seen from the above expression.

5.3. Exogenous pricing with backordering

In all the above models, backorders were not allowed. The customers were assumed to be impatient, and the cost of lost sales was assumed to be high enough so that demand in each period was met in entirety. Now we introduce a case in which the customers are willing to wait when their demand is not satisfied. To improve customer loyalty, many video rental stores offer free rental coupons that are good for renting videos that are currently out of stock in a later period. We consider s_t as the unit cost of backordering in period t. Also, let $(X)^+ = \max(X, 0)$ and $(X)^- = \max(-X, 0)$. All other notation is carried forward from the base model. The retailer's decision problem in this case is given by the following mathematical program:

$$\Pi = \min_{I_0, I_t, Y_t} \left\{ c_0 I_0 + \sum_{t=1}^{T} h_t(I_t)^+ + \sum_{t=1}^{T} s_t(I_t)^- - \sum_{t=1}^{T} p_t Y_t + \sum_{t=1}^{T} F_t \delta(Y_t) - \sum_{t=1}^{T} \rho D_t \right\} \tag{18}$$

subject to: $Y_t, I_0 \geqslant 0,$

where

$$I_t = I_{t-1} - Y_t - \hat{D}_t, \quad \forall t \geqslant 1. \tag{19}$$

All the terms in the objective function are the same as in the base model except an additional third term, which represents the total backordering costs. Also, I_t is no longer constrained to be non-negative as backorders are allowed. An important point to mention is the interpretation of $\hat{D}_t$. In the case without backordering, we had interpreted it as the effective demand, i.e., rental demand net of all returns. In order to extend that interpretation to this case, we have to make an additional assumption that the customers also cancel the backorders and the cancelation of backorders is also in the same proportion as the returns. In other words, k_t also denotes the proportion of the backorders canceled in period t after their booking. While our model is general and can accommodate different cancelation parameters, this choice simplifies the exposition to a great extent and helps us derive qualitative insights. As in the previous cases, we again begin by trying to characterize the properties of the optimal solution that would help us to devise a dynamic programming algorithm.

Proposition 13. *There exists an optimal solution that is loopless.*

This is an important result as it allows us to extend the method of analysis from the base case (without backordering) to the one with backordering. We can show that a direct corollary of this result is the existence of $s = \arg\min\{t : I_t^* = 0\}$. We can prove this using contradiction. Suppose this is not true, i.e., $I_t^* > 0$ or $I_t^* < 0$, $\forall t$. Then, as we require $I_T^* = 0$ and $X_1^* > 0$, we have $Y_k^* > 0$ for some $1 < k \leqslant T$, i.e., we would have to sell in some future period. Then, we immediately have a loop $(X_1^*, I_1^*, \ldots, I_{k-1}^*, Y_k^*)$, which would contradict the proposition. Having shown the existence of zero inventory periods, the next step is to identify properties of the optimal solution between periods of zero inventory.

Lemma 14. *For any optimal solution that has $I_i^* = I_j^* = 0,\ j > i$ and $I_t > 0,\ i < t \leqslant j$, the optimal disposal policy possesses the following properties*:

(i) $\hat{D}_t < 0$, *for some* $t; i < t \leqslant j$;
(ii) *If* $\hat{D}_t < 0,\ \forall t; i < t \leqslant j$, *then* $Y_j^* = -\sum_{t=i+1}^{j} \hat{D}_t > 0, Y_t^* = 0,\ \forall t; i+1 \leqslant t < j$; *and*
(iii) *If* $\hat{D}_t > 0$ *for some* t; $i < t \leqslant j$, *then there exists at most one* k; $i < k \leqslant j$ *such that* $Y_k^* = -\sum_{t=i+1}^{j} \hat{D}_t > 0, Y_t^* = 0,\ \forall t \neq k; i+1 \leqslant t \leqslant j$.

The first case implies that between two periods i and j with zero inventory, there has to be at least one period with a negative effective demand. This is because if all periods had positive effective demand, i.e., more rentals as compared with returns, backorders would only increase and we would not be able to reach zero inventory again. If all the periods between i and j had a negative effective demand, then it would be optimal for the retailer to sell in period j as selling before this period would mean either selling again or ending up with a non-zero inventory in period j. On the other hand, if at least one period has a positive effective demand, then there is a possibility that the sum of all effective demands is zero, i.e., the total returns between i and j exactly balance the rentals during that interval. In such a case, it is optimal not to sell in any period. Thus in the third case there is at most one period between i and j with a positive disposal quantity.

5.3.1. Dynamic programming formulation

Now we are ready to formulate the dynamic programming algorithm to obtain the optimal solution for this case. We proceed in a manner similar to the previous case. The retailer's decision problem is equivalent to

$$\Pi = \min_{s=1,2,\ldots,T} \{f(1,s) + V(s)\},$$

where

$$V(i) = \min_{j > i+1} \{c(i,j) + V(j)\}.$$

Thus, the formulation is very similar to the base model, with minor changes in calculating the cost parameters as shown below:

$$f(1,s) = c_0 X_1^* + \sum_{t=1}^{s-1} h_t (I_t^*)^+ + \sum_{t=1}^{s-1} s_t (I_t^*)^-,$$

where

$$I_t^* = X_1^* - \sum_{i=1}^{t} \hat{D}_i, \quad X_1^* = \sum_{t=1}^{s} \hat{D}_t.$$

The term $c(i,j)$ can be calculated using Lemma 14 in the case of a feasible solution. In case of an infeasible solution, the cost is ∞. Thus, we have

$$c(i,j) = \begin{cases} F_j - p_j \sum_{t=i+1}^{j} \hat{D}_t + \sum_{t=i+1}^{j} h_t (I_t^*)^+ + \sum_{t=i+1}^{j} s_t (I_t^*)^-, & \hat{D}_j < 0, I_t^* > 0, \quad \forall t, \\ \min_k \{F_k - p_k \sum_{t=i+1}^{j} \hat{D}_t + \sum_{t=i+1}^{j} h_t (I_t^*)^+ + \sum_{t=i+1}^{j} s_t (I_t^*)^-\}, & \hat{D}_j > 0, I_t^* > 0, \quad \forall t, \\ \infty & \text{otherwise.} \end{cases}$$

Again, the complexity of the algorithm is $O(T^3)$ as in the base case as calculation of $c(i,j)$ again involves a minimization over k periods.

6. Conclusion and future research

In this paper, we have presented various models to obtain optimal procurement and disposal policies of a rental retailer with specific focus on the video rental business. We considered cases with exogenous and endogenous prices and with and without backordering. We showed that our methodology can be easily extended to the case with multiple procurement opportunities. Besides video rentals, our model can be applied to other rental or remanufacturable products with a more general time-varying demand pattern. See Bayiz and Tang (2004) for a potential application.

In our model, the rental demand is time-varying and deterministic. The rental demand in practice is not deterministic. However, a good forecast can be obtained using box-office collections and DVD sales. Future research could explore extension of the model to uncertain demand. One possibility is to model the demand using an independent normal demand where the mean of the demand is decreasing over time. We can extend the model as presented in Tang and Deo (2004) to model the disposal decision. In addition, the return parameters have been assumed to be constant in our model. We believe that the return parameters would be fairly constant across various titles within the "new releases" category and can be estimated using previous return data in the store.

One limitation of the model is that we assumed that the rental price is constant over the entire time horizon that we consider. This is because we only consider the time period during which the title is in the "new releases" section. In reality, after a certain time period, the retailer moves the title from the "new releases" to the "catalog" section and reduces the rental price and also possibly increases the rental duration. For example, Blockbuster charges over \$4 for a four-day rental for a "new release" title, whereas it charges only around \$3 for a seven-day rental for a "catalog" title. In our future work, we aim to explore the optimal switching policy for the retailer and study the impact of rental duration, demand elasticity and demand pattern on this switching policy. Another limitation is that we have assumed no cannibalization of the rental revenue as a result of the disposal policy of the rental retailer. In reality, this need not be true. A possible extension to the model could involve cannibalization of rental revenue.

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Appendix A: Proofs

In all the proofs, we use X_1 to denote the initial order quantity. Thus, $X_1 = I_0$.

Proof of Proposition 1. We prove using contradiction. Suppose there exists an optimal solution with loops. There are two types of loops possible:

- an outer loop that takes the form of $(X_1^*, I_1^*, \ldots, I_{j-1}^*, Y_j^*)$ for some j.
- an inner loop that takes the form of $(Y_i^*, I_i^*, \ldots, I_{j-1}^*, Y_j^*)$ for some i and j.

We consider both cases separately. In each case, we construct a variant solution that is feasible and results in reduction of the total cost.

Case I. Consider an optimal solution that contains the following smallest loop $(X_1^*, I_1^*, \ldots, I_{j-1}^*, Y_j^*)$ for some $j \leqslant t$. Let Π^* be the total cost associated with this optimal solution. Next, we construct a variant of this solution:

$$(\tilde{X}_1, \tilde{I}_1, \ldots, \tilde{I}_{j-1}, \tilde{Y}_j) = (X_1^* - 1, I_1^* - 1, \ldots, I_{j-1}^* - 1, Y_j^* - 1). \tag{A1}$$

In other words, we buy one unit less in period 1, hold one unit less in periods 1 through $j-1$ and consequently sell one unit less in period j as compared with the original optimal solution. Let $\tilde{\Pi}$ be the total cost associated with this variant. As the original optimal solution is also feasible, we have the following relationships:

$$\begin{aligned}
I_1^* &= X_1^* - \hat{D}_1, \\
I_2^* &= I_1^* - \hat{D}_2, \\
&\vdots \\
I_{j-1}^* &= I_{j-2}^* - \hat{D}_{j-1}, \\
I_j^* &= I_{j-1}^* - Y_j^* - \hat{D}_j .
\end{aligned}$$

Now using (A1), we obtain

$$\tilde{I}_1 = I_1^* - 1 = (X_1^* - 1) - \hat{D}_1 = \tilde{X}_1 - \hat{D}_1,$$

Similarly,

$$\begin{aligned}\tilde{I}_2 &= \tilde{I}_1 - \hat{D}_2,\\ &\vdots\\ \tilde{I}_{j-1} &= \tilde{I}_{j-2} - \hat{D}_{j-1},\\ \tilde{I}_j &= \tilde{I}_{j-1} - \tilde{Y}_j - \hat{D}_j = I_{j-1}^* - Y_j^* - D_j = I_j^*.\end{aligned}$$

So the variant solution is also feasible and does not change the original optimal solution beyond period j. Then

$$\tilde{\Pi} - \Pi^* = -c_0 + p_j - \sum_{i=1}^{j-1} h_i.$$

As $c_0 \geqslant p_t$, $\forall t$; $c_0 \geqslant p_j$. Also $h_i > 0$, $\forall i$. Thus, $\tilde{\Pi} - \Pi^* < 0$. But this contradicts the assumption that the original solution is optimal.

Case II. We consider another optimal solution with the following smallest loop: $(Y_i^*, I_1^*, \ldots, I_{j-1}^*, Y_j^*)$ for some i and j. Let Π^* be the total cost associated with this solution. We again construct a variant:

$$(\tilde{Y}_i, \tilde{I}_i, \ldots, \tilde{I}_{j-1}, \tilde{Y}_j) = (Y_i^* + 1, I_1^* - 1, \ldots, I_{j-1}^* - 1, Y_j^* - 1). \tag{A2}$$

In this case, we sell one unit more in period i, thus resulting in holding one unit less in periods i through $j-1$ and consequently selling one unit less in period j. Let $\tilde{\Pi}$ be the cost associated with this variant solution. As the original solution is also feasible, we have the following relationships:

$$I_i^* = I_{i-1}^* - \hat{D}_i - Y_i^*,$$

$$I_t^* = I_{t-1}^* - \hat{D}_t, \quad i < t < j,$$

$$I_j^* = I_{j-1}^* - \hat{D}_j - Y_j^*.$$

Similar to the previous case, we can show that the following relationships also hold:

$$\tilde{I}_i = I_i^* - 1 = I_{i-1}^* - \hat{D}_i - (Y_i^* + 1) = I_{i-1}^* - \hat{D}_i - \tilde{Y}_i,$$

$$\tilde{I}_t = \tilde{I}_{t-1} - \hat{D}_t, \quad i < t < j,$$

$$\tilde{I}_j = I_{j-1}^* - \hat{D}_j - Y_j^* = (I_{j-1}^* - 1) - \hat{D}_j - (Y_j^* - 1) = \tilde{I}_{j-1} - \hat{D}_j - \tilde{Y}_j.$$

Thus, we have shown that the variant is feasible as well and does not change the solution beyond period j. Then,

$$\tilde{\Pi} - \Pi^* = -p_i + p_j - \sum_{t=i}^{j-1} h_t.$$

As $p_t \geqslant p_{t+1}$, $\forall t$; $p_i \geqslant p_j$. Also $h_t > 0$, $\forall t$. Thus, $\tilde{\Pi} - \Pi^* < 0$. But this contradicts the assumption that the original solution is optimal. Thus, we conclude that the optimal solution cannot contain any loop.

Proof of Lemma 3

(i) We prove this using contradiction. Suppose the claim is not true. Then $s \geqslant \tau + 1$. But using the definition of τ in (3) we obtain $\hat{D}_t < 0$, $\forall t \geqslant \tau + 1$. Hence $\hat{D}_s < 0$. Also, by the definition of s we have $I_t^* > 0$, $\forall t < s$. Now we focus on period s and write the material balance to obtain

$$I_s^* = I_{s-1}^* - \hat{D}_s - Y_s^*.$$

Using $I_s^* = 0$ we obtain $Y_s^* = I_{s-1}^* - \hat{D}_s$. Combining this with the above two equations, we can show that $Y_s^* > 0$. But now $(X_1^*, I_1^*, \ldots, I_{s-1}^*, Y_s^*)$ form a loop, which contradicts Proposition 1. Hence $s \leqslant \tau$.

(ii) This is easy to observe. If this is not true and $Y_t^* > 0$ for some $t \leqslant s$, there would be a loop $(X_1^*, I_1^*, \ldots, I_{t-1}^*, Y_t^*)$, which contradicts Proposition 1.

(iii) Conducting material balance from period 1 through s we obtain

$$I_s^* = I_0^* - \sum_{t=1}^{s} \hat{D}_t - \sum_{t=1}^{s} Y_t^*.$$

Using the previous result that $I_s^* = 0$, we obtain $I_0^* = \sum_{t=1}^{s} \hat{D}_t + \sum_{t=1}^{s} Y_t^*$. Also note that $Y_t^* = 0$, $\forall t \leqslant s$ from the above result. Thus we show that $I_0^* = \sum_{t=1}^{s} \hat{D}_t$.

Proof of Lemma 4

(i) Suppose that $\hat{D}_{i+1} > 0$. Taking material balance across period i+1 we obtain

$$\begin{aligned} I_{i+1}^* &= I_i^* - \hat{D}_{i+1} - Y_{i+1}^* \\ &= -\hat{D}_{i+1} - Y_{i+1}^*. \end{aligned}$$

Since $\hat{D}_{i+1} > 0$ by supposition and $Y_t^* \geqslant 0$, $\forall t$ by definition, we obtain $I_{i+1}^* < 0$. But this leads to infeasibility as we need $I_t^* \geqslant 0$, $\forall t$. Thus we obtain the desired result.

(ii) Consider the material balance over period j. We obtain

$$I_{j-1}^* - \hat{D}_j - Y_j^* = I_j^*.$$

If $I_j^* = 0$, then $Y_j^* = I_{j-1}^* - \hat{D}_j$. Also, as $I_{j-1}^* > 0$ and $\hat{D}_j < 0$; we obtain $Y_j^* > 0$. Now suppose that $Y_t^* > 0$ for some $t; i+1 \leqslant t \leqslant j-1$. Then this creates a loop $(Y_t^*, I_t^*, \ldots, I_{j-1}^*, Y_j^*)$ which contradicts Proposition 1. Hence, $Y_t^* = 0; i+1 \leqslant t \leqslant j-1$. Now taking the material balance from periods i+1 through j we have

$$I_i^* - \sum_{t=i+1}^{j} \hat{D}_j - Y_j^* = I_j^*$$

Since $I_i^* = I_j^* = 0$ by assumption, we obtain the result $Y_j^* = -\sum_{t=i+1}^{j} \hat{D}_t$.

(iii) This case is different from the previous case as it is possible that $Y_j^* = 0$. Let us suppose that there are two periods $t = k_1, k_2; i+1 \leqslant k_1 < k_2 \leqslant j$ such that $Y_{k_1}^* > 0$ and $Y_{k_2}^* > 0$. But this implies that $(Y_{k_1}^*, I_{k_1}^*, \ldots, I_{k_2-1}^*, Y_{k_2}^*)$ form a loop and contradict Proposition 1. Hence, there is at most one k, $i+1 \leqslant k \leqslant j$ such that $Y_k^* > 0$. From the material balance over periods i through j we have

$$Y_k^* = -\sum_{t=i+1}^{j} \hat{D}_t$$

$$= \sum_{t=i+1}^{j} [-\hat{D}_t]^+ - \sum_{t=i+1}^{j} [\hat{D}_t]^+ \geqslant 0.$$

Thus, the optimal disposal quantity is the difference between the sum of negative and positive effective demands. Also note that this implies that if $I_i^* = 0$, $I_j^* = 0$, then $\sum_{t=i+1}^{j} [-\hat{D}_t]^+ \geqslant \sum_{t=i+1}^{j} [\hat{D}_t]^+$ for feasibility. In case of equality we have $Y_k^* = 0$. Hence we have "at most one" k and not "exactly one" k such that $Y_k^* > 0$.

Proof of Lemma 5

From the definition of $\hat{D}_t$, we know that

$$\hat{D}_t = D_t - \sum_{i=1}^{t-1} k_i D_{t-i}, \quad t \leqslant r+1.$$

Similarly, writing the same expression for $t+1$, we have

$$\hat{D}_{t+1} = D_{t+1} - \sum_{i=1}^{t} k_i D_{t-i}, \quad t \leqslant r+1.$$

Subtracting the latter from the earlier, we obtain

$$\hat{D}_t - \hat{D}_{t+1} = (1+k_1)D_t - D_{t+1} + \sum_{i=1}^{t} (k_{i+1} - k_i) D_{t-i}.$$

Since $D_t \geqslant D_{t+1}$, we have $(1+k_1)D_t > D_{t+1}$. Combining this observation with the supposition that $k_{i+1} > k_i$, we have $\hat{D}_t - \hat{D}_{t+1} > 0, t \leqslant r+1$.

Proof of Lemma 6

(i) We prove by contradiction. We know that for feasibility $I_\tau^* \geqslant 0$. Suppose $I_\tau^* > 0$. Then using the fact that $\hat{D}_t < 0, t > \tau$ in the material balance equations for period $t > \tau$, we obtain $I_t^* > 0$, $\forall t > \tau$ if $Y_t^* = 0$. But we know that $I_T^* = 0$. Hence the excess inventory has to be disposed of (i.e. $Y_k^* > 0$) in some period k; $\tau < k \leqslant T$. But this forms a loop $(X_1^*, I_1^*, \ldots, I_{k-1}^*, Y_k^*)$, which contradicts Proposition 1. Hence $I_\tau^* = 0$.

(ii) Suppose the statement is not true. Then, $Y_k^* > 0$ for some $k \leqslant \tau$. Also, $\hat{D}_t > 0$, $\forall t \leqslant \tau$. Hence, $I_t^* > 0$, $\forall t \leqslant \tau$ for feasibility. Hence $(X_1^*, I_1^*, \ldots, I_{k-1}^*, Y_k^*)$ form a loop and hence contradict Proposition 1. Thus we obtain the stated result.

(iii) Consider the material balance over periods 1 through τ. We obtain

$$I_0^* - \sum_{t=1}^{\tau} \hat{D}_t - \sum_{t=1}^{\tau} Y_t^* = I_\tau^*.$$

Using the previous two results we have $I_\tau^* = 0$ and $Y_t^* = 0;\ \forall t \leqslant \tau$. Substituting these into the above equation, we obtain $I_0^* = \sum_{t=1}^{\tau} \hat{D}_t$. Thus, the initial order quantity is the sum of all positive effective demands.

Proof of Lemma 7
We prove using contradiction. Suppose that the statement of the lemma is not true. Then $\exists i$ such that $X_i^* Y_i^* > 0$, i.e., $X_i^* > 0$ and $Y_i^* > 0$ in the optimal solution. Consider a variant of this solution such that $\tilde{X}_i = X_i^* - 1$ and $\tilde{Y}_i = Y_i^* - 1$. In other words, we buy and sell one unit less. One can see from (19) that if the original solution is feasible, then the variant is also feasible. If Π^* and $\tilde{\Pi}$ are the costs associated with the original and the variant solution, then we have $\Pi^* - \tilde{\Pi} = p_i - c_i < 0$. But this contradicts the supposition that the original solution is optimal.

Proof of Proposition 8
While the proof is an extension of the one for the base model, there are more cases to consider as there are four different types of loops possible. In each case, we first suppose that the optimal solution has a loop with associated cost Π^*. Then we construct a variant, which is feasible and has associated cost $\tilde{\Pi}$. Finally, we show that $\tilde{\Pi} < \Pi^*$ for each of the cases and hence reach a contradiction.

Case I. Let the optimal solution have a smallest loop of the form $(X_i^*, I_i^*, \ldots, I_{j-1}^*, Y_j^*)$. Next, consider a variant $(\tilde{X}_i, \tilde{I}_i, \ldots, \tilde{I}_{j-1}, \tilde{Y}_j) = (X_i^* - 1, I_i^* - 1, \ldots, I_{j-1}^* - 1, Y_j^* - 1)$. Similar to Proposition 1, it is easy to show that if the original solution is feasible, then so is the variant. Then the difference between the costs associated with two solutions is given by $\tilde{\Pi} - \Pi^* = -c_i - \sum_{t=i}^{j-1} h_t + p_j$ where we know that $c_i \geqslant p_i \geqslant p_j,\ \forall j > i$ and $h_t > 0,\ \forall t$. Hence $\tilde{\Pi} - \Pi^* < 0$. Thus, the original solution cannot be optimal.

Case II. Let the optimal solution have a smallest loop of the form $(Y_i^*, I_i^*, \ldots, I_{j-1}^*, Y_j^*)$. Next, consider a variant $(\tilde{Y}_i, \tilde{I}_i, \ldots, \tilde{I}_{j-1}, \tilde{Y}_j) = (Y_i^* + 1, I_i^* - 1, \ldots, I_{j-1}^* - 1, Y_j^* - 1)$. Again, if the original solution is feasible, so is the variant. Then $\tilde{\Pi} - \Pi^* = -p_i - \sum_{t=i}^{j-1} h_t + p_j$. Also because $p_i \geqslant p_j,\ \forall j > i$ and $h_t > 0,\ \forall t$, we have $\tilde{\Pi} - \Pi^* < 0$. Thus the original solution cannot be optimal.

Case III. Let the optimal solution have a smallest loop of the form $(X_i^*, I_i^*, \ldots, I_{j-1}^*, X_j^*)$. Next, consider a variant $(\tilde{X}_i, \tilde{I}_i, \ldots, \tilde{I}_{j-1}, \tilde{X}_j) = (X_i^* - 1, I_i^* - 1, \ldots, I_{j-1}^* - 1, X_j^* - 1)$. Again if the original solution is feasible, so is the variant. Then $\tilde{\Pi} - \Pi^* = -c_i - \sum_{t=i}^{j-1} h_t + c_j$. Again

$c_i \geqslant c_j$; $\forall j > i$ and $h_t > 0$, $\forall t$ and hence $\tilde{\Pi} - \Pi^* < 0$. Thus the original solution cannot be optimal.

Case IV. Let the optimal solution have a smallest loop of the form $(Y_i^*, I_i^*, \ldots, I_{j-1}^*, X_j^*)$ with a minimum sum of $(Y_i + \sum_{t=i}^{j-1} I_t + X_j)$. Next, consider a variant $(\tilde{Y}_i, \tilde{I}_i, \ldots, \tilde{I}_{j-1}, \tilde{X}_j) = (Y_i^* - 1, I_i^* + 1, \ldots, I_{j-1}^* + 1, X_j^* - 1)$. Again, if the original solution is feasible, so is the variant. Then given that the original solution is optimal, we have

$$\tilde{\Pi} - \Pi^* = p_i + \sum_{t=i}^{j-1} h_t - c_j \geqslant 0,$$
$$c_j \leqslant p_i + \sum_{t=i}^{j-1} h_t. \tag{A3}$$

Now consider another variant $(\overline{Y}_i, \overline{I}_i, \ldots, \overline{I}_{j-1}, \overline{X}_j) = (Y_i^* + 1, I_i^* - 1, \ldots, I_{j-1}^* - 1, X_j^* - 1)$. Let $\overline{\Pi}$ be the cost associated with this variant. Again, as in the previous case, using the fact that the original solution is optimal we obtain

$$\overline{\Pi} - \Pi^* = -p_i - \sum_{t=i}^{j-1} h_t + c_j \leqslant 0,$$

where we have used (A3). Thus the second variant is optimal. But it reduces the sum$(Y_i + \sum_{t=i}^{j-1} I_t + X_j)$ and hence we have a contradiction. Thus, we have shown that in all cases, there exists an optimal solution that is loopless.

Proof of Corollary 9

First note that there cannot exist a $k; i < k \leqslant j$ such that both $Y_k^* > 0$ and $X_k^* > 0$ as it contradicts (7). This implies that either $Y_k^* > 0$ or $X_k^* > 0$ or $X_k^* = Y_k^* = 0, \forall k, i < k \leqslant j$. Now we need to show that there cannot be more than one k such that either $Y_k^* > 0$ or $X_k^* > 0$. We use contradiction for this purpose. Let us suppose that there $\exists k_1, k_2; i < k_1 < k_2 \leqslant j$ such that either $Y_{k_1}^* > 0$ or $X_{k_1}^* > 0$ and $Y_{k_2}^* > 0$ or $X_{k_2}^* > 0$. Consider the first case where $X_{k_1}^* > 0$ and $Y_{k_2}^* > 0$. Then, $(X_{k_1}^*, I_{k_1}^*, \ldots, I_{k_2-1}^*, Y_{k_2}^*)$ form a loop and contradict (8). A similar contradiction can be shown for all other cases. The only thing left to be shown is that it is possible that $Y_k^* = X_k^* = 0$, $\forall i < k \leqslant j$. Consider a situation where $\sum_{t=i+1}^{j} [-\hat{D}_t]^+ = \sum_{t=i+1}^{j} [\hat{D}_t]^+$, i.e., the effective demand coming in exactly equals effective demand going out from period $i+1$ through j. In this case, we do not need to sell or buy anything in the optimal solution. This proves the result.

Proof of Corollary 10

(i) Material balance over period $i+1$ yields $I_i^* + X_{i+1}^* - Y_{i+1}^* - \hat{D}_{i+1} = I_{i+1}^*$. We know that $I_i^* = 0$ and hence the retailer cannot sell anything in period $i+1$. Hence, $Y_{i+1}^* = 0$. Also, $I_{i+1}^* > 0$. Thus, $X_{i+1}^* > \hat{D}_{i+1} > 0$. From Lemma 7 and Corollary 9, we know that $X_k^* = Y_k^* = 0$, $\forall k \neq i+1$. Now taking material balance over periods i through j, we obtain $I_i^* + X_{i+1}^* - \sum_{t=i+1}^{j} \hat{D}_t = I_j^*$. Using $I_i^* = 0$ and $I_j^* = 0$, we obtain the desired result.

(ii) Material balance over period j yields $I_{j-1}^* + X_j^* - Y_j^* - \hat{D}_j = I_j^*$. We know that $I_{j-1}^* > 0$ and $I_j^* = 0$. Thus, $Y_j^* - X_j^* > -\hat{D}_j > 0$. From Lemma 7 we know that either X_j^* or Y_j^* should be zero. Clearly, if $Y_j^* = 0$, then $X_j^* < 0$, leading to infeasibility. Hence $X_j^* = 0$ and $Y_j^* > 0$. Then from Corollary 9 we know that $X_k^* = Y_k^* = 0,\ \forall k \neq j$. Now taking material balance over periods i through j we obtain $I_i^* - Y_j^* - \sum_{t=i+1}^{j} \hat{D}_t = I_j^*$. Using $I_i^* = 0$ and $I_j^* = 0$ we obtain the desired result.

(iii) That there is at most one k such that $X_k^* + Y_k^* > 0$ follows directly from Corollary 9. If $\sum_{t=i+1}^{j} \hat{D}_t > 0$, then $X_k^* = \sum_{t=i+1}^{j} \hat{D}_t$ and if $\sum_{t=i+1}^{j} \hat{D}_t < 0$, then $Y_k^* = -\sum_{t=i+1}^{j} \hat{D}_t$ from material balance over periods i through j. Combining the two, we obtain the required result.

Proof of Lemma 12

(i) As $i < j < \hat{t}$, clearly $Y_t^* = 0; i+1 \leqslant t \leqslant j$. Then using the material balance equation over periods i+1 through j, we obtain $I_i^* - \sum_{t=i+1}^{j} \hat{D}_t = I_j^*$ and hence $\sum_{t=i+1}^{j} \hat{D}_t = 0$. Thus, the positive and negative effective demands have to cancel out in this case.

(ii) As $j > \hat{t} > r+1; \hat{D}_j < 0$ using Lemma 2. Then material balance over period j gives $I_{j-1}^* - \hat{D}_j - Y_j^* = I_j^*$. Thus, $Y_j^* = I_j - 1^* - \hat{D}_j > 0$. Also, there cannot be more than one t such that $Y_t^* > 0$ as that would create a loop and hence contradict Proposition 11. Next, from material balance over periods i+1 through j we obtain $I_i^* - \sum_{t=i+1}^{j} \hat{D}_t - Y_j^* = I_j^*$ and hence $Y_j^* = -\sum_{t=i+1}^{j} \hat{D}_t$. This completes the proof.

Proof of Proposition 13

Similar to the previous propositions, we proceed via contradiction. Here, we consider two cases depending on the nature of the loop. We first assume that an optimal solution has a loop and then construct a variant that is loopless and has a weakly lower cost than the optimal solution. In both cases, let Π^*(loop) be the cost associated with the original solution within the loop and $\tilde{\Pi}$(loop) and $\overline{\Pi}$(loop) be the corresponding cost associated with the variants.

Case I. Consider an optimal solution with a smallest loop of the form $(X_1^*, I_1^*, \ldots, I_{j-1}^*, Y_j^*)$ for some j. Of all the solutions of this form, we choose the one that has a minimum sum of $(X_1 + \sum_{t=1}^{j-1} I_t + Y_j)$. We shall use this property for tie breaking, i.e., to choose between solutions with the same total cost. We did not need this condition in the base case, but need it here, because I_t^* is unconstrained in sign because of backordering. We consider a variant of the form $(\tilde{X}_1, \tilde{I}_1, \ldots, \tilde{I}_{j-1}, \tilde{Y}_j) = (X_1^* - 1, I_1^* - 1, \ldots, I_{j-1}^* - 1, Y_j^* - 1)$. It can be shown that this variant is feasible, given that the original solution is feasible just as in the base case. Then

$$\Pi^*(\text{loop}) = c_0 X_1^* + \sum_{t=1}^{j-1} h_t (I_t^*)^+ + \sum_{t=1}^{j-1} s_t (I_t^*)^- - p_j Y_j^*,$$

$$\tilde{\Pi}(\text{loop}) = c_0 (X_1^* - 1) + \sum_{t=1}^{j-1} h_t (I_t^* - 1)^+ + \sum_{t=1}^{j-1} s_t (I_t^* - 1)^- - p_j (Y_j^* - 1),$$

Thus

$$\Pi^*(\text{loop}) - \tilde{\Pi}(\text{loop}) = -p_j + c_0 + \sum_{t=1}^{j-1} \left\{ h_t[(I_t^*)^+ - (I_t^* - 1)^+] + s_t[(I_t^*)^- - (I_t^* - 1)^-] \right\}.$$

Also, since the original solution is optimal, we have $\Pi^*(\text{loop}) - \tilde{\Pi}(\text{loop}) \leqslant 0$ as the variant does not change the original solution outside the loop. Hence, we obtain the following relationship:

$$p_j - c_0 \geqslant \sum_{t=1}^{j-1} \left\{ h_t[(I_t^*)^+ - (I_t^* - 1)^+] + s_t[(I_t^*)^- - (I_t^* - 1)^-] \right\}. \tag{A4}$$

Next, we construct another variant $(\overline{X}_1, \overline{I}_1, \dots, \overline{I}_{j-1}, \overline{Y}_j) = (X_1^* + 1, I_1^* + 1, \dots, I_{j-1}^* + 1, Y_j^* + 1)$ with the associated cost $\overline{\Pi}(\text{loop})$. Working through the same calculations, in this case we obtain

$$p_j - c_0 \leqslant \sum_{t=1}^{j-1} \left\{ h_t[(I_t^* + 1)^+ - (I_t^*)^+] + s_t[(I_t^* + 1)^- - (I_t^*)^-] \right\}. \tag{A5}$$

Using (A4) and (A5), we obtain

$$\sum_{t=1}^{j-1} h_t[(I_t^* + 1)^+ - 2(I_t^*)^+ + (I_t^* - 1)^+] + \sum_{t=1}^{j-1} s_t[(I_t^* + 1)^- - 2(I_t^*)^- + (I_t^* - 1)^-] = 0.$$

For $I_t^* > 0$ and $I_t^* < 0$, respectively, from the above equation, we obtain

$$(I_t^* + 1)^+ - 2(I_t^*)^+ + (I_t^* - 1)^+ = 0,$$

$$(I_t^* + 1)^- - 2(I_t^*)^- + (I_t^* - 1)^- = 0.$$

Thus, both (A4) and (A5) have to hold with equality. To observe this, consider that (A4) holds with strict inequality. Then, adding (A4) and (A5), we obtain $\sum_{t=1}^{j-1} h_t[(I_t^* + 1)^+ - 2(I_t^*)^+ + (I_t^* - 1)] + \sum_{t=1}^{j-1} s_t[(I_t^* + 1)^- - 2(I_t^*)^- + (I_t^* - 1)^-] > 0$. But this contradicts (A6). A similar result can be shown if (A5) holds with strict inequality. Thus, we have shown that $\Pi^*(\text{loop}) = \tilde{\Pi}(\text{loop}) = \overline{\Pi}(\text{loop})$. However, in the first variant, $(\tilde{X}_1 + \sum_{t=1}^{j-1} \tilde{I}_t + \tilde{Y}_j) < (X_1^* + \sum_{t=1}^{j-1} I_t^* + Y_j^*)$. This contradicts the initial supposition.

Case II. Consider an optimal solution with a smallest loop of the form $(Y_i^*, I_i^*, \dots, I_{j-1}^*, Y_j^*)$ for some i, j. Of all the solutions of this form, we choose the one that has the minimum sum of $(Y_i^* + \sum_{t=i}^{j-1} I_t^* + Y_j^*)$. We consider a variant of the form $(\tilde{Y}_i, \tilde{I}_i, \dots, \tilde{I}_{j-1}, \tilde{Y}_j) = (Y_i^* - 1, I_i^* - 1, \dots, I_{j-1}^* - 1, Y_j^* - 1)$. It can be shown that this variant is feasible, given that the original solution is feasible. Then

$$\Pi^*(\text{loop}) = -\,p_i Y_i^* + \sum_{t=i}^{j-1} h_t(I_t^*)^+ + \sum_{t=i}^{j-1} s_t(I_t^*)^- - p_j Y_j^*,$$

$$\tilde{\Pi}(\text{loop}) = -\,p_i(Y_i^* - 1) + \sum_{t=i}^{j-1} h_t(I_t^* - 1)^+ + \sum_{t=1}^{j-1} s_t(I_t^* - 1)^- - p_j(Y_j^* - 1),$$

Thus

$$\Pi^*(\text{loop}) - \tilde{\Pi}(\text{loop}) = -p_j + p_i + \sum_{t=i}^{j-1} \left\{ h_t[(I_t^*)^+ - (I_t^* - 1)^+] + s_t[(I_t^*)^- - (I_t^* - 1)^-] \right\}.$$

Also, since the original solution is optimal, we have $\Pi^*(\text{loop}) - \tilde{\Pi}(\text{loop}) \leqslant 0$, as the variant does not change the original solution outside the loop. Hence, we obtain the following relationship:

$$p_j - p_i \geqslant \sum_{t=i}^{j-1} \left\{ h_t[(I_t^*)^+ - (I_t^* - 1)^+] + s_t[(I_t^*)^- - (I_t^* - 1)^-] \right\}. \tag{A6}$$

Next, we construct another variant $(\overline{Y}_i, \overline{I}_i, \ldots, \overline{I}_{j-1}, \overline{Y}_j) = (Y_i^* + 1, I_i^* + 1, \ldots, I_{j-1}^* + 1, Y_j^* + 1)$ with the associated cost $\overline{\Pi}(\text{loop})$. Working through the same calculations, in this case we obtain

$$p_j - p_i \leqslant \sum_{t=i}^{j-1} \left\{ h_t[(I_t^* + 1)^+ - (I_t^*)^+] + s_t[(I_t^* + 1)^- - (I_t^*)^-] \right\}. \tag{A7}$$

Following the same steps as in the previous case, we can show that both (A6) and (A7) have to hold with equality and hence $\Pi^*(\text{loop}) = \tilde{\Pi}(\text{loop}) = \overline{\Pi}(\text{loop})$. However, in the first variant, $(\tilde{X}_1 + \sum_{t=1}^{j-1} \tilde{I}_t + \tilde{Y}_j) < (X_1^* + \sum_{t=1}^{j-1} I_t^* + Y_j^*)$. This contradicts the initial supposition. Thus, in both cases, we have shown that there exists an optimal solution that is loopless.

Proof of Lemma 14

(i) We prove by contradiction. Suppose that $\hat{D}_t > 0,\ \forall t; i < t \leqslant j$. Then taking a material balance from periods i through j, we have $I_i^* - \sum_{t=i+1}^{j} \hat{D}_t - \sum_{t=i+1}^{j} \hat{Y}_t^* = I_j^*$. Thus $\sum_{t=i+1}^{j} \hat{Y}_t^* = -\sum_{t=i+1}^{j} \hat{D}_t < 0$, which is clearly infeasible. Thus, $\hat{D}_t < 0$ for some $t; i < t \leqslant j$.

(ii) Using the same material balance as above $\sum_{t=i+1}^{j} Y_t^* = -\sum_{t=i+1}^{j} \hat{D}_t > 0$. Also, material balance for period j gives $Y_j^* = I_{j-1}^* - \hat{D}_j > 0$. If $\exists k \neq j$ such that $Y_k^* > 0$ then $(Y_k^*, I_k^*, \ldots, I_{j-1}^*, Y_j^*)$ form a loop and contradict Proposition 13. Hence $Y_t^* = 0,\ \forall t \neq j$. Thus, $Y_j^* = -\sum_{t=i+1}^{j} \hat{D}_t > 0$.

(iii) This is a more general case as compared with the previous one. However, similar to the above case, it is not possible to have two periods k_1 and k_2 such that $Y_{k_1}^* > 0$ and $Y_{k_2}^* > 0$. However now we can have a case where $\sum_{t=i+1}^{j} \hat{D}_t = 0$, in which case, $Y_t^* = 0,\ \forall t; i+1 \leqslant t < j$.

Appendix B: Solution of convex program with tree constraints

First we ignore the tree constraint and solve the optimization problem. Note that without the tree constraint, the problem is separable in t and each optimization problem is of the form

$$\min_{\overline{\lambda}_t > 0} \left\{ H_t(\overline{\lambda}_t, \mu_t) = \frac{(\beta_t + \mu_t - \overline{\lambda}_t)^2}{4\alpha_t} + \overline{\lambda}_t u_t \right\}.$$

On differentiating, we obtain

$$\frac{\delta H_t(\overline{\lambda}_t, \mu_t)}{\delta \overline{\lambda}_t} = -\frac{(\beta_t + \mu_t - \overline{\lambda}_t)}{2\alpha_t} + u_t. \tag{B1}$$

Note that if $u_t < 0$ and $\beta_t + \mu_t - \overline{\lambda}_t = 0$, we have $\frac{\delta H_t(\overline{\lambda}_t, \mu_t)}{\delta \overline{\lambda}_t} < 0$ and hence $\overline{\lambda}_t = \infty$, i.e., the problem is unbounded below. In this case we directly obtain $Y_t^* = \frac{\beta_t + \mu_t - \overline{\lambda}_t}{2\alpha_t} = 0$. Hence we now focus only on cases where $u_t > 0$. Then setting (B1) to zero, we obtain the unconstrained optimum $\overline{\lambda}_t^* = \mu_t + \beta_t - 2\alpha_t u_t$. But depending on the value of u_t we have the following:

$$\overline{\lambda}_t^* = \begin{cases} \mu_t + \beta_t - 2\alpha_t u_t, & \mu_t \geqslant 2\alpha_t u_t - \beta_t, \\ 0, & \text{otherwise.} \end{cases}$$

The next step is to verify whether $\overline{\lambda}_t$ satisfy all the tree constraints. If they do, then we are done and we have to solve for the optimal μ_t. In this simple case, in order to solve for μ_t^*, we have to consider two cases depending on the value of $\overline{\lambda}_t$.

Case I. $\mu_t \leqslant 2\alpha_t u_t - \beta_t$ or $\overline{\lambda}_{\mathbf{t}}^* = 0$

The outer optimization problem (from (15)) is given by

$$\min_{\mu_t} \frac{(\beta_t + \mu_t)^2}{4\alpha_t}$$

$$\text{subject to:} \quad \mu_t \geqslant 0,$$

$$\mu_t \leqslant 2\alpha_t u_t - \beta_t.$$

Here, for feasibility $u_t \geqslant \beta_t/2\alpha_t$. In that case, clearly, $\mu_t^* = 0$.

Case II. $\mu_t \geqslant 2\alpha_t u_t - \beta_t$ or $\overline{\lambda}_{\mathbf{t}}^* = \mu_t + \beta_t - 2\alpha_t u_t$

The outer optimization problem (from (15)) is given by

$$\min_{\mu_t} \; 2\alpha_t u_t^2 + (\beta_t + \mu_t)^2 u_t$$

$$\text{subject to:} \quad \mu_t \geqslant 0,$$

$$\mu_t \geqslant 2\alpha_t u_t - \beta_t.$$

Clearly, the objective function is linear in u_t with a positive coefficient and hence the optimal solution is given by

$$\mu_t^* = \begin{cases} 2\alpha_t u_t - \beta_t, & u_t \geqslant \beta_t/2\alpha_t, \\ 0, & 0 < u_t \leqslant \beta_t/2\alpha_t. \end{cases}$$

We summarize the result in the case when the tree constraints are fulfilled in first iteration.

1.	$u_t < 0$	$\bar{\lambda}_t^* = \mu_t^* = \infty$	$Y_t^* = 0$
2.	$0 < u_t \leqslant \beta_t/2\alpha_t$	$\bar{\lambda}_t^* = \beta_t - 2\alpha_t u_t; \mu_t^* = 0$	$Y_t^* = \mu_t$
3.	$u_t > \beta_t/2\alpha_t$	$\bar{\lambda}_t^* = \mu_t^* = 0$	$Y_t^* = \beta_t/2\alpha_t$

This result has an easy intuitive explanation. Recall that u_t was interpreted as an "effective supply" in period t. So in periods when the "effective supply" is negative, it is optimal not to sell. In periods, when the "effective supply" is relatively small, it is optimal to sell up to the bound and in periods when the "effective supply" is relatively high, it is optimal to sell at the unconstrained optimum.[1] However, if the tree constraints are not fulfilled in the first iteration, we need to follow the reduction procedure explained in Tang (1990).

Let us suppose that the tree constraint is violated only for periods t and $t+1$, i.e. ($\bar{\lambda}_t \leqslant \bar{\lambda}_{t+1}$). Then, according to the reduction method in Tang (1990), the optimal solution is not altered for other periods, and the optimal solution for these two periods is recalculated in the second iteration. According to Tang (1990), the reduction scheme results in $\bar{\lambda}^* = \bar{\lambda}_t^* = \bar{\lambda}_{t+1}^*$. We also assume that $\alpha_t = \alpha_{t+1} = \alpha$ to simplify the exposition. As the application of the procedure is straightforward and our primary objective is to gain insights into the nature of the optimal policy, we only present the result in the case when all the tree constraints are satisfied in the second iteration.

1.	$u_t + u_{t+1} < 0$	$\lambda^* = \mu_t^* = \mu_{t+1}^* = \infty$	$Y_t^* = Y_{t+1}^* = 0$
2.	$0 < u_t + u_{t+1} \leq \frac{\beta_t+\beta_{t+1}}{2\alpha}$	$\lambda^* = (\beta_t + \beta_{t+1}) - 2\alpha(u_t + u_{t+1})$	$Y_t^* = \frac{u_t+u_{t+1}}{2} + \frac{\beta_t-\beta_{t+1}}{4\alpha}$
	$\mu_t^* = \mu_{t+1}^* = 0$	$Y_{t+1}^* = \frac{u_t+u_{t+1}}{2} + \frac{\beta_{t+1}-\beta_t}{4\alpha}$	
3.	$u_t + u_{t+1} > \frac{\beta_t+\beta_{t+1}}{2\alpha}$	$\lambda^* = \mu_t^* = \mu_{t+1}^* = 0$	$Y_t^* = \frac{\beta_t}{2\alpha}, Y_{t+1}^* = \frac{\beta_{t+1}}{2\alpha}$

First, it can be seen that the result reduces to the previous case if all the parameters in both periods are identical. The most important feature of the solution is that the optimal selling policy depends on both u_t and u_{t+1}. Thus, a myopic policy is no longer optimal and the retailer has to look ahead for one more period. A similar analysis can be continued until all the tree constraints are satisfied. Once all the Y_t^* are obtained, the original outer optimization problem to find I_0^* is straightforward. One can see from (9) and (10) that

$$I_0^* = \max_t \sum_{k=1}^{t} (Y_k^* + \hat{D}_k).$$

Thus, the optimal selling and procurement policy of the retailer can be found using this method. Also, having obtained Y_t^*, the optimal price can be obtained very easily using $p_t^* = (a_t - Y_t^*)/b_t$.

[1]Note that the form of the optimal solution is similar to that obtained by solving the primal problem. However, the expression is not exactly identical because we solve the primal problem for a special case of $T = 2$.