Firm investment and the composition of debt*

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Abstract

This paper analyzes optimal debt structure when firms simultaneously choose the size of the project to be financed (investment), and the composition of debt between intermediated debt (bank loans) and arm's length debt (bonds). The key distinction between the two forms of debt is that, when liquidation looms, intermediated debt is easier to restructure. Absent deadweight losses in liquidation, debt structure is irrelevant to the investment choices of the entrepreneur. With liquidation losses, I show that investment is financed by a combination of bank and market finance so long as 1) banks have higher intermediation costs than markets and 2) internal resources of entrepreneurs are sufficiently small. The share of bank finance in total investment then depends non-monotonically on internal resources: firms with very limited internal resources are increasingly reliant on bank finance to expand investment, while medium-sized firms reduce the contribution of bank finance as their internal resources increase. The model's predictions finds support in cross-sectional data on the debt structure of US manufacturing firms.

Keywords : Debt, banks, restructuring, renegotiation, bankruptcy, investment

JEL Classification Numbers: G21, G33

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1 Introduction

The most prominent feature of the corporate debt structure in the US is how dramatically it varies, both in the cross-section of firms and over time. Firms use various types of debt instruments, issued to different types of lenders, with diverse covenants and organized within rich priority structures. In the cross-section, recent work by Rauh and Sufi (2010) analyzes the holdings of bank debt, bonds, program debt (such as commerical paper) and mortgage debt of a sample of publicly traded firms. They find that 70% of firm-year observations hold at least two different types of debt instruments on their balance sheets; moreover, the two most prominently used types of debt are bonds (convertible and non-convertible), and bank debt.[1] Over time, aggregates of corporate debt also display dramatic variation. Adrian, Colla, and Shin (2012), echoing the findings of Kashyap, Stein, and Wilcox (1993) on bank debt and commercial paper, document the fact that over the course of the 2007-2009 recession, the fall in aggregate bank debt outstanding was mirrored by a rise in the issuance of corporate bonds. This suggests that the overall debt structure of firms also changes with business cycle conditions.

Little attention has however been paid to the interaction between debt heterogeneity and the real decisions of firms. On the one hand, the litterature on the link between financial constraints and firm-level investment typically treats all corporate debt as homogeneous. Empirical work on the topic (Fazzari, Hubbard, and Petersen (1988), Hoshi, Kashyap, and Scharfstein (1991), Whited (1992), Kaplan and Zingales (1997), Hubbard (1998)) focuses on measures of total leverage. Likewise, theoretical work on financial constraints and firm-level investment (Cooley and Quadrini (2001), Albuquerque and Hopenhayn (2004), Clementi and Hopenhayn (2006)) explores single borrower-lender relationships, thereby also abstaining from issues of debt heterogeneity. On the other hand, there is an extensive literature on debt heterogeneity (Diamond (1991), Diamond (1993), Rajan (1992), Bolton and Scharfstein (1996), Bolton and Freixas (2000), Park (2000), DeMarzo and Fishman (2007)). This literature explores optimal capital structure with multiple potential lenders, but typically in the context of a firm that must finance a project of fixed size. Each of these strands of the literature is silent on the question of whether, or how, the scale of borrowing and investment could be related to the forms of debt financing that are available to a firm.

The goal of this paper is to provide a simple theoretical framework in which to analyze this question. Specifically, I ask how access to different forms of debt financing affects the borrowing and investment choices of firms. I study a static model in which an entrepreneur can finance investment by borrowing from two potential sources, which I label "bank" and "market" lenders. The entrepreneur uses her own internal finance, in combination with these sources of debt, to finance investment in fixed capital. Production

[1] See the first column of their table 1.

possibilities are standard. Capital is the input into a decreasing returns to scale technology. This technology is subject to idiosyncratic productivity risk, which is realized only after the entrepreneur has raised debt. The entrepreneur has limited liabity; she can default on any debt claims, in which case the project is liquidated. Borrowing is constrained by the fact that liquidation entails inefficient losses.

The model postulates two main differences between bank and market debt. On the one hand, I assume that bank debt can be easily renegotiated in times of financial distress, while market debt cannot. The view that bank debt is more flexible than market debt underlies much of the literature on debt heterogeneity. Gertner and Scharfstein (1991) and Bolton and Scharfstein (1996) provide microfoundations for this view, building on the idea that dispersed holding of market debt creates a hold-out problem in which individual debt holders have limited incentives to agree to a debt restructuring plan.[2] In the model, this flexibility implies that bank debt typically commands a smaller liquidation risk premium than market debt. On the other hand, I assume that the marginal cost of debt issuance is larger for banks that for market lenders. The debt structure chosen by the entrepreneur balances the benefit of bank debt flexibility with the difference in marginal cost of debt issuance between banks and market lenders.

The first prediction of the model is that the optimal debt structure of most firms features a mix of bank and market finance. This is consistent with the fact documented by Rauh and Sufi (2010) that firms simultaneously use different types of debt instruments, but it stands in contrast with much of theoretical literature on debt heterogeneity, which typically finds that the optimal debt structure involves borrowing either from one source or from the other, depending on firms' characteristics. Here, the fact that investment and scale are endogenous is crucial. Indeed, the very largest firms, which have sufficient internal finance relative to the optimal investment scale and face little liquidation risk, do not find it beneficial to combine bank with market debt, and instead prefer to use only market debt. For the other firms, with more limited internal resources and higher liquidation risk, the marginal benefit of bank flexbility is not negligible relative to the "spread" between bank and market marginal lending costs. This leads to an interior solution for the debt structure.

The second prediction of the model is that the share of bank debt as a fraction of total debt is a non-monotonic function of firms' internal ressources. Among those firms that use a mixed debt structure, the share of bank debt is increasing when internal finance is very small, but starts decreasing when internal finance becomes sufficiently large (and eventually drops to 0 for the largest firms). This non-monotonic pattern arises because the very smallest firms choose an investment policy which exhausts their bank borrowing capacity. For these firms, any marginal increase in internal finance expands their bank borrowing capacity, and leads

[2] Empirical evidence in support of the view that bank debt is easier to restructure than bonds is discussed in detail in section 2.

to an expansion in bank debt issuance. On the other hand, when firms have accumulated sufficient internal finance, their bank borrowing constraint need not bind. These firms choose an interior debt structure. In that case, a marginal increase in internal finance results in a substitution away from bank debt, and into market debt.

The third prediction relates to comparative statics. Specifically, the model helps to distinguish between parameters that leave that affect overall borrowing, but leave the composition of borrowing unchanged, from parameters that affect borrowing composition. For example, an increase in the spread between bank and market lending costs causes firms with large internal finance to substitute market debt for bank debt. These firms therefore increase market debt issuance, but reduce bank debt issuance, when the spread increases. For smaller firms, with less internal finance, the increase in the spread leads to a decline in issuance of both market and bank debt. By contrast, a fall in average productivity lowers the overall level of borrowing of all firms, small and large, and leaves the composition of borrowing mostly unchanged. This provides a rationale for the intuition underlying much of the empirical work on the identification of monetary policy shocks through their differential effects on debt choices of small vs. large firms, such as Kashyap, Stein, and Wilcox (1993) and Gertler and Gilchrist (1994).

Finally, I use evidence from the balance sheet of US manufacturing firms, both in the cross-section and over time, to shed light the key predictions of the model.[3] In the cross-section of this sample of firms, much as in the model, the bank share of total debt is non-monotonic in firm size: it increases for the very smallest firms, and decreasing thereafter.[4] Additionally, over the last three recessions, and especially over the most recent one, the level and composition of debt of small and large firms evolved in very different ways. The stock of outstanding debt of small firms declined substantially, mostly due to a decline in the stock of oustanding bank debt, while large firms substituted market debt for bank debt and did not experience a significant fall in their overall liabilities. From the standpoint of the model, this pattern is consistent with the view that asymmetric shocks afffecting banks' lending costs played an important role in the in the early stages of the 2007-2009 recession.

This paper contributes to the theoretical literature on debt heterogeneity in two ways. First, while it builds on the "trade-off" theory of debt structure (Gertner and Scharfstein (1991), Rajan (1992), Bolton and Scharfstein (1996)), the paper extends it by endogenizing the choice of investment scale. In this respect, the closest setup to this paper is Hackbarth, Hennessy, and Leland (2007), who likewise rely on the assumption that bank debt is more flexible than market debt in restructuring, but do not allow for an endogenous

[3] This evidence, which I document in a sample including both private and publicly traded firms, echoes and supplements the findings of Rauh and Sufi (2010) and Adrian, Colla, and Shin (2012), who focus on publicly traded, and therefore large firms.

[4] The data I use focuses on total assets as a measure of size, rather than internal finance; I discuss this issue in more detail in section 3.

determination of investment scale. Second, it shows that debt heterogeneity can help explain why credit supply shocks have different effects on the investment behavior of small and large firms, as documented by Kashyap, Stein, and Wilcox (1993) or Gertler and Gilchrist (1994).

The rest of the paper is organized as follows. Section 2 describes the model and derives the set of feasible debt structures for a firm with a given level of internal resources. Section 3 studies the optimal debt structure. Section 4 studies some comparative statics of the model. Section 5 relates the findings of section 3 and 4 to data on the debt composition of a sample of US manufacturing firms during the last three recessions. Finally, section 6 concludes.

2 A model of bank and market financing

This section describes a static model in which an entrepreneur with internal funds e finances a project by borrowing from two lenders: a bank, and the market. The only friction of the model is that there is limited liability; the entrepreneur can choose to default on her debt obligations. However, doing may involve output losses, if the project is liquidated. This motivates the key distinction between bank and market lenders: bank loans can be restructured in times of financial distress, in order to avoide inefficient liquidation losses. Market debt, on the other hand, cannot. I come back to the interpretation of the model's assumptions below after describing the key elements of the model.

2.1 Production structure and timing

The entrepreneur owns the firm and operates a technology which takes physical assets k as an input, and produces output:

$$y = \phi k^{\zeta}$$

Here, ϕ, the productivity of the technology employed by the entrepreneur, is a random variable, the realization of which is unknown to both the entrepreneur and the lenders at the time when investment in physical assets is carried out. In what follows, I denote the CDF of ϕ by $F(.)$. ζ governs the degree of returns to scale of the technology operated by the entrepreneur. Assets depreciate at rate $\delta \in [0,1]$. After production has been carried out and depreciation has taken place, the following ressources are available to the entrepreneur to either consume or repay creditors:

$$\pi(\phi) = \phi k^{\zeta} + (1-\delta)k \tag{1}$$

I make the following assumptions about the production structure:

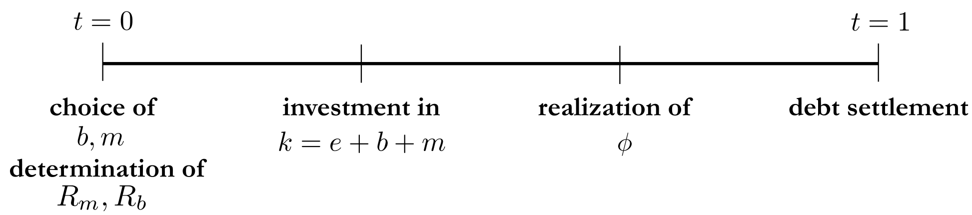


Figure 1: Timing of model.

Assumption 1 *The firm's production technology has the following characteristics:*

- *Production has decreasing returns to scale:* $\zeta < 1$*;*
- *The productivity shock* ϕ *is a positive, continuous random variable with density* f*. Moreover,* $f(0) = 0$ *and the hazard rate of* ϕ *is strictly increasing.*

The first part of the assumption is standard in models of firm investment, and guarantees that firms have a finite optimal scale of operation. The second part of the assumption consists in restrictions on the distribution of productivity shocks. Restricting the shock ϕ to be a postive random variable implies that there is a positive lower bound on ressources, $(1-\delta)k$, so that riskless lending may occur, to the extent that $\delta < 1$. The increasing hazard rate is a technical assumption which guarantees the unicity of lending contracts.[5]

The entrepreneur finances investment in physical assets, k, from three sources: internal finance e, with which it is initially endowed; bank debt, b, and market debt m. The resulting balance sheet constraint is thus simply:

$$k = e + b + m.$$

The timing of actions and events, for an entrepreneur with internal finance e, is summarized in figure 1. The model has two periods. At $t = 0$, the entrepreneur, the bank lender and the market lender agree about a debt structure (b, m), and promised repayments, R_b to the bank, and R_m for the market lender. Investment in k then takes place, and the productivity of the firm, ϕ, is realized. At time $t = 1$, debt payments are settled; that is, the firm can choose to make good on promised repayments, restructure its debt, or proceed to bankruptcy.

Finally, in this section, I only assume that all agents are utility maximizers and have preferences that are weakly increasing in their monetary payoffs. In the next section, I will focus on optimal choices in the case of a risk-neutral entrepreneur; however, all the results presented in this section on the settlement of debt are entirely independent of the assumption of risk-neutrality, and hold for general preference specifications so long as preferences are increasing in payoffs. In particular, the set of feasible debt structures characterized in this section is identical across preference specifications.

[5] See appendix A for details.

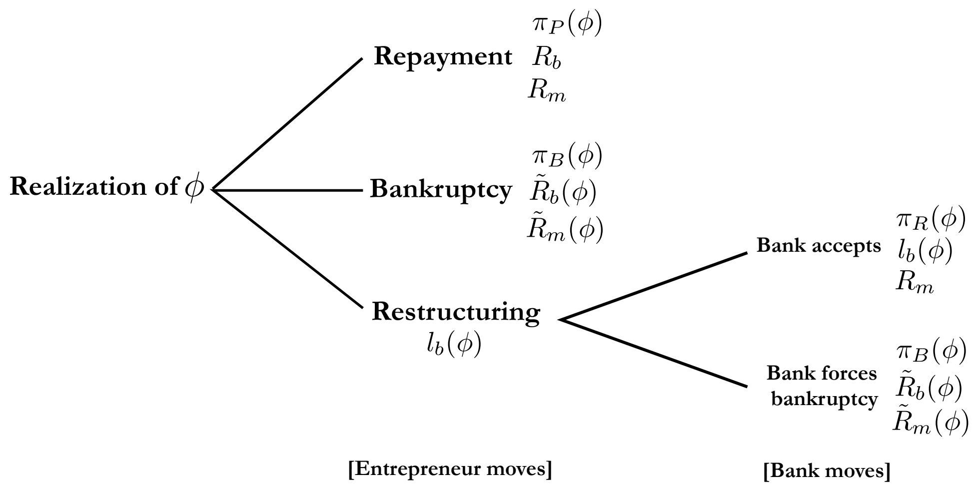


Figure 2: Debt settlement.

2.2 Debt settlement

The debt settlement stage takes place once the productivity of the firm has been observed by all parties. I model the debt settlement process as a two-stage game. In the first stage, the entrepreneur can choose between three alternatives, summarized in figure 2: repay in full both its bank and market creditors; make a restructuring offer to the bank; or file for bankruptcy. If the entrepreneur chooses to repay in full both its creditors, her payoff is:

$$\pi_P(\phi) = \pi(\phi) - R_m - R_b \tag{2}$$

while the payoff to the bank and market lender are, respectively, R_b and R_m, as initially promised. I next turn to describing each party's payoff under the two other alternatives, bankruptcy and restructuring.

2.2.1 Bankruptcy

If the entrepreneur chooses to file for bankruptcy, the project is terminated and liquidated, and the proceeds from liquidation are distributed to creditors. Once bankruptcy is declared, the entrepreneur has no claim to liquidation proceeds; that is, her liquidation payments are assumed to be 0, so that the monetary payoff to the entrepreneur in bankruptcy is:[6]

$$\pi_B(\phi) = 0. \tag{3}$$

[6]This is without loss of generality. Allowing for the entrepreneur to be a residual claimant in bankruptcy would not alter the results, since in the debt settlement stage, bankruptcy would never be declared in states in which the entrepreneur has sufficient resources to repay both lenders. I omit this possibility to alleviate notation.

I make the following assumption about the impact of liquidation on output:

Assumption 2 (Liquidation losses) *Under bankruptcy, the proceeds $\tilde{\pi}(\phi)$ to be distributed to creditors and the entrepreneur are a fraction χ of the project's value:*

$$\tilde{\pi}(\phi) = \chi\pi(\phi) \quad , \quad 0 \leq \chi < 1.$$

Liquidation leads to inefficient losses of output; that is, the liquidation value of the project is strictly smaller than the value of the project under restructuring or repayment. Specifically, liquidation losses are equal to $(1-\chi)\pi(\phi)$. Consistent with the evidence in Bris et al. (2006) discussed below, this assumption captures fact that bankruptcy proceedings are typically costly, both administratively and because they halt production activities. Moreover, asset values of firms after cash auction proceedings are typically only a fraction of pre-bankruptcy values. This is the key friction in the static model with risk-neutrality: absent bankruptcy losses, lending would be unconstrained, as I will discuss below.

I assume that bankruptcy proceeds are distributed among creditors according to an agreed-upon priority structure, in line with the Absolute Priority Rule (APR) that governs chapter 7 proceedings in the US.[7] In this section, I assume that bank debt is senior to market debt.[8] Under this priority structure, the payoff to bank lenders and market lenders, are:

$$\tilde{R}_b^K(\phi) = \min\left(R_b, \chi\pi(\phi)\right), \tag{4}$$

$$\tilde{R}_m^K(\phi) = \max\left(\chi\pi(\phi) - R_b, 0\right). \tag{5}$$

The first line states that the bank's payoff in bankruptcy is at most equal to its promised repayment R_b. The second payoff states that market lenders are residual claimants. Note that this formulation does not, a priori, preclude cases in which $\tilde{R}_m(\phi) \geq R_m$, that is, market lenders receiving a residual payment larger than their initial claim. I will however show that this never occurs in the equilibrium of the debt settlement game.

2.2.2 Restructuring

Instead of filing for bankruptcy, I assume that the entrepreneur can enter a private workout process with her creditors. Because going bankrupt implies losses of output, it may sometimes be in the interest of

[7] See White (1989) for institutional details on the APR.

[8] I come back to the issue of the optimality of bank seniority in conclusion. It is likely that, in this model, bank seniority is the optimal priority structure from the standpoint of the firm; numerically, it is the case for all the versions of the model which I have explored.

creditors and the entrepreneur to arrive at a compromise. I make the following restriction to the workout process.

Assumption 3 (Bank debt flexibility) *The entrepreneur may only restructure debt payments owed to the bank, R_b; payments to the market lender, R_m, cannot be renegotiated.*

This is the key distinction between bank and market lending in the model; I delay its discussion to the next paragraph, and first describe its implications for the debt settlement process. I assume that the private workout operates as follows: the entrepreneur makes a one-time offer to the bank which takes the form of a reduction in promised repayments $l_b \leq R_b$. In case the offer is accepted, the bank obtains l_b, and the entrepreneur obtains:

$$\pi_R(\phi) = \pi(\phi) - R_m - l_b(\phi). \tag{6}$$

If, on the other hand, the bank refuses the entrepreneur's offer, the private workout fails, and the project is liquidated. In this case, the payoff to the bank is given by equation (4). The participation constraint of the bank is thus:

$$l_b \geq \tilde{R}_b^K(\phi).$$

The entrepreneur will choose her restructuring offer to maximize her net payoff under restructuring, subject to the participation constraint of the bank. Formally:

$$\begin{aligned} \pi_R(\phi) &= \max_{l_b} \quad \pi(\phi) - R_m - l_b \quad \text{s.t.} \quad l_b \geq \tilde{R}_b^K(\phi) \\ &= \begin{cases} \pi(\phi) - R_b - R_m & \text{if } R_b \leq \chi\pi(\phi) \\ (1-\chi)\pi(\phi) - R_m & \text{if } R_b > \chi\pi(\phi) \end{cases} \end{aligned} \tag{7}$$

Intuitively, this result indicates that the entrepreneur will choose to make a restructuring offer only when its cash on hand is small enough, relative to promised repayments to the bank. Note that the larger the value of χ, the higher the restructuring threshold; that is, potential bankruptcy losses effectively allow the entrepreneur to extract concessions from the bank.

2.2.3 Debt settlement equilibria

Given the realization of ϕ, the entrepreneur chooses whether to repay, restructure or file for bankrucpy, by comparing her payoffs $\pi_P(\phi)$, $\pi_B(\phi)$ and $\pi_R(\phi)$ under each option. The following proposition describes the resulting perfect equilibria in pure strategies of the debt settlement game described in figure 2. There is a unique equilibrium for each realization of ϕ; however, the set of possible equilibria, parametrized by ϕ,

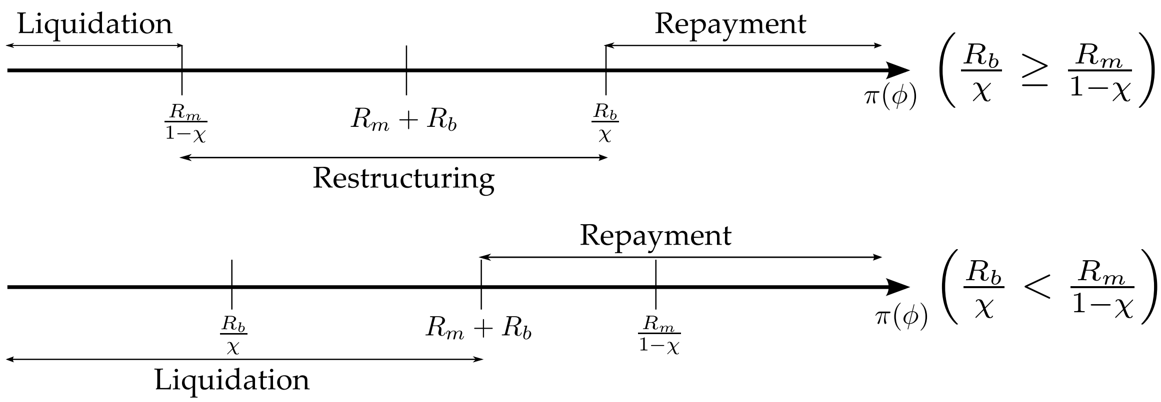


Figure 3: Debt settlement equilibria.

depends on the terms of the debt contracts.

Proposition 1 (Debt settlement equilibria)

*If $\frac{R_m}{1-\chi} < \frac{R_b}{\chi}$ (**R-contracts**), there are some realizations of ϕ for which the entepreneur chooses to use her restructuring option. Specifically, the entrepreneur chooses to repay her creditors when $\pi(\phi) \geq \frac{R_b}{\chi}$; to restructure debt when $\frac{R_m}{1-\chi} \leq \pi(\phi) < \frac{R_b}{\chi}$; and to file for bankruptcy when $\pi(\phi) < \frac{R_m}{1-\chi}$.*

*If $\frac{R_m}{1-\chi} \geq \frac{R_b}{\chi}$ (**K-contracts**), there are no realizations of ϕ such that the entrepreneur attempts to restructure debt with the bank. Instead, she chooses to repay her creditors when $\pi(\phi) \geq R_m + R_b$, and otherwise, she files for bankruptcy.*

Moreover, in bankruptcy or restructuring, market and bank lenders never obtain more than their promised repayments: $\tilde{R}_m(\phi) \leq R_m$ and $\tilde{R}_b(\phi) \leq R_b$, regardless of whether the debt contract is an $R-contract$ or a $K-contract$.

The proof for this and all following propositions are reported in appendix A. Figure 3 illustrates sets of equilibria for each type of contract. In the case of a K-contract ($\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$), no restructuring ever occurs, and bankruptcy losses cannot be avoided when the cash on hand of the firm, $\pi(\phi)$, falls below the threshold at which the firm prefers declaring bankruptcy over repayment, $R_m + R_b$. This occurs because the stake of the flexible creditors, R_b, is too small for restructuring to bring about sufficient gains for the entrepreneur to avoid default on market debt.

On the other hand, in the case of an R-contract, ($\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$), the flexibility of bank debt sometimes allows the entrepreneur to make good on its payments on market debt (this corresponds to restructuring region below $R_m + R_b$ in figure 3). Some R-equilibria will see the entrepreneur exert a degree of bargaining power over the bank: indeed, the bank will be forced to accept a settlement, even though the firm has enough cash on hand to make good on both its promises (this corresponds to the restructuring region above

$R_m + R_b$ in figure 3). This region corresponds to strategic restructurings on the part of the entrepreneur, who takes advantage of the fact that the bank can never extract from her more than its reservation value under restructuring, $\chi\pi(\phi)$, in any private workout.

2.3 Discussion

The model's fundamental distinction between bank lending and market lending is that bank lenders are capable of flexibility in times when the firm is not able (or willing) to repay her debt. A natural question is then whether this assumption is borne out in the data. Gilson, Kose, and Lang (1990) examine a sample of 169 financially distressed firms. They show that about half (80 or 47% of the total) firms successfully restructure outside of formal judicial proceedings, while the other half (89, or 53% of the total) file for bankruptcy. They show that successful restructurings out of court involve, in 90% of cases, a firm that has outstanding bank loans, while only 37.5% of successful restructuring involves firms with oustanding public debt. Moreover, they show that the existence of bank loans is the single most important determinant of whether firms successfully restructure out of court, more so than other firm characteristics such as firm size, age or leverage. A theoretical justification of the observation that bank debt is easier to restructure is developped by Gertner and Scharfstein (1991), who argue that when coordination problems among dispersed holders of public debt lead to a failure to efficiently restructure that debt. Bolton and Scharfstein (1996) also study how free-riding problems can lead to inefficiencies during debt restructuring involving a large number of creditors. The assumption of bank debt flexibility can thus be thought of as a reduced-form manner of capturing the coordination and free-riding problems discussed elsewhere in the literature.

Additionally, the model makes the assumption that the formal liquidation of a project − which occurs only if restructuring has failed − leads to inefficient losses. In the model, this occurs when $\chi < 1$; when $\chi = 1$, liquidation leads to a transfer of ownership but not losses in values. Bris, Welch, and Zhu (2006) study a sample of 61 chapter 7 and 225 chapter 11 filings between 1995 and 2001. In particular, they report measures of the ratio of pre to post-bankruptcy asset values (excluding legal fees). For chapter 11 proceedings, which provides a legal framework for debt restructuring but does not involve liquidation, the mean of the ratio of pre to post-bankruptcy asset values in their sample is 106.5%; that is, on average, this ratio increases after chapter 11 proceedings. For chapter 7 proceedings, this is not the case: asset values decline as a result after the bankruptcy. Here, Bris, Welch, and Zhu (2006) offer two measures of post-bankruptcy asset values. The first measure is liquidation value of the firm after collateralized lenders have seized the assets to which they had a lien outside of bankruptcy proceedings. With this measure, asset values post-bankruptcy are 17.2% of pre-bankruptcy asset values, on average. The second measure tries attempts to include the value of collateralized assets; average post-bankruptcy values are then estimated to represent 80.0% of pre-bankruptcy values; the median ratio is 38.0%.

While the model's debt settlement stage does not clearly distinguish between private and formal (chapter 11) workouts, it assumes that they are costless, whereas liquidation (chapter 7) is assumed to be costly. In this respect, the model's assumptions are thus consistent with the results of Bris, Welch, and Zhu (2006). In general, assuming that debt renegotiation (private or under a chapter 11 filing) is costly should not alter the key qualitative results of the model. So long as renegotiation costs are strictly smaller than those associated to liquidation, renegotiation will be beneficial to the firm provided that promised repayment to bank lenders are sufficiently large, as in figure 3.

2.4 Debt pricing and the lending menu

I now turn to describing the pricing of bank and market debt contracts at $t = 0$, before the realization of the shock ϕ and the ressources $\pi(\phi)$.

2.4.1 Expected lending returns

Let:

$$\mathbb{E}\left[\Pi_i\left(e, b, m, R_b, R_m\right)\right] = \int_0^{+\infty} \tilde{R}_i(\phi) dF(\phi), \quad i = b, m$$

denote the gross expected returns for each lender (market or bank) at time $t = 0$. Appendix A details the expressions of lenders' payoff functions $\tilde{R}_b(\phi)$ and $\tilde{R}_m(\phi)$ for the debt settlement equilibrium associated to each realization of ϕ. They depend on $\pi(\phi) = \phi k^\zeta + (1-\delta)k = \phi(e+b+m)^\zeta + (1-\delta)(e+b+m)$. Moreover, their expression also depends on whether the debt contract is an R- or a K-contract. For example, assume that the contract is a K-contract, that is, $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$. In this case, the bank will face three possible outcomes, conditional on the realization of $\pi(\phi)$:

- If $\phi \geq \overline{\phi} \equiv \frac{R_b + R_m - (1-\delta)(e+b+m)}{(e+b+m)^\zeta}$, that is, above the bankruptcy threshold, the bank will be repayed in full;
- if $\overline{\phi} > \phi \geq \underline{\phi} \equiv \frac{R_b - (1-\delta)(e+b+m)}{\chi(e+b+m)^\zeta}$, the entrepreneur will file for bankruptcy; but the bank, because of its seniority in the priority structure of the bank, will still be repayed in full (market lenders will only receive partial repayments);
- if $\underline{\phi} > \phi$, the bank is only partially repayed, and the market lenders receive no payments.

Thus, the expected return function of the bank lender will be given by:

$$\mathbb{E}\left[\Pi_b\left(e,b,m,R_b,R_m\right)\right] = \underbrace{\left(1-F\left(\overline{\phi}\right)\right)R_b}_{\text{no bankruptcy}} \quad + \quad \underbrace{R_b\int_{\underline{\phi}}^{\overline{\phi}} dF(\phi)}_{\text{bankruptcy, full repayment}} \quad + \quad \underbrace{\chi\int_{0}^{\underline{\phi}} \pi(\phi)dF(\phi)}_{\text{bankruptcy, partial repayment}}$$

Expressions of the expected returns functions of both types of lenders for each type of contract are also reported in appendix A. Importantly, expected lending returns for the bank are independent of R_m, and thus independent of whether the contract is an R- or a K- contract. This result follows from the assumption that the bank may only accept or reject the offer of the firm. Because of this, the bank always accepts its reservation value, the bankruptcy payoff, as restructuring settlement. When the bank is senior in the priority structure, the bankruptcy payoff is therefore independent of R_m, so that the expected returns from bank lending are independent of the value of R_m, and therefore of the condition $\frac{R_m}{1-\chi} \gtrless \frac{R_b}{\chi}$.

2.4.2 The lending menu

I make the assumption that both kinds of financial intermediaries are perfectly competitive, so that debt is priced by equating the gross expected return from lending to the lenders' gross cost of funds.

I assume that lenders have constant marginal costs of funds, and I denote lender i's cost of funds by $1+r_i$, with $r_i > 0$. For now, I make no assumptions on which type lender has the higher cost of funds; I come back to this issue in the next section.

A contract (R_m, R_b) corresponding to debt structure (b, m) will be available to a firm with internal finance e if it satisfies the zero profit condition of both lenders.

Definition 1 (Lending contracts) *The set of lending contracts proposed to a firm with internal finance e that desires to implement a debt structure (b, m) is the set:*

$$\mathcal{L}(b,m,e) \equiv \left\{ (R_b, R_m) \in \mathbb{R}_+^2 \;\middle|\; \begin{array}{l} \mathbb{E}\left[\Pi_b\left(e,b,m,R_b,R_m\right)\right] = (1+r_b)b \\ \mathbb{E}\left[\Pi_m\left(e,b,m,R_b,R_m\right)\right] = (1+r_m)m \end{array} \right\}$$

Since $\mathcal{L}(b, m, e)$ is a subset of $\mathbb{R}^2$, it is a partially ordered set when endowed by the product order $\leq_\times$.[9] $(\mathcal{L}(b, m, e), \leq_\times)$ therefore has at most one least element.[10]. This justifies the following definition:

Definition 2 (The dominating contract) *The dominating contract for debt structure (b, m) and internal finance e is the least element of the partially ordered set $(\mathcal{L}(b, m, e), \leq_\times)$ (if it exists).*

[9]This is the partial order defined by $(x_1, y_1) \leq_\times (x_2, y_2) \iff x_1 \leq y_1$ and $x_2 \leq y_2$.
[10]This is $(R_b, R_m) \in \mathcal{L}(b, m, e)$ such that $\forall(\tilde{R}_b, \tilde{R}_m) \in \mathcal{L}(b, m, e)$, $(R_b, R_m) \leq_\times (\tilde{R}_b, \tilde{R}_m)$.

Finally, the lending menu for a firm with internal finance e is the set of all debt structures (b, m) such that there exists a dominating contract at (b, m) for equity e:

Definition 3 (The lending menu) *The lending menu for a firm with internal finance e is the set:*

$$\mathcal{S}(e) \equiv \left\{ (b,m) \in \mathbb{R}_+^2 \;\middle|\; \begin{array}{l} \mathcal{L}(b,m,e) \neq \emptyset \\ (\mathcal{L}(b,m,e), \leq_\times) \text{ has a least element} \end{array} \right\}$$

Several intuitive elements of these formal definitions are worth emphasizing. First, the lending menu $\mathcal{S}(e)$ of definition 3 is the set of feasible debt structures for the entrepreneur with internal finance e. There are two requirements for a debt structure to be part of the lending menu: first, there must exist lending contracts for that debt structure; second, one of them must be a dominating contract, in the sense of definition 2. Intuitively, the dominating contract has the property that it is (weakly) cheaper, in both dimensions (R_b, R_m), than any other contract that satisfies the lenders' zero-profit conditions.

Note that, a priori, there is no reason to think that $(\mathcal{L}(b,m,e), \leq_\times)$ generically has a least element. It could well be that, for a certain level of internal finance e and a certain debt structure (b, m), the set of lending contracts contains two elements (R_b, R_m) and $(R_b^{'}, R_m^{'})$ such that $R_b > R_b^{'}$ but $R_m < R_m^{'}$, which cannot be ordered by $\leq_\times$. In that case, $(\mathcal{L}(b,m,e), \leq_\times)$ would have no least element. Definition 3 would then exclude (b, m) from the firms' feasible set, the lending menu $\mathcal{S}(e)$, despite the fact that $\mathcal{L}(b,m,e)$ would be non-empty. This would seem to arbitrarily restrict the set of feasible contracts.

Fortunately, this is never the case: $(\mathcal{L}(b,m,e), \leq_\times)$ always has a least element when $\mathcal{L}(b,m,e) \neq \emptyset$, so that the lending menu never contains debt structures associated to several possible contracts.[11] I discuss this further in the next subsection.

Finally, while the definition of competitive contracts only require promised repayments to be positive, they in fact also satisfy $R_m \geq (1 + r_m)m$ and $R_b \geq (1 + r_b)b$; that is, lenders never ask for promised repayments which imply a yield below their marginal cost of funds. Intuitively, since the expected value of total (bankruptcy and non-bankruptcy) claims has to equal the cost of funds for each lenders, it cannot be the case that both are strictly smaller or strictly larger that that cost of funds. As bankruptcy claims are strictly smaller than non-bankruptcy claims, it must therefore be the case that non-bankruptcy claims exceed the costs of funds.

[11] This result is proved in appendix A, as part of the proof of proposition 2.

2.4.3 Some general properties of the lending menu

The following proposition describes in more detail the structure of the lending menu, emphasizing the fact that it can be explicitly partitioned betwen contracts leading to K-equilibria and contracts leading to R-equilibria.

Proposition 2 (A partition of the lending menu in the general case) *The lending menu $\mathcal{S}(e)$ can be partitioned as:*

$$\mathcal{S}(e) = \mathcal{S}_R(e) \cup \mathcal{S}_K(e) \quad , \quad \mathcal{S}_R(e) \cap \mathcal{S}_K(e) = \emptyset,$$

where:

$$\mathcal{S}_R(e) = \tilde{\mathcal{S}}_R(e) \quad , \quad \mathcal{S}_K(e) = \tilde{\mathcal{S}}_K(e) \setminus \left(\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e)\right),$$

and

$$\tilde{\mathcal{S}}_K(e) \equiv \left\{ (b,m) \in \mathbb{R}^2_+ \;\middle|\; \begin{array}{lr} 0 \leq \frac{(1+r_b)b+(1+r_m)m}{(e+b+m)^\zeta} \leq \hat{M}(e+b+m) + (1-\delta)(e+b+m)^{1-\zeta} & (\text{c}-\text{joint}) \\ R_l(b,m,e) > \frac{R_b(b,m,e)}{\chi} & (\text{frontier}-\text{K}) \end{array} \right\},$$

$$\tilde{\mathcal{S}}_R(e) \equiv \left\{ (b,m) \in \mathbb{R}^2_+ \;\middle|\; \begin{array}{lr} 0 \leq \frac{(1+r_b)b}{\chi(e+b+m)^\zeta} \leq \mathbb{E}(\phi) + (1-\delta)(e+b+m)^{1-\zeta} & (\text{c}-\text{bank}) \\ 0 \leq \frac{(1+r_m)m}{(1-\chi)(e+b+m)^\zeta} \leq \hat{I}(e+b+m) + (1-\delta)(e+b+m)^{1-\zeta} & (\text{c}-\text{market}) \\ \frac{R_{m,l}(b,m,e)}{1-\chi} \leq \frac{R_b(b,m,e)}{\chi} & (\text{frontier}-\text{R}) \end{array} \right\}.$$

The sets $\mathcal{S}_R(e)$ and $\mathcal{S}_K(e)$ are non-empty, compact and connected subsets of $\mathbb{R}^2_+$. Moreover, for a firm with internal finance e:

- *The dominating contract associated to (b,m) is an R-contract, if and only if, $(b,m) \in \mathcal{S}_R(e)$;*
- *The dominating contract associated to (b,m) is a K-contract, if and only if, $(b,m) \in \mathcal{S}_K(e)$;*

Expressions for the functions $\hat{M}(e+b+m)$, $\hat{I}(e+b+m)$, $R_l(b,m,e)$, $R_{m,l}(b,m,e)$ and $R_b(b,m,e)$ are given in appendix A.

There are two important elements in this proposition. First, the lending menu is the union of two subsets, $\tilde{S}_K(e)$ and $\tilde{S}_R(e)$, which correspond, respectively, with the set of debt structures for which there exists a K-contract (the set $\tilde{S}_K(e)$) and the set of debt structures consistent for which there exists an R-contract (the set $\tilde{S}_R(e)$). The intersection of these two sets is however not empty; that is, there are debt structures for which there exists both a K- and an R-contract. However, the proposition establishes that for such debt structures, the R-contract is always the dominating contract. The intuition for this result is straightforward.

Imagine that there are two contracts, (R_b, R_m) and $(\tilde{R}_b, \tilde{R}_m)$, associated to a debt structure (b, m): an R-contract, $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, and a K-contract, $\frac{\tilde{R}_b}{\chi} < \frac{\tilde{R}_m}{1-\chi}$. As discussed previously, because of bank debt seniority, banks' gross expected returns from lending are the same for both contracts, so that so $R_b = \tilde{R}_b$. Therefore, $\frac{\tilde{R}_m}{1-\chi} > \frac{\tilde{R}_b}{\chi} = \frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, so that the two contracts can be ordered, and the R-contract dominates.

Second, each of the two subsets $\tilde{S}_K(e)$ and $\tilde{S}_R(e)$ can be described as the intersection of the sets defined by three inequality constraints. The inequalities $(\mathrm{c}-\mathrm{bank})$, $(\mathrm{c}-\mathrm{market})$ and $(\mathrm{c}-\mathrm{joint})$ correspond, respectively, to borrowing constraints with respect to market, bank lending and total lending. The borrowing constraint with respect to bank lending, for example, can be rewritten as:

$$0 \leq (1+r_b)b \leq \chi \left(\mathbb{E}(\phi)(e+b+m)^{\zeta} + (1-\delta)(e+b+m)\right) = \chi \int_0^{+\infty} \pi(\phi) dF(\phi).$$

The first inequality states that firms are note allowed to lend. The second inequality states that gross lending costs $(1+r_b)b$ cannot exceed the maximum expected return which the bank can achieve by lending to the firm. This maximum is attained when the threshold for ϕ below which the firm renegotiates, which is implicitly defined by $\frac{R_b}{\chi} = \pi(\phi)$ as per proposition 1, goes to infinity. This implies that when the bank borrowing constraint is binding (that is, when the debt structure is such that $(\mathrm{c}-\mathrm{bank})$ holds with equality), the entrepreneur always renegotiates her loan with the bank, so that the banks' effective repayment is $\chi\pi(\phi)$, for all ϕ. The banks' expected repayment is then a fraction χ of total expected output. Thus, when the firm is at her bank borrowing constraint, the bank effectively holds a claim to a constant fraction of the project's value – an equity-like stake.

This is in contrast to the market borrowing constraint $(\mathrm{c}-\mathrm{market})$. This borrowing constraint also states that gross lending costs of markets cannot exceed the maximum expected repayment which market lenders can achieve. Contrary to the banks' case, this expected repayment is not monotonic in the threshold for ϕ below which the entrepreneur goes bankrupt, which is implicitly defined by $\frac{R_m}{1-\chi} = \pi(\phi)$. This is a direct consequence of the fact that there are losses associated with going bankrupt, that is, $\chi < 1$. These losses (along with the technical part of assumption 1) imply that expected returns of market lenders are increasing in R_m when R_m is small (so that bankruptcy probability and bankruptcy losses are small), but decreasing in R_m when R_m is large (so that bankruptcy probabilities and bankruptcy losses are large).[12] The value $\tilde{I}(e+b+m) < \mathbb{E}(\phi)$ correspond to the bankrucpy threshold which maximizes expected returns for market lenders, which trades off bankruptcy losses with higher promised repayments. If, instead, there were no bankruptcy losses, $\chi = 1$, expected returns would never decrease in promised repayments R_m, and the market borrowing constraint would take the same form as the banks'; that is, we would have

[12] Appendix A contains a detailed proof of this statement.

$\hat{I}(e+b+m) = \mathbb{E}(\phi)$. Likewise, in the case of a K-contract, renegotiation is never used for any realization ϕ; bankruptcy losses are therefore unavoidable. For the same reasons as in the case of the R-contract, this implies that the total surplus from lending has an interior maximum in the bankruptcy threshold, which corresponds to the value $\hat{M}(e+b+m) < \mathbb{E}(\phi)$. Thus, the ability to renegotiate debt is a crucial determinant of borrowing constraints in this model.

Proposition 2 does not provide a full characterization of the lending menu. However, some general results can be established about the type of debt structures which are to be found in each of the subsets $\mathcal{S}_K(e)$ and $\mathcal{S}_R(e)$. This is the object of the following corollary to proposition 2.

Corollary 1 (Debt structure and the lending menu) *For a given level of internal finance e and a debt structure $(b,m) \in \mathcal{S}(e)$, let $s = \frac{b}{b+m}$ denote bank debt as a fraction of total debt. Define:*

$$\underline{s}_R = \frac{1}{1 + \frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}},$$

and let $\overline{s}_K(e)$, $1 \geq \overline{s}_K(e) \geq \underline{s}_R$, be defined as in appendix A. Then:

- *if $1 \geq s \geq \overline{s}_K(e)$, then $(b,m) \in \mathcal{S}_R(e)$;*
- *if $\underline{s}_R > s \geq 0$, then $(b,m) \in \mathcal{S}_K(e)$.*

This corollary indicates that the set of debt structures $\mathcal{S}_R(e)$ contains no debt structures (b,m) with an excessively low fraction of bank debt $s = \frac{b}{b+m}$: there can indeed be no debt structures such that $s < \underline{s}_R$ in that set. In particular, when $s = 0$ (a pure market contract), the associated contract must be a K-contract. This is intuitive: in that case, there can be no renegotiation from the part of the entrepreneur, since no bank has taken part in lending. The interesting result, however, is that the debt structure of firms must contains a *sufficient fraction* of bank debt for renegotiation to be a possibility in the equilibrium of the debt settlement game. Otherwise, the gains associated with the renegotiation of bank liabilities are never sufficient to repay in full the market lenders in case of bankruptcy, so that bankruptcy can never be avoided. Accordingly, the threshold s_R below which the debt structure ceases to allow for renegotiation is increasing in χ: the larger the bankruptcy losses (the smaller χ), the larger the renegotiation gains, and the more market debt the firm can take on as a fraction of total debt. On the contrary, in the limit where $\chi = 1$, there are no renegotiation gains, and $s_R = 1$, so that there are no debt structures associated with R-contracts.

Likewise, the corollary also indicates that the set $S_K(e)$ contains no debt structures particularly tilted towards bank debt; a pure bank contract, $s = 1$, must be associated to an R-contract, since in the absence of market debt, it is always beneficial for the entrepreneur to renegotiate down debt payments (provided her productivity ϕ is sufficiently low). Thus, the model indicates that debt structures tilted towards bank debt tend to be associated with contracts leading to debt renegotiations, whereas debt structures tilted towards market debt tend to be associated with contracts where debt renegotiations are not an equilibrium outcome.

2.4.4 An analytical characterization of the lending menu when $\delta = 1$

I next turn to a particular case in which the lending menu can be characterized analytically.

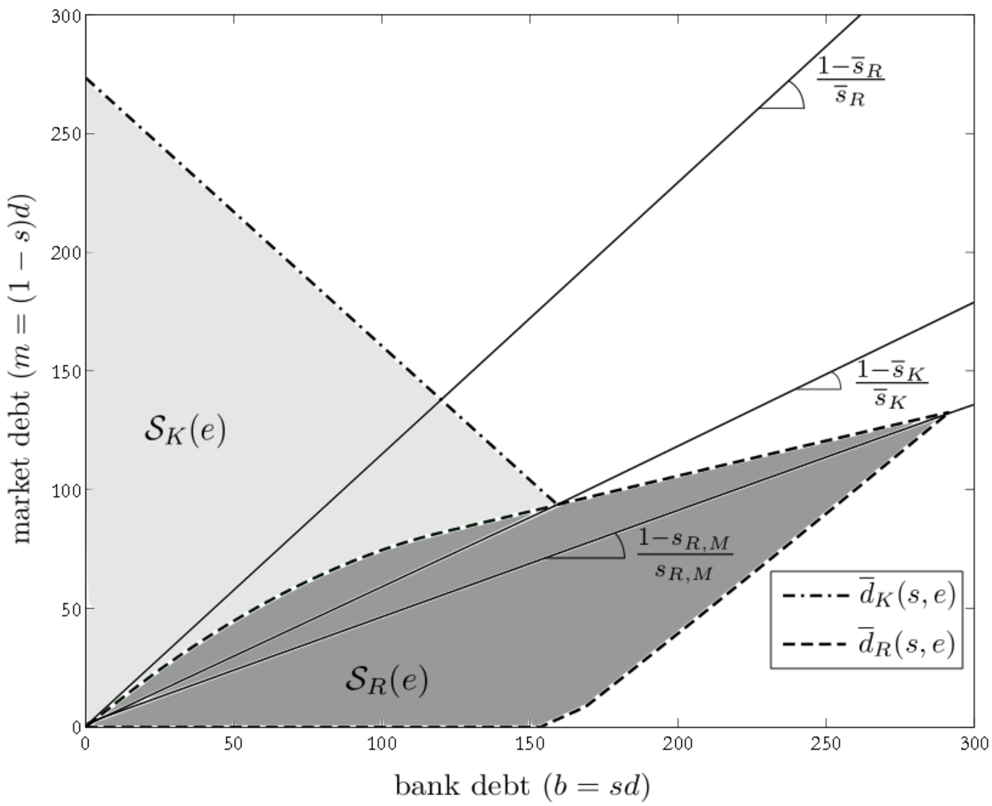


Figure 4: The lending menu $\mathcal{S}(e)$ when $\delta = 1$.

Proposition 3 *Let $(d, s) \in \mathbb{R}^+ \times [0, 1]$ denote the debt structure, with $d = b + m$ and $s = \frac{b}{b+m}$. The sets $\mathcal{S}_R(e)$ and $\mathcal{S}_K(e)$ can be parametrized as:*

$$\mathcal{S}_R(e) = \left\{ (d, s) \in \mathbb{R}_+ \times [\underline{s}_R, 1] \;\middle|\; 0 \le d \le \overline{d}_R(s, e) \right\},$$

$$\mathcal{S}_K(e) = \left\{ (d, s) \in \mathbb{R}_+ \times [0, \overline{s}_K] \;\middle|\; \underline{d}_K(s, e) \le d \le \overline{d}_K(s, e) \right\}.$$

Here, $\underline{s}_R$ is defined as in corollary 1, and:

$$\overline{s}_B = \frac{1}{1 + \left(\frac{1}{\chi}\upsilon - 1\right)\frac{1+r_b}{1+r_m}}$$

Expression for the borrowing limits $\overline{d}_R(s,e)$, $\overline{d}_K(s,e)$ and $\underline{d}_K(s,e)$ and the constant $\upsilon > \chi$ are given in appendix A.

Moreover, $\frac{\partial \overline{d}_K}{\partial s}(s,e) < 0$, while $\frac{\partial \overline{d}_R}{\partial s}(s,e) \geq 0$, if and only if, $\underline{s}_R \leq s \leq s_{R,M} < 1$, where:

$$s_{R,M} = \frac{1}{1 + \frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}\frac{\hat{I}}{\mathbb{E}(\phi)}}.$$

Figure 3 depicts the lending menu $\mathcal{S}(e)$ when $\delta = 1$, using the results of proposition 6.[13] The lending menu is plotted in (b,m) rather than (d,s) space, but there is a simple correspondence between the two: each value of s corresponds to a straight line passing through 0 and with slope $\frac{1-s}{s}$, and along each of these lines, total debt d increases.

Note first that the bank share of external liabilities $s = \frac{b}{b+m}$ spans the lending menu, as it varies from $s = 0$ (which corresponds to the the vertical axis) to $s = 1$ (which corresponds to the horizontal axis). As emphasized in the general case, the lending menu contains only elements in $\mathcal{S}_K(e)$ if $s \leq \overline{s}_R$, that is, to the left of the solid black line with slope $\frac{1-\overline{s}_R}{\overline{s}_R}$); and only elements in $\mathcal{S}_R(e)$ is $s > \overline{s}_K(e)$, that is, to the right of the solid black line with slope $\frac{1-\overline{s}_K}{\overline{s}_K}$).

The set $\mathcal{S}_K(e)$ corresponds to the light gray area of the graph. This set comprises debt structures such that the bank share of external liabilities $s = \frac{b}{b+m}$ ranges from 0 to $\overline{s}_K$ (corresponding to the solid black line with slope $\frac{1-\overline{s}_K}{\overline{s}_K}$). For this range of bank shares, the upper bound on borrowing using a K-contract, $\overline{d}_R(s,e)$, is the dotted line at the boundary of $\mathcal{S}_K(e)$. This frontier corresponds to the pairs (b,m) such that condition $(\mathrm{c-joint})$ is binding, and it is downward sloping. Intuitively, this indicates that bank and market borrowing, for this type of contract, are "substitutes", in the sense that a marginal increase in bank borrowing tightens the market borrowing constraint (and conversely). This is because for K-contracts, given the seniority of bank debt, a marginally larger amount of bank liabilities makes it less likely that market liabilities will be repayed. Note that for $s < \underline{s}_R$, the lower bound on total debt for debt structures belonging to $\mathcal{S}_K(e)$ is 0, while for $\underline{s}_R \leq s \leq \overline{s}_K$, it coincides with the upper bound on total debt of the other set of debt structures, $\mathcal{S}_R(e)$. Appendix A shows that this boundary corresponds to the conditions $(\mathrm{frontier-R})$ and $(\mathrm{frontier-K})$, which exactly coincide in the range $\underline{s}_R \leq s \leq \overline{s}_K$. For this range of debt structures, a particular bank share s can therefore correspond to either a K-contract (if total borrowing is small enough, ie below $\overline{d}_R(s,e)$), or to an R-contract (if total borrowing is large enough, ie above $\overline{d}_R(s,e)$).

The set $\mathcal{S}_R(e)$ corresponds to the dark gray area of the graph. In this region, the upper bound on total borrowing is associated to the dashed blacked line $\overline{d}_R(s,e)$. For debt structures such that $\overline{s}_K \leq s \leq s_{R,M}$,

[13] Other model parameters, and in particular the shock distribution, are identical to those of the baseline calibration of the model, discussed in section 4. In this example, $e = 10$; model parameters are choosen in such a way that a firm with internal finance $e = 100$ is indifferent between borrowing from market lenders and having no debt.

this dotted line corresponds to cases in which the constraint (c − market) is binding, while for debt structures such that $s \geq s_{R,M}$, this line corresponds to the bank borrowing constraint (c − bank). The slope of the frontiers thus indicate that and increase in the bank share loosens the market borrowing constraint so long as the bank borrowing constraint is not binding, and thus leads to higher total borrowing limits. In that region, the lending menu thus exhibits "complementarity", contrary to the part of the lending menu associated with K-contracts.

In this section, I have proposed a model of market and bank debt pricing based on the view that bank debt is easier to renegotiate than market debt, and I have derived the set of feasible debt contracts for an entrepreneur with internal finance e implied by the model. This set was derived under the relatively weak assumptions that credit markets were perfectly competitive and that entrepreneurs' utility was increasing in monetary payoffs. The key insight from the analysis of feasible debt structures is that, regardless of entrepreneur's internal ressources e, renegotiation of debt in times of financial distress is only desirable for firms that choose to hold a sufficient amount of bank debt relative to market debt. I next turn to drawing the implications of these findings for the optimal choice of debt structure.

3 The optimal choice of debt structure

This section adresses the question of which debt structure, among those that are feasible, an entrepreneur with internal finance e will choose to finance investment. Furthermore, I explore how both the optimal share of borrowing financed by banks $s = \frac{b}{b+m}$ as well as the optimal total amount of borrowing $b + m$ vary with own equity e.

Throughout the section, I maintain the two following assumptions:

Assumption 4 *The enterpreneur is risk-neutral, and her assets completely depreciate after productivity is realized and output is produced:* $\delta = 1$.

I first assume that assets fully depreciate at the end of period 1, that is, $\delta = 1$. Given the static nature of the model, this is a natural assumption, and it furthermore simplifies the analytical characterization of the optimal debt structure. It is however is not crucial to any of the results below. The second assumption I maintain in this section is that the entrepreneur is risk-neutral. While risk-neutrality can be viewed as a benchmark case, it influences strongly the results, by linking closely profit maximization and maximization of total surplus from investment, as I analyze below.

3.1 The entrepreneur's profit function

Under assumption 4, the optimal debt structure of an enterpreneur with own equity e solves:

$$\hat{\pi}(e) = \max_{b,m \in \mathcal{S}(e)} \mathbb{E}\left[\tilde{\pi}(\phi; e, b, m)\right],$$

where $\tilde{\pi}(\phi; e, b, m)$ denotes the profits accruing to the entrepreneur, conditional on the debt structure (b, m) and therefore the associated contract (R_b, R_m), and the realization of the shock ϕ.

Proposition 4 *For $(b, m) \in \mathcal{S}(e)$, the objective function of the entrepreneur is given by:*

$$\mathbb{E}\left[\tilde{\pi}(\phi; e, b, m)\right] = \underbrace{\mathbb{E}(\pi(\phi)) - (1 + r_b)b - (1 + r_m)m}_{\text{total expected surplus from investment}} - \underbrace{(1 - \chi)\int_0^{\underline{\phi}(e,b,m)} \pi(\phi)dF(\phi)}_{\text{expected liquidation losses}}, \tag{8}$$

where:

$$\underline{\phi}(e, b, m) = \begin{cases} \frac{R_K(b,m,e)}{(e+b+m)^{\zeta}} & \text{if } (b, m) \in \mathcal{S}_K(e) \\ \frac{R_{m,l}(b,m,e)}{(1-\chi)(e+b+m)^{\zeta}} & \text{if } (b, m) \in \mathcal{S}_R(e) \end{cases}$$

This result is a consequence of risk-neutrality of the lenders and the entrepreneur: under risk-neutrality, profit maximization for the entrepreneur is equivalent to the maximization of total expected surplus, net of the losses incurred in case liquidation is carried out. In particular, in the absence of bankruptcy costs (that is, when $\chi = 1$), profit maximization for the firm is equivalent to maximization of total surplus. In this case, it is clear that the optimal debt structure is always a corner solution, with the entrepreneur borrowing only from the lender with the smallest cost of funds, as described in the lemma below.

Lemma 2 (The optimal debt structure in the absence of liquidation losses) *Assume there are no liquidation losses, that is: $\chi = 1$. Then, the solution to the entrepreneur's problem is given by:*

$$\forall e, \quad \hat{k}_i \equiv \left(\frac{\mathbb{E}(\phi)}{1 + r_i}\right)^{\frac{1}{1-\zeta}}, \quad \hat{\pi}(e) = (1 + r_{min})\left(\frac{1 - \zeta}{\zeta}k^* + e\right),$$

where $r_{min} = \min(r_b, r_m)$. Moreover:

- *If $r_b < r_m$, the optimal debt structure is entirely bank-financed: $\forall e$, $\hat{b}(e) = \hat{k}_b - e$, $\hat{m}(e) = 0$.*
- *If $r_b > r_m$, the optimal debt structure is entirely market-financed: $\forall e$, $\hat{b}(e) = 0$, $\hat{m}(e) = \hat{k}_m - e$.*
- *If $r_b = r_m = r$, the entrepreneur is indifferent between all debt structures such that $\forall e$, $\hat{b}(e) + \hat{m}(e) + e = \hat{k}_m = \hat{k}_b$.*

3.2 Lenders' cost of funds

Lemma 2 emphasizes that the relative cost of funds of lenders is a crucial determinant of the optimal debt structure; when $\chi = 1$, the entrepreneur only borrows from the lender with the smallest cost of funds.

When $\chi < 1$, the entrepreneur's expected profits depend not only on the relative cost of funds, but also on the magnitude of expected liquidation losses, the second term in the right hand side of equation (8). Debt structure plays a role in determining the magnitude of these losses, because it affects how often the project must be liquidated at the debt settlement stage. Generically, for a comparable total amount of lending $b + m$, debt structures $(b, m) \in \mathcal{S}_R(e)$ will be liquidated less often. Indeed, the very point of an R-contract is that, by restructuring bank debt, it offers the entrepreneur a means of avoiding liquidation even when her ressources fall below "natural" liquidation threshold, that is, when productivity is such that $\pi(\phi) < R_m + R_b$ (see figure 1). Thus, by choosing an R-contract, the entrepreneur will reduce expected liquidation losses. At the same time, corollary 1 indicates that debt structures $(b, m) \in \mathcal{S}_R(e)$ typically feature more bank debt; that is, $s = \frac{b}{b+m}$ is closer to 1. By borrowing more from bank lenders, the entrepreneur thus reduces her expected liquidation losses.

Thus, intuitively, the optimal debt structure should limit expected liquidation losses, while at the same time it should feature as much debt as possible from the lender that has the smallest cost of funds, as suggested by lemma 2. It is clear that these objectives need not conflict with one another. If $r_b \leq r_m$, borrowing more from bank lenders allows the entrepreneur both to minimize her liquidation losses, and to gain from the smaller cost of funds of banks. Given a particular scale of total borrowing $b+m$, when $r_b \leq r_m$, the entrepreneur should always try to use as much bank finance as possible. This intuition is formalized in the following proposition.

Proposition 5 (The optimal debt structure when $r_b \leq r_m$) *Assume that banks have a lower marginal cost of funds than markets, that is, $r_b \leq r_m$. Then, the optimal debt structure of an entrepreneur with internal finance e either features no borrowing from markets, $m^*(e) = 0$, or, is such that the bank borrowing constraint,* $(\mathrm{c-bank})$*, is binding.*

Thus, when $r_b \leq r_m$, the *only* reason for which an entrepreneur would want to issue liabilities to market lenders is that she has already exhausted her borrowing capacity from bank lenders. In that case, only firms with little internal finance e will issue market liabilities, since for large levels of e, the bank borrowing constraint is less likely to bind. In turn, this would imply that larger firms are less likely to issue market debt; in particular, the very largest firms would issue only bank debt. When $r_b \leq r_m$, the model thus leads to a counterfactually negative relationship between internal finance and size, on the one hand, and bank debt's fraction of total external liabilities, on the other. This motivates the following assumption.

Assumption 5 (Lending costs) *Bank lenders have a larger marginal cost of funds:*

$$r_b > r_m.$$

A possible interpretation for this assumption is the following. Lenders both have the same marginal cost of funds, $1 + r$, but banks incur an additional costs $c(b)$ per dollar lent. This cost could arise for two reasons. First, there might be due diligence costs associated to obtaining information that allows the bank to restructure the entrepreneur's liability when profits are low, as in Rajan (1992). Although this information acquisition, and the problems of information revelation associated with it, are not modelled here, the cost function $c(b)$ could be seen as a reduced-form way of modelling them. Second, the bank may face different balance sheet restrictions than market lenders; in particular, banks may face tighter capital requirement. The function $c(b)$ may then stand for the costs of issuing additional bank equity in order to meet those capital requirements when making a additional loan in the amount b. With these additional lending costs $c(b)$, the bank's cost of funds becomes $(1 + r)b + c(b)b$. In particular, if $c(b) = c$, this formulation of the model is equivalent to the one developed above, with $r_m = r$ and $r_b = r + c$. Overall, assumption 5 is not only necessary for the model to deliver relevant comparative statics; it is also consistent with the view that bank loans is more costly process, per unit of debt issue, than the issuance of market debt.

3.3 Which type of debt structure does the entrepreneur choose?

With the assumption that $r_b > r_m$, there is tension between the two objectives that an optimally chosen debt structure should pursue, that is, limiting liquidation losses while at the same time borrowing as much as possible from lenders with the smallest possible cost of funds. The optimal debt structure chosen by an entrepreneur with internal finance e reflects this tension.

The trade-off between flexibility and cost does not necessarily lead to a debt structure in which the entrepreneur borrows both from market lenders and from bank lenders. The following proposition emphasizes that this depends on the extent of her internal funds. When the entrepreneur has sufficiently large e, the gains associated to bank debt flexibility are always outweighed by the gains from borrowing from the cheaper source of funds, the market.

Proposition 6 (Market finance vs. mixed finance) *Assume that banks have a larger marginal cost of funds than markets, that is, $r_b > r_m$. Let $(\hat{b}(e), \hat{m}(e))$ denote the optimal debt structure of an entrepreneur with equity e. There exists $e^* > 0$ such that:*

- *<u>if $e > e^*$,</u> $(\hat{b}(e), \hat{m}(e)) \in \mathcal{S}_K(e)$; moreover, the optimal debt structure features "pure market finance":*

$$\hat{m}(e) > 0 \quad , \quad \hat{b}(e) = 0;$$

- *<u>if $e < e^*$,</u> $(\hat{b}(e), \hat{m}(e)) \in \mathcal{S}_R(e)$; moreover, the optimal debt structure features "mixed finance":*

$$\hat{m}(e) \geq 0 \quad , \quad \hat{b}(e) > 0.$$

This proposition states that there exists a switching point e^* in equity e, which dictates the type of debt structure that the entrepreneur will choose. For those entrepreneurs with $e < e^*$, it is optimal to choose a debt structure in $\mathcal{S}_R(e)$, which allows for restructuring. On the other hand, for entrepreneurs with $e \geq e^*$, it is optimal to choose a debt structure in $\mathcal{S}_K(e)$; and in fact, it is optimal to choose a debt structure with only market debt.

There are two intuitions behind this proposition. First, among debt structures $(b, m) \in \mathcal{S}_K(e)$, the entrepreneur always prefers those with no bank debt at all. This is because, under a K-contract, restructuring never occurs; so the additional costs of borrowing from banks come with no added benefits of flexibility in restructuring. The two types of debt are equivalent in terms of their effect on liquidation losses, so that part of the trade-off is irrelevant, and the entrepreneur simply issues all her liabilities with whichever lender has the smallest marginal cost of funds. Under assumption 5, market lenders have the cheapest cost of funds, so that the optimal debt structure always has $b = 0$, that is, lies on the y-axis in the lending menu depicted in figure 3. If instead $r_b \leq r_m$, then the optimal debt structure within $\mathcal{S}_K(e)$ would have been located at the frontier between the two sets $\mathcal{S}_K(e)$ and $\mathcal{S}_R(e)$, and would be locally dominated by some debt structure $(b, m) \in \mathcal{S}_R(e)$ allowing for restructuring.

The second aspect of the proposition is the fact that entrepreneurs with small e prefer debt structures which allow for restructuring, that is, prefer debt structure $(b, m) \in \mathcal{S}_R(e)$. Here, the assumption of decreasing returns to scale plays an important role. To understand this, it is useful to think of the case of no liquidation costs. In that case, decreasing returns imply that projects have an optimal size $e + b + m = \hat{k}_m$, using the notation of lemma 3. In that case, leverage ratios $\frac{b+m}{e} = \frac{\hat{k}_m - e}{e}$ are decreasing with e. In turn, a higher leverage ratio implies a higher probability of liquidation. This is because, for two different levels of equity e and $e' > e$, the distribution of profits is identical (and given by the distribution of $\phi(\hat{k}_m)^\zeta$), but promised repayment R_b and R_m are larger for e than for e', because more total borrowing is needed to with less internal funds, that is, $\hat{k}_m - e > \hat{k}_m - e'$. Thus, absent liquidation costs, the smaller an entrepreneur's internal funds, the higher her probability of being liquidated.

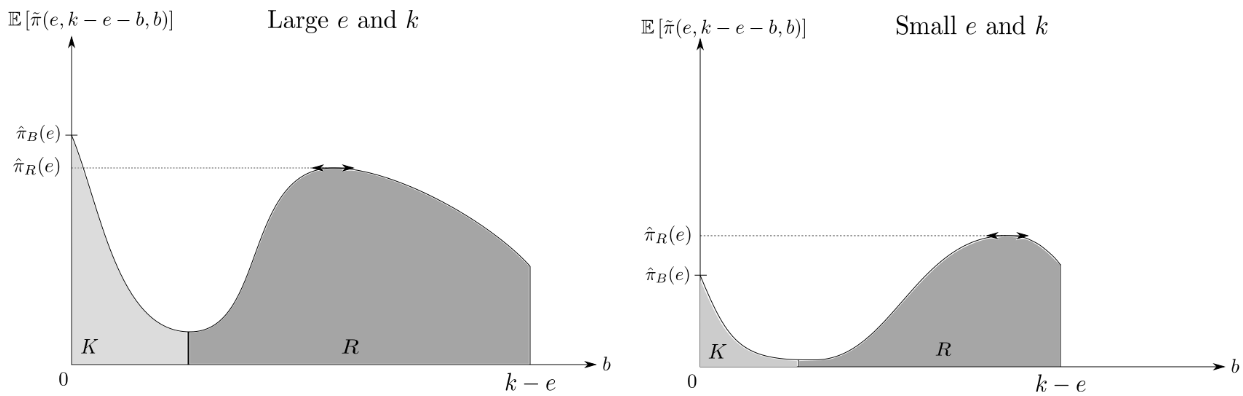


Figure 5: Cross section of the profit function. Two different levels of internal funds e and total investment k are represented: small e and k (left), large e and k (right).

With liquidation costs, the optimal investment scale $\hat{k}_m$ cannot necessarily be reached for any level of e, so that two entrepreneurs with different levels of e need not choose the same total scale $k = e + b + m$; in particular, entrepreneurs with smaller e may be limited to debt structures allowing only total a maximum total scale smaller than $\hat{k}_m$. However, to the extent that the elasticity of total borrowing to internal funds e is sufficiently small (in particular, strictly smaller than 1), it will still be the case that leverage ratios decrease with e. Thus, a logic similar to the case without liquidation costs applies, and an entrepreneur with little internal funds has a higher probability of being liquidated at the debt settlement stage.

Thus, for entrepreneurs with small e, the gains from choosing a debt structure that allows for debt restructuring, that is, a debt structure which is an element of $\mathcal{S}_R(e)$, are large, relative to the costs of using this type of debt structure (which arise because $r_b > r_m$). On the contrary, with large e, the entrepreneur needs little ouside funding; lending is therefore close to risk-free, the benefits of debt flexibility are negligible, and debt structures allowing for restructuring become unappealing to the firm. The switching point e^* thus corresponds to the level of internal funds such that the optimal debt structure within $\mathcal{S}_R(e)$ (which allows for restructuring) and the optimal debt structure within $\mathcal{S}_K(e)$ (which does not) leave the entrepreneur indifferent.

This logic is further illustrated in figure 5. The left panel represents a section of the profit function of the entrepreneur along a line corresponding to the optimal total borrowing $\hat{d}(e) = \hat{b}(e) + \hat{m}(e)$ associated with equity level e. This corresponds, graphically, to a line with slope -1 in the lending menu depicted in figure 3. The region marked with a K corresponds to debt structures in $\mathcal{S}_K(e)$. In this region, debt structures are tilted towards market debt, and the optimum $\hat{\pi}_B(e)$ within that region is attained for $b = 0$, the leftmost point on the graph; this corresponds to the first point discussed above. However, the global maximum of the entrepreneur's profit function, $\hat{\pi}_B(e)$ corresponds to the local maximum in region marked with an R, which contains the debt structures in $\mathcal{S}_R(e)$ along the line $b + m = \hat{d}(e)$. In this case, e is sufficiently small that the benefits of flexiblity outweigh the costs of borrowing from banks rather than markets; the resulting debt structure is mixed, between bank and market finance. The right panel of the figure looks at a similar cross-section, for a value of internal funds $e' > e$. In this case, the leftmost point in the region of K-contracts is the global optimum; the firm then chooses a pure market debt structure. The switching point e^* corresponds to the case in which the two local maxima of the regions associated with K and R contracts are equal: $\hat{\pi}_K(e) = \hat{\pi}_R(e)$.

Summarizing, the first important prediction of the model is that firms with large amounts of internal funds e will choose to finance themselves through market debt, while firms with small amounts of internal finance will rely on a mix of market and bank debt dominated by bank debt.

3.4 The optimal debt structure under R-contracts

Next, I turn to characterizing more precisely the optimal debt structure when $e < e^*$. In that case, following proposition 6, the optimal debt structure $(\hat{b}(e), \hat{m}(e))$ is an element of $\mathcal{S}_R(e)$, and is therefore associated to an R-contract. The key results are summarized in the following proposition.

Proposition 7 (The optimal debt structure when $e \leq e^*$) *Assume $r_b > r_m$. Consider an entrepreneur with internal funds $e < e^*$ and let $\hat{s}(e) = \frac{\hat{b}(e)}{\hat{b}(e)+\hat{m}(e)}$ denote the fraction of total liabilities that are bank debt in her optimal debt structure, and let $\hat{d}(e) = \hat{b}(e) + \hat{m}(e)$ denote total borrowing. Then, there exists $\tilde{e} < e^*$ such that:*

- *For $0 \leq e < \tilde{e}$, the bank borrowing constraint is binding at the optimal debt structure, $\frac{\partial \hat{s}}{\partial e} > 0$ and $\frac{\partial \hat{d}}{\partial e}$;*
- *For $\tilde{e} \leq e \leq e^*$, the optimal debt structure of the firm satisfies:*

$$\hat{s}(e) = 1 - \frac{\Gamma}{1+r_m} \frac{(\hat{k}_{int})^{\zeta}}{\hat{k}_{int} - e}, \quad \hat{d}(e) = \hat{k}_{int} - e \tag{9}$$

where the expression of Γ and $\hat{k}_{int}$ are given in appendix B. In particular, $\frac{\partial \hat{s}}{\partial e} \leq 0$ and $\frac{\partial \hat{d}}{\partial e} \leq 0$.

3.4.1 The bank share

Proposition 7 states that the optimal debt structure is such that the optimal bank share $\hat{s}(e)$ is non-monotonic in the entrepreneur's own equity, e. This is illustrated in figure 6, which plots $\hat{s}(e)$ as a function of e.[14]

[14] Parameter values corresponding to the solution depicted in figure 6 are identical to those of the baseline calibration explored in the next section; see table 1 for details.

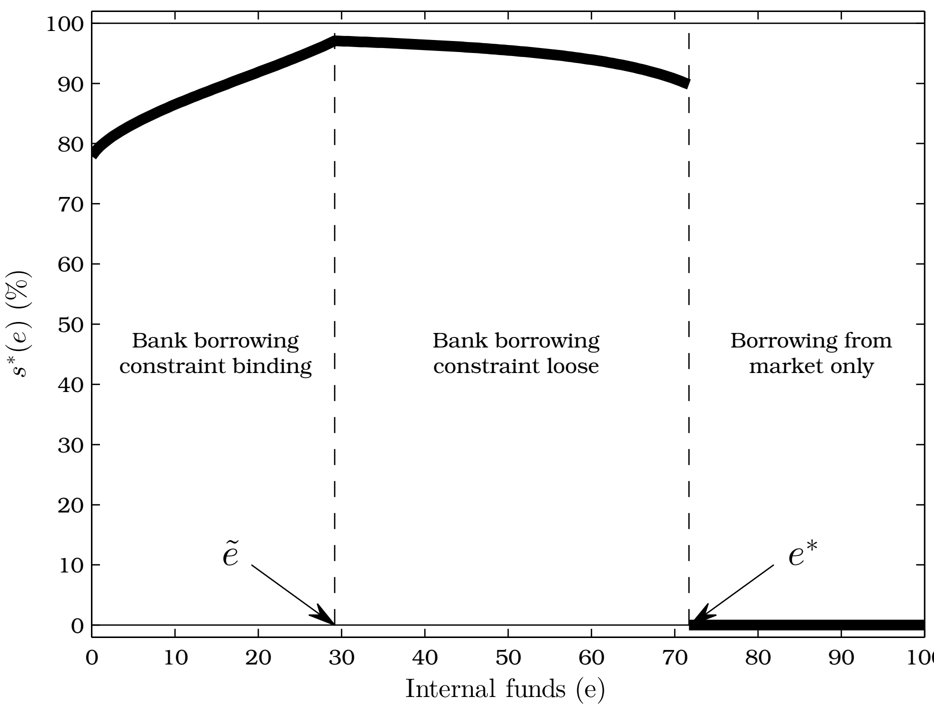


Figure 6: Optimal bank share of total debt.

To understand why the banking share of liabilities has a non-monotonic relationship with internal funds, it is useful to note that the derivative of the objective function of the entrepreneur with respect to the share of bank debt, $s = \frac{b}{b+m}$, using equation (8), is given by:

$$\frac{\partial \mathbb{E}(\tilde{\pi})}{\partial s} = (1-\chi)\pi\left(\underline{\phi}\right) f(\underline{\phi}) \left(-\frac{\partial \underline{\phi}}{\partial s}\right) - (r_b - r_m)$$

The first term in this expression, $(1-\chi)\pi\left(\underline{\phi}\right) f(\underline{\phi}) \left(-\frac{\partial \underline{\phi}}{\partial s}\right)$, represents the marginal decrease in expected liquidation losses associated with an increase in the share of bank borrowing, keeping total borrowing fixed. Note that $\frac{\partial \underline{\phi}}{\partial s} < 0$, that is, an increase in bank borrowing keeping total borrowing fixed leads to a reduction in expected liquidation losses, as previously discussed. The second term in this expression, $(r_b - r_m)$, represents the increase in lending costs associated with a marginal increase in bank borrowing, again keeping total borrowing fixed. By borrowing one dollar more from banks and one less from markets, the firm incurs an additional cost of $(r_b - r_m) > 0$.

When internal funds are sufficiently small ($e < \tilde{e}$), the bank share is increasing in e because the bank borrowing constraint binds at the optimal debt structure; an increase in e, by loosening the bank borrowing constraint, will necessarily lead to more bank borrowing. The bank borrowing constraint itself is binding because, when e is small, marginal reductions in expected liquidation losses associated to more bank borrowing are large. In particular, when $e < \tilde{e}$, any debt structure $(b, m) \in \mathcal{S}_R(e)$ is such that:

$$(1-\chi)\pi\left(\underline{\phi}\right) f(\underline{\phi}) \left(-\frac{\partial \underline{\phi}}{\partial s}\right) > r_b - r_m.$$

Under this condition, at any debt structure, the marginal value of more bank debt is strictly larger than the associated costs, $r_b - r_m$.The internal funds level $\tilde{e}$ in fact solves $(r_b - r_m) = -(1-\chi)\pi\left(\underline{\phi}\right) f(\underline{\phi})\frac{\partial \underline{\phi}}{\partial s}$, with the right hand side is evaluated at $s = s_{R,M}$ and $d = \overline{d}_R(s_{R,M}, \tilde{e})$; see appendix B for details. However, note that this need not imply that $s = 1$; even though the firm exhausts it bank borrowing capacity, it may still find it profitable to borrow from the market.

Note that, as discussed previously, when the bank borrowing constraint is binding, bank debt is always renegotiated by the firm, regardless of the realization of the productivity ϕ. In this case, the bank always receives a fraction χ of the output produced by the firm. Thus, the optimal debt structure, when e is sufficiently small, is such that the bank contract essentially has the feature of an equity contract. Namely, the bank effectively agrees to receive a constant fraction of the firms' output. This contract is beneficial to the entrepreneur because it allows her to avoid very frequent liquidation, given her high leverage.

For a sufficiently large level of internal funds, $e \geq \tilde{e}$, the bank borrowing constraint becomes loose at the optimal debt structure. This is the second case described in the proposition. In that case, the optimal debt structure of the firm satisfies:

$$(1-\chi)\pi\left(\underline{\phi}\right) f(\underline{\phi}) \left(-\frac{\partial \underline{\phi}}{\partial s}\right) = r_b - r_m$$

The marginal benefits of bank and market lending are exactly equalized, and the optimal debt structure is an interior point of the set $\mathcal{S}_K(e)$. Expression (9) is the analytical solution to this first order condition; in particular, it indicates that bank's share of total debt is decreasing, as a function of internal funds. This result comes from the fact that an increase in internal funds reduces the marginal impact of bank lending on total expected liquidation losses; formally, $\frac{\partial^2 \underline{\phi}}{\partial e \partial s} < 0$. As internal funds increase, gains from bank borrowing relative to market borrowing thin out, as the optimal leverage ratio falls and liquidation becomes less likely. Note, additionally, that for a given level of internal funds e, the bank's share of total borrowing is increasing in r_m.

3.4.2 Total borrowing

The second important result from proposition 7 is that total borrowing, $\hat{d}(e)$, is non-monotonic in own equity e. This is reported in the two top panels of figure 7, which plot optimal bank borrowing $\hat{b}(e)$ and optimal market borrowing $\hat{m}(e)$. The two middle panels of figure 7 plot total borrowing $\hat{d}(e) = \hat{b}(e) + \hat{m}(e)$

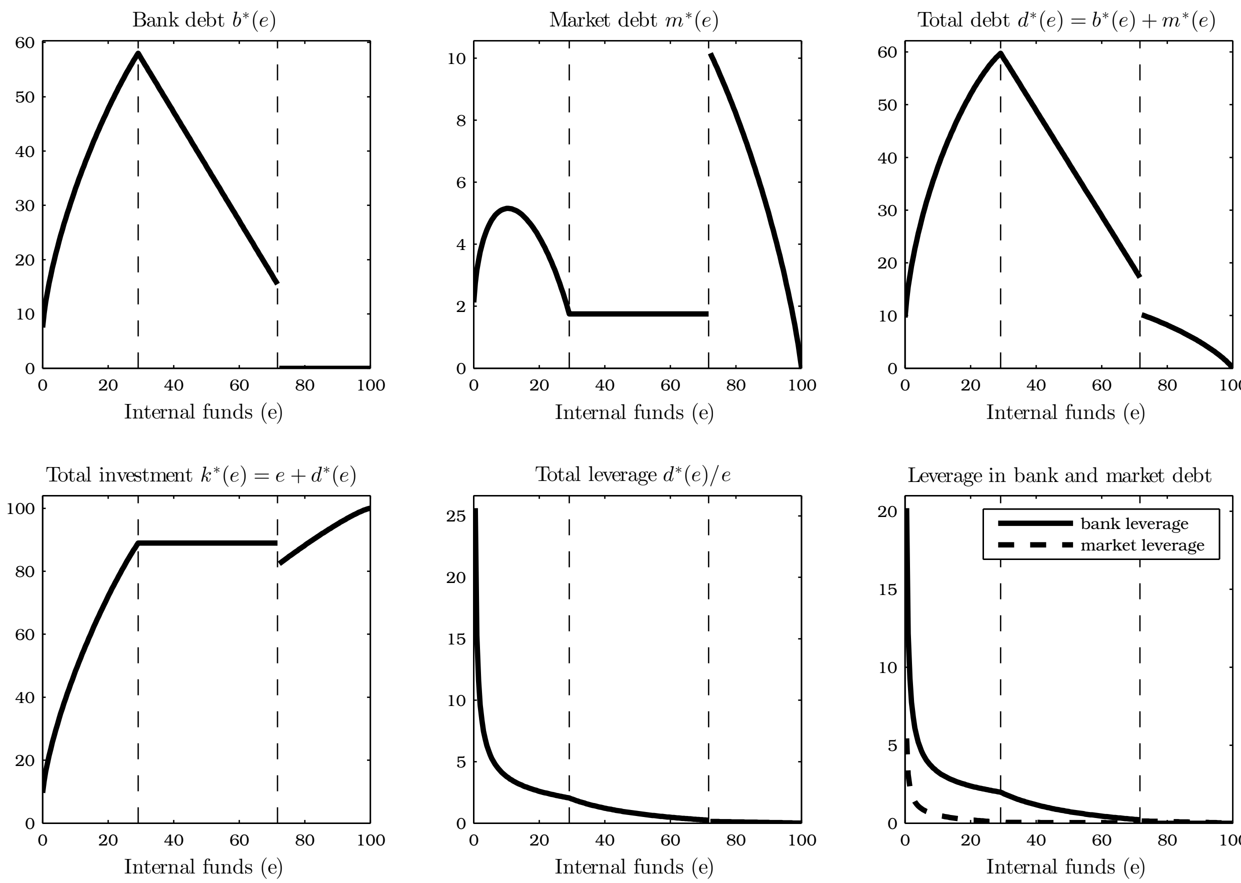


Figure 7: Optimal borrowing and investment policies.

and total investment $\hat{k}(e) = e + \hat{b}(e) + \hat{m}(e)$ as a function of e, while the two bottom ones report leverage ratios.

For small values of e, total borrowing is increasing, so that total assets are also increasing. Like in the case of the bank share, total borrowing is increasing for small values of internal funds because in that range, the bank borrowing constraint binds. Total borrowing increases mostly as a result of the increase in bank borrowing associated to the loosening of the borrowing constraint. In that range of e, most borrowing originates from the bank. Moreover, since total borrowing increases with e, total assets also do. In that region, the leverage ratio of the firm is increasing with e.

On the other hand, for $\tilde{e} \leq e \leq e^*$, borrowing falls one for one with equity, so total assets are constant and equal to $\hat{k}_{int}$. On this range, for every dollar increase in own equity, the firm chooses to retire one dollar of bank debt; the amount of borrowing from market lenders, on the other hand, is unchanged. The leverage ratio of the firm is now decreasing with internal funds. Using the results of proposition 7, the amount of market borrowing is constant, and given by:

$$\forall e, \quad \hat{m}(e) = \frac{\Gamma(\chi)}{1+r_m}(\hat{k}_{int})^{\zeta}$$

This result, as others reported in this section, does not depend on whether there is full depreciation of assets or not; that is, when $\delta = 1$, total borrowing is still constant over a certain range of equity. However, there is no simple analytical characterization of borrowing and the debt structure in that case.

Summarizing, I have shown that the model outlined in section 2, in which debt structure is the result of a trade-off between bank flexibility and the relative cost of funds of lenders, predicts a broadly decreasing relationship between a entrepreneur's internal funds e and the share of bank debt employed in her optimal debt structure. The bank share is in fact increasing for the very smallest firms, after which the entrepreneur retires bank debt as her internal ressources increase, first progressively, then – as e reaches the threshold e^* – abruptly switching to market debt only.

I next turn to analyzing the dependence of debt structure and borrowing on other factors than internal funds, and in particular, the cost of funds of borrowers, and the distribution of entrepreneur's productivity shocks, $F(.)$.

4 Comparative statics of the debt structure

In this section, I ask how the optimal debt structure of firms changes in response to changes in fundamentals. I focus, in particular, on the effects of changes in the cost of funds of lenders, r_b and r_m, and of changes in the distribution of productivity shocks $F(.)$.

While I provide some general results on these comparative statics, most of the discussion focuses on the comparison between numerical solutions to the model. In all calibrations, the baseline case is as follows. First, I assume full depreciation, $\delta = 1$. Second, I choose a degree of returns to scale of $\zeta = 0.92$, in line with the gross output estimates of Basu and Fernald (1997) for the US manufacturing sector. Third, I use the estimates of Bris et al. (2006) for the gap between pre and post-bankruptcy asset values. Their mean and median estimates of post- to pre-bankruptcy asset values, are, respectively, 38.0% and 80.0%; I choose an intermediate value and set $\chi = 0.60$, implying a post- to pre-bankruptcy value of assets of 60.0%. Finally, I assume that ϕ follows a Weibull distribution.[15] In the baseline calibration, the location parameter of the distribution is normalized so that a firm with internal finance $e = 100$ is indifferent between borrowing from

[15] Unlike the log-normal distribution commonly used in the literature, this distribution has the advantage of having an increasing hazard rate, in accordance with assumption 1.

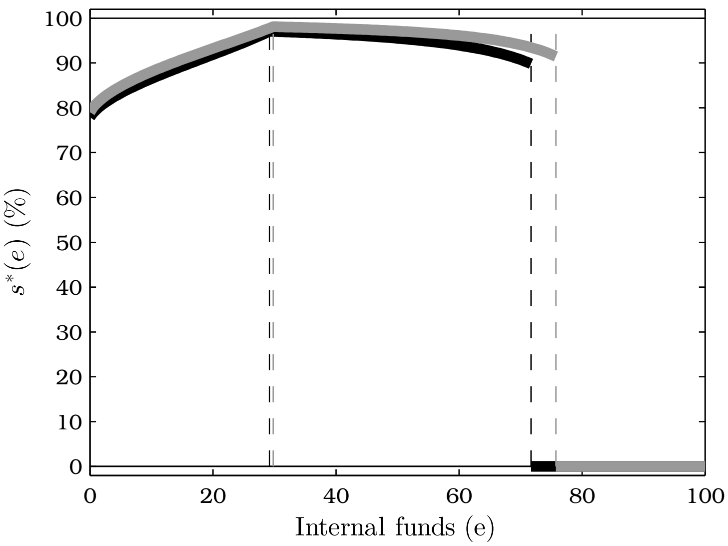


Figure 8: The effect of an increase in the spread $r_b - r_m$ on bank share's of total debt. The black line is the bank share in the baseline calibration; the grey line is the bank share after an increase in the spread $r_b - r_m$.

the market and using only its internal funds to invest in physical assets. Finally, in the baseline calibration, the cost of funds of bank and market lenders are set to $r_m = 5\%$ and $r_b = 6\%$, respectively. This calibration is summarized in 1, along with the other calibrations discussed in this section.

4.1 The effect of an increase in banks' cost of lending

I first ask what patterns of change in debt structure the model predicts, when banks' cost of funds increase relative to market lenders'. The following proposition provides a general result on the effect of an increase in the spread $r_b - r_m$.

Proposition 8 (The threshold between mixed and market finance) *The threshold e^* is a decreasing function of the lending cost of banks r_b and of the spread $r_b - r_m$.*

This proposition implies that any reduction in the spread between bank and market will induce a switch from mixed to market finance for firms with sufficiently high e. Specifically, for any (r_b, r_m) and $(\hat{r}_b, \hat{r}_m)$ such that $r_b - r_m < \hat{r}_b - \hat{r}_m$, all firms with $e^*_{\hat{r}_b - \hat{r}_m} < e < e^*_{r_b - r_m}$ will choose a mixed debt structure under the spread $r_b - r_m$, but will move to borrowing only from market lenders under the higher spread $\hat{r}_b - \hat{r}_m$.

To illustrate this, figure 8 reports the optimal bank share as a function of internal finance, in the baseline case $r_m = 5\%$ and $r_b = 6\%$, and in a case where $\hat{r}_m = r_m = 5\%$ but $\hat{r}_b = 7\%$. The red line corresponds to the optimal debt structure under the high spread, while the green line corresponds to the optimal debt structure under the low spread. In this case, the increase in spread occurs because of an increase in banks' cost of funds; the markets' cost of funds, however is unchanged. For this reason, any firm with $e > e^*_{r_b - r_m}$

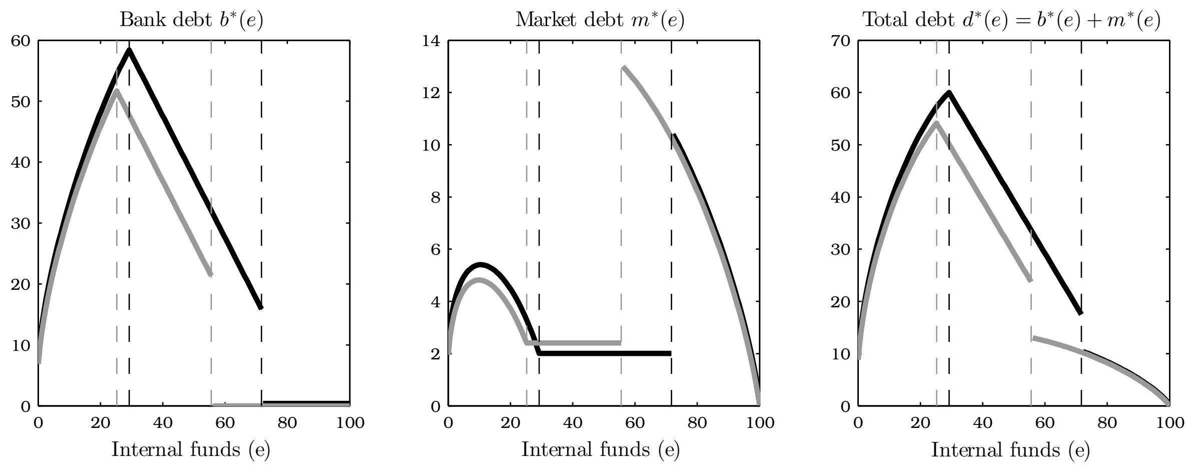


Figure 9: The effect of an increase in the spread $r_b - r_m$ on borrowing. The black line corresponds to the baseline calibration; the grey line corresponds to a higher lending spread (see table 1 for details on the calibrations used).

is unaffected by the change in borrowing costs. For firms with $e < e^*_{r_b - r_m}$, several features of the change in optimal debt structure are noticeable.

First, as indicated by proposition 8, the threshold for switching to bank finance decreases and consequently, firms with a sufficently large internal funds switch to a debt structure with only market debt. Second, for firms with internal finance below $e_{\hat{r}_b - \hat{r}_m}$, the main effect of the increase in spreads is a large reduction in bank borrowing. This is visible in figure 9, which reports the borrowing policy functions of firms under the low and high spread calibrations. Firms that are not at their bank borrowing constraint reduce significantly there is very little change in market debt. In fact, market borrowing somewhat falls. Constrained firms, however, experience a smaller reduction in bank credit. Market borrowing changes very little. It increases slightly for firms that are not at their bank borrowing constraint, while, for firms whose borrowing constraint tightens as a result of the increase in banks' borrowing costs, it falls.

It is natural to ask whether the model predicts that a joint increase in the lending costs of both types of lenders has similar consequences. Figures 13 and 14 in appendix C adress this question. There, I document the effect of an increase of r_m from 5% to 6% and of r_b from 6% to 7%. Figure 13 corresponds to the calibration reported under "High cost levels (1)" in table 1. In this calibration, to keep things comparable to the baseline case, I also reduce the location parameter of the productivity distribution in such a way that the largest internal finance level is unchanged under the new calibration.[16] In this case, the joint increase in the level of borrowing costs leaves borrowing and the bank share almost completely unchanged. By contrast, figure 14 reports the debt structure when the location parameter of productivity is kept fixed, according to the calibration reported under "High cost levels (2)" in table 1. In this case, the maximum level of internal

[16] That is, the level of internal finance at which firms are indifferent between borrowing from the market and lending risk-free.

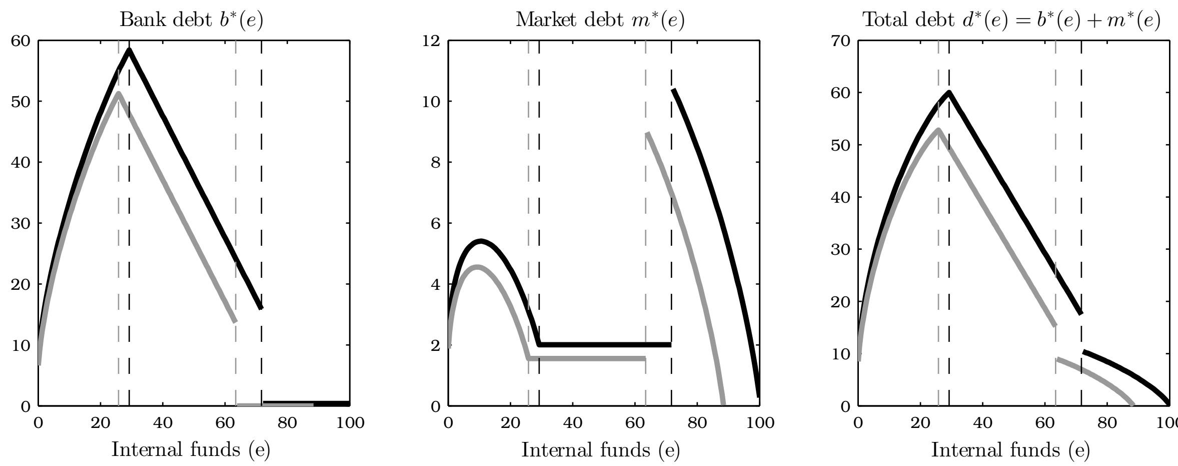


Figure 10: The effect of a fall in average productivity on borrowing. The black line corresponds to the baseline calibration; the grey line corresponds to lower average productivity (see table 1 for details on the calibrations used).

finance falls substantially, as the unconstrained size of the entrepreneur's project declines (see lemma 2). As a result, borrowing in both types of debt falls. Particularly, overall borrowing falls for the largest firms – in contrast with both fact 3 and with the model's prediction in the case of an increase in the spread $r_b - r_m$.

This exercise suggests that while joint increases in lenders' costs of fund depresses the level of overall borrowing, it does not alter its composition. This is furthermore true across the spectrum of internal finance levels. Some degree of asymmetry in the increase in lending costs, that is, some variation in the spread $r_b - r_m$, is thus needed to account for changes in the composition as well as the scale of borrowing in the cross-section of firms.

4.2 The effect of changes in the distribution of firms' productivity

I next turn to discussing the relationship between the distribution of productivity shocks, $F(.)$, and the optimal debt structure.

4.2.1 A fall in average productivity

I first look at the impact of a fall in the average productivity of firms, $\mathbb{E}(\phi)$. Specifically, I consider a change in the location and scale parameters of Weibull distribution such that average productivity, $\mathbb{E}(\phi)$, falls by 1% relative to the baseline calibration, while the second moment of productivity is unchanged.[17]

The resulting changing in the optimal debt structure is reported in figure 10.[18] Much as a joint increase in lenders' cost of funds, the fall in average productivity has similar effects on borrowing both from banks and from the markets: optimal borrowing in both types of debt is strictly smaller, at all levels of internal

[17] The corresponding calibration is reported in table 1, under the column "Low average productivity".
[18] Figure 15 in appendix C reports changes in borrowing in more detail.

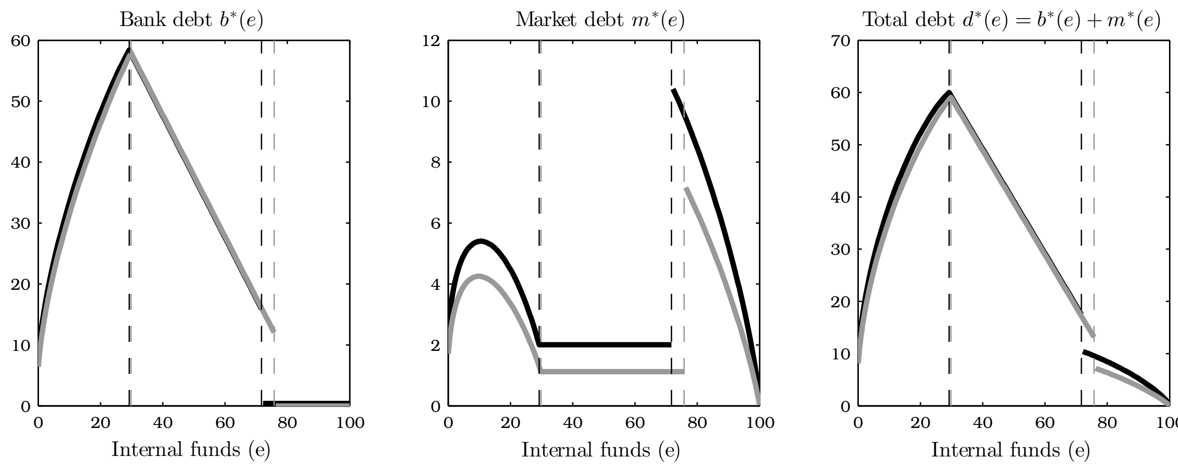


Figure 11: The effect of an increase in the dispersion of productivity. The black line corresponds to the baseline calibration; the grey line corresponds to higher productivity dispersion (see table 1 for details on the calibrations used).

funds. Note, additionally, that the switching threshold e^* moves to the left, since the maximum size of the firm also falls. As a result, the change in average productivity has a pure scale effect, but does not alter the composition of debt across firms, similarly to joint changes in lending costs.[19]

4.2.2 An increase in the dispersion of productivity

Next, I look at the impact of an increase in the second central moment of the distribution of productivity, keeping the first moment constant. In figure 11, I report the result of increasing the coefficient of variation of the distribution by 10 %, relative to the baseline calibration, while keeping the first moment constant. The corresponding calibration is reported in table 1, under the column "High productivity dispersion".

Note that, contrary to the change in the first moment of the distribution, this change does not affect the maximum size of the firm, so that the range of relevant levels of internal funds for the firm is unchanged. The result of the increase in dispersion is different for market and bank lending: market lending decreases uniformly, across values of e, while bank debt is relatively unchanged. As higher dispersion generically increases expected liqudation losses, the threshold for switching towards market finance increases. The model thus suggests that an increase in the dispersion of productivity should lead to a shift toward more bank-financing for firms with small and intermediate levels of internal finance, mirroring the results on debt composition obtained for the spread between bank and market finance. Note, however, that in both cases, total borrowing and total investment of firms is lower than in the baseline calibration.

[19]This point is made even clearer in figure 15, which reports the optimal bank share as a function of the level of internal funds. This schedule shifts left, but its shape is unchanged.

Summarizing, I have shown the comparative statics of the optimal debt structure are markedly different depending on whether one focuses on parameters that affect symmetrically the two types of debt (common lending costs, average productivity), or on parameters that have asymmetric effects on their costs or on their return to the firm (the lending spread, productivity dispersion). With a higher spread or higher productivity dispersion, small firms borrow less but remain bank-dependent. Medium-sized firms are more likely to use purely market-financed debt structures with a higher spread, while they are *less* likely to do to so with higher productivity dispersion. These parameters thus have effects on both borrowing scale *and* borrowing composition. By contrast, higher lending costs and lower average productivity are associated with changes in the scale of borrowing, but not in borrowing composition.

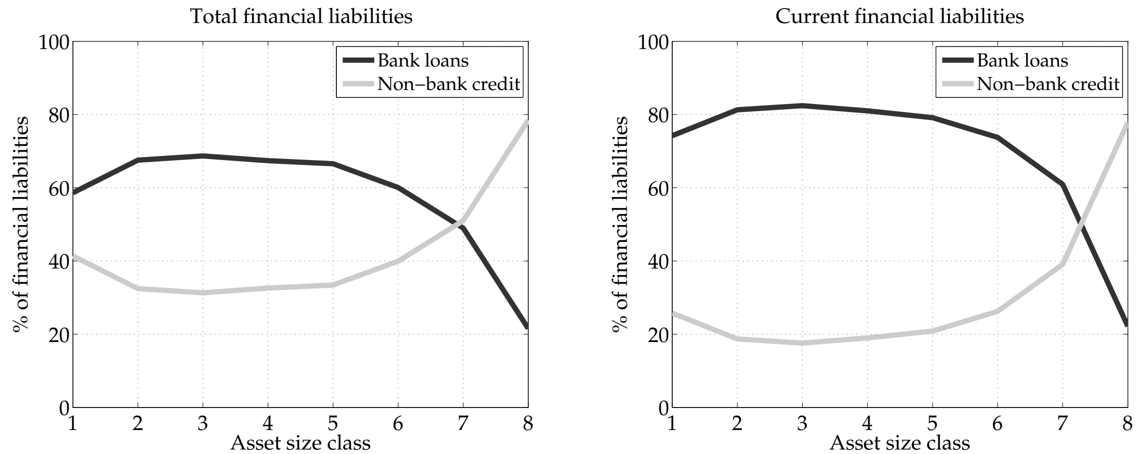
5 An empirical assessment of the model

I now relate the findings of sections 4 and 5 with empirical evidence on the debt structure of US firms. In particular, I ask to what extent the cross-sectional relationship between size and debt composition described in section 4 is borne out in the data, and explore whether the comparative statics of section 5 can help shed light on changes in debt structure of US firms in recent recessions.

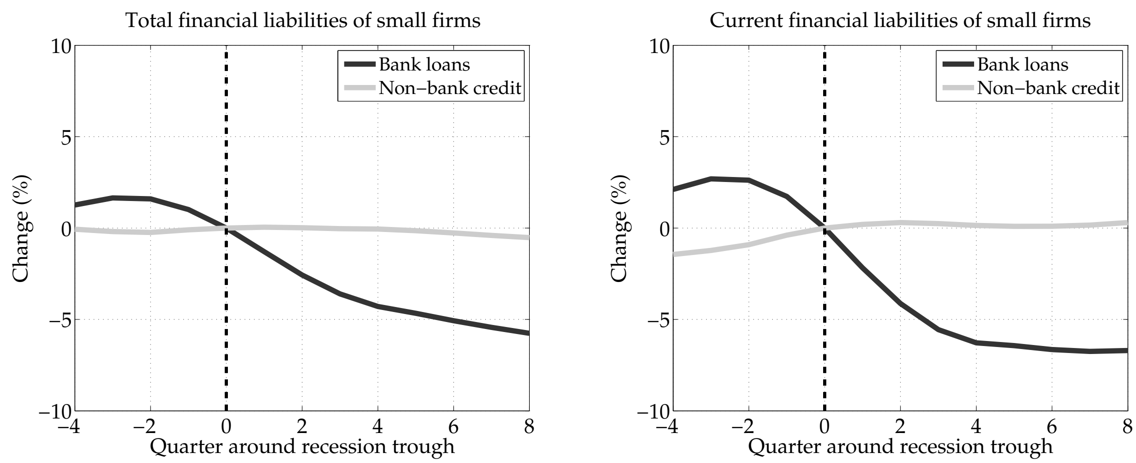
5.1 Data

I focus on data from the Quarterly Financial Report of Manufacturing firms (QFR). The QFR is a survey of manufacturing firms that reports aggregates of balance sheets and income statements across size bins. The QFR's measure of size is assets at book value. One important advantage of the QFR is that it contains information on firms of all sizes, and in particular smaller, privately owned companies. Moreover, it is a quarterly dataset. The drawback is that data is only available in semi-aggregated form. An alternative is to use firm-level data, such as Compustat. The Compustat sample contains more detailed information on all debt instruments used by firms. It is also not restricted to manufacturing firms. However, it contains only annual balance sheet information for publicly traded companies. Thus, it excludes most smaller, non-traded companies, and has lower frequency of observation.[20] More information on the QFR, as well as a description of the construction of the data used below, is given in appendix D.2.

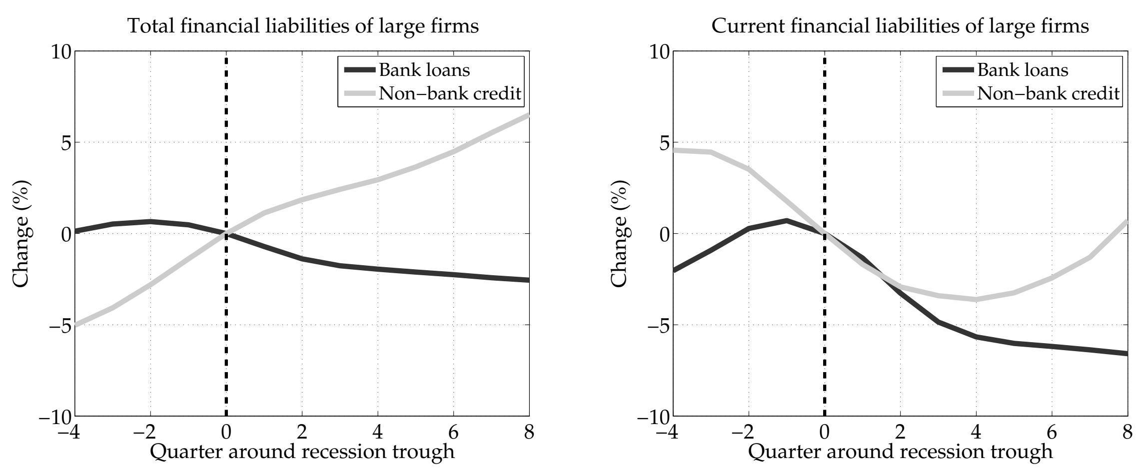
[20]The Compustat sample is focus of the empirical work of Rauh and Sufi (2010), for cross-sectional variation, and Adrian, Colla, and Shin (2012) for business-cycle changes. My results are in line with their findings, and extend them to the sample of private and smaller firms included in the QFR.



(a) The composition of debt in the cross-section



(b) The debt structure of small firms during recent recessions



(c) The debt structure of large firms during recent recessions

Figure 12: Key features of the debt structure of firms in the QFR. In figure 12(a), the left panel graphs current liabilities: the ratios $CB/CFIN$ (bank loans, green line) and $CNB/CFIN$ (non-bank credit, red line). The right panel plots total liabilities: the ratios $TB/TFIN$ and $TNB/TFIN$. In figures 12(b) and 12(c), the right panels plot current liabilities and the left panels plot total liabilities; see text for details. Series in figures 12(b) and 12(c) are smoothed with a 2 by 4 MA smoother before computing growth rates.

5.2 The composition of debt in the cross-section

Figure 12(a) illustrates the differences in debt structure between small and large firms, by reporting the average bank share of total (bank and non-bank) debt across different asset size bins. Roughly 70% of the debt of small firms is held in the form of bank loans, while 80% of the debt of the largest firms is held in the form of non-bank credit. Of these 80%, on average over the last 5 years, 60% were held in the form of commercial paper and bonds. This holds whether one focuses on total or current debt. Thus, the majority of the debt of small firms are bank loans, while the majority of the debt of large firms are non-bank credit instruments. Note, however, that the relationship between size and the bank share is not strictly monotonic; for the very smallest firms, the bank share is slightly increasing in size.

In the model, the bank share is, broadly speaking, high for entrepreneurs with little internal funds, and low for firms with large internal funds. However, for firms with internal funds below $e \leq e^*$, the bank share is non-monotonic, but overall relatively large, and in fact sufficiently so for the bank borrowing constraint to bind for entrepreneurs with very small e. When $e > e^*$, on the other hand, any debt issuance takes the form of market borrowing, as entrepreneurs find that the costs associated to borrowing from banks completely outweigh benefits of debt flexibility.

This paints a picture which is, at first glance, consistent with the cross-sectional debt structure reported in figure 12(a). However, the model predicts a relationship between internal finance e and the bank share of liabilities, while that figure relates total assets to the bank share of liabilities. As discussed previously, the model does not predict a strictly monotonic relationship between size $k(e) = e + \hat{b}(e) + \hat{m}(e)$ and internal funds e. One cannot therefore infer from the predicted relationship between internal funds and the bank share, that a similar link between total assets and the bank share exists in the model.

This need not be viewed as a shortcoming of the model. First, in this model b and m represent new debt issuance, which are to be retired after a single period. In particular, the entrepreneur does not inherit legacy assets or long-lived debt that needs to be serviced. In reality, much of the balance sheet of the firms in the sample analyzed in section 5 is made up of liabilities maturing over long periods of time. It is also this lack of history-dependence which accounts for the fact that above the level e^*, the firm switches to a debt structure entirely composed of market borrowing. On the face of it, this is not a realistic feature of the model, but it should be interpreted as indicating that new debt issuances will take the form of market debt once internal funds have reached the threshold e^*.

Second, the assumption of risk neutrality is crucial to obtain the result that total borrowing is constant. With risk aversion, a marginal increase in internal funds would not necessarily need to a marginal fall in bank borrowing, to the extent that bank lending not only reduces expected liquidation losses, but also reduces the variance of the payoff to the entrepreneur. In a general version of the model with risk aversion, the fact that borrowing is constant on a range of levels is internal funds is likely to disappear.

Third, the patter of figure 12(a) concerns the complete distribution of firms across asset sizes. The fact that firms are distributed across different asset sizes may originate from other reasons than initial differences in internal funds. Besides history-dependence, one source of firm size heterogeneity that has received particular attention in the literature are differences in the (average) marginal productivity of capital, $\mathbb{E}(\phi)$. Firms with different long-run productivity levels both operate at different scales, and face different borrowing constraints. Taking these long-run differences into account in the model may help shed better light on the asset size / debt structure link, as opposed to the internal funds / debt structure link.

5.3 Understanding cyclical changes in debt structure

The comparative statics reported in section 4 emphasizes that size is a key determinant of the relationship between debt structure and aggregate conditions. In studying how debt structure changed during the most recent recessions, I will therefore focus on the differences between small and large firms. For creating aggregates of small and large firms, I directly use the fixed bins provided by the QFR. I define small firms as those with $1bn\$$ in assets or less (the first seven size bins of the QFR sample; see table 2), and large firms as those with more than $1bn\$$ in assets (the last bin of the QFR sample).[21]

The series reported in figures 12(b) and 12(c) show averages of cumulative growth rates of total financial liabilities of small and large firms, broken down between bank and non-bank debt, over the three recessions in the sample. Cumulative growth rates are computed around NBER recession troughs. I focus on a window of 1 year before the recession trough to 2 years after the recession trough.[22]

Figure 12(b) graphs the growth rate in bank loans and non-bank credit for small firms.[23] For these firms, outstanding bank debt falls and non-bank debt does not change, resulting in a fall in total liabilities. This holds regardless of whether one focuses on total liabilities (left panel) or only current liabilities (right panel).

Figure 12(c) reports the same series as figure 12(b), for ”large” firms (the last asset size bin of the QFR). The left panel of figure 12(c) shows that for large firms, total outstanding banks debt falls, while non-bank

[21] This definition of small and large firm categories is transparent, but does not adress the fact that, because of trend growth in the book value of assets, large firm bins become more populated over time. In appendix D.3, I consider an alternative definition of small and large firm groups, proposed by Gertler and Gilchrist (1994), which keeps the fraction of total sales accounted for by the small firm group category constant. I show that these facts, which contrast changes in debt structure of small and large firms, are robust to this alternative definition of size classes.

[22] Appendix D.4 reports the levels of all the series used.

[23] For each type of liability, the growth rate reported is weighted by its share as a fraction of total liabilities at the recession trough. For example, for total bank loans, the series reported is the average of $\gamma_{TB,t} = \frac{TB_0}{TB_0+TNB_0}\left(\frac{TB_t}{TB_0} - 1\right)$ over the three recessions, where the time index is 0 for the quarter corresponding to the trough of a recession. Weighting the growth rates in this fashion conserves additivity: for example, for small firms, $\gamma_{TB,t} + \gamma_{TNB,t} = \gamma_{TFIN,t}$, where $\gamma_{TFIN,t}$ is the growth rate of total financial liabilities of small firms around the recession trough.

debt increases.[24] Note, importantly, that this pattern is driven by new and/or long-dated liabilities. Indeed, no increase in non-bank current liabilities (those maturing in one year or less) is visible, as reported in the right panel of figure 12(c).

Since this sample contains only three recessions, one of which, 2007-2009, was substantially larger in magnitude than the others, it is natural to ask whether the patterns documented here are driven mostly by the Great Recession. This question is adressed in appendix D.3.3, where I report figures similar to 12(b)-12(c), excluding the last recession. There are two differences: first, the overall reduction in bank lending is smaller over these recessions, both for small and large firms; second, substitution towards non-bank credit for large firms is more muted than when including the last recession. It is during the 2007-2009 recession that the changes in the debt structure of large firms were most different from changes in the debt structure of small firms.

The comparative statics of section 6 indicate that only variation in the lending spread and in productivity dispersion can account for changes in both the scale and composition of borrowing. Moreover, increases in productivity dispersion are associated with a substitution toward bank borrowing. The model thus suggests that the patterns documented in figures 12(b)-12(c) are a telltale sign of an asymmetric change in banks' lending cost, an result implicitly underlying much of the empirical work on the bank lending channel, such as for example Kashyap, Stein, and Wilcox (1993) and Gertler and Gilchrist (1994).

There is some tension between the model and the patterns of debt substitution for large firms documented in figure 12(c). In the model, in response to an increase in the spread, total borrowing for these firms falls; the substitution between bank and market debt is less than one for one. Figure 12(c), on the other hand, suggests that the substitution of bank for market credit is at least one for one. As discussed section 5, it is likely that, in the data, the magnitude of debt substitution for large firms is overstated by upwards reclassification. Additionally, the model omits important alternative forms of financing than could also serve as substitutes to bank debt, in particular short-term instruments such as commercial paper, which is an important channel of adjustement for firms in times of tight money (see Acharya and Schnabl (2010)), and are therefore likely to account for a share of substitution away from bank debt.[25]

[24] Note that, since the increase in non-bank debt is larger than that of bank debt, total financial liabilities are in fact slightly increasing. The increase in total assets may be the artifact of the accumulation of firms in the larger size bins over time; in fact, as reported in appedix D.3, total liabilities for large firms do not increase for alternative definitions of the large firm category, but the debt substitution pattern remains.

[25] Because commercial paper is no separated from other non-bank liabilities for all asset size classes in the QFR, it was included in non-bank liabilities in the construction of the previous graphs. Excluding commercial paper from non-bank liabilities does not alter these graphs substantially; see appendix D.3 for details.

6 Conclusion

In this paper, I proposed a static model of the joint determination of investment and debt structure for a firm with access to both bank debt and market debt. The model builds on the trade-off theory of the debt structure, according to which a firm's debt structure reflects a trade-off between the flexibility afforded by bank debt in times of financial distress, and the higher marginal costs of lending of banks. The model extends existing studies by allowing the entrepreneur to choose the scale of the project she operates, and thus explicitly modelling the link between investment and debt structure choices.

I showed that this model is consistent with the fact that small firms borrow mostly from banks, while the debt structure of large firms consists predominantly of market debt. Additionally, the model predicts that in response to an increase in banks' cost of funds, relative to market lenders', firms with sufficient internal finance should optimally substitute their bank liabilities for market debt, while firms with less access to internal finance reduce bank borrowing without increasing market liabilities. This pattern is consistent with changes in the debt structure of US firms, small and large, over the last three recessions, and particularly after the 2007-2009 recession.

This paper suggests three avenues for future research. First, the model emphasizes the importance of lenders' costs of funds r_b and r_m as determinants of the debt structure in the cross-section, and suggests that an increase in the spread $r_b - r_m$ may account well for changes in the debt structure of US firms. An important question is then whether it is possible to construct an empirical counterpart to this spread, and whether it changes substantially over the business cycle. As emphasized by the model, interest rates on loans or bond yields are clearly only upper bounds on these costs since they incorporate premia (liquidation risk premia, in the case of the model). Second, the model postulates that bank debt is senior to market debt. While, empirically, bank debt tends to be placed on top of firms' priority structures, the rationale for this a subject of debate in much of the literature on debt heterogeneity; in fact, in some setups, such as Rajan (1992), the optimal debt structure makes bank debt junior to non-monitored debt. In the context of the model developed in this paper, numerical results suggest the alternative priority structure, whereby bank debt is junior, would not be preferred by the firm, if it were it be given the ability to choose. An important question is whether this result can be generally established, and in particular whether it always holds for any level of internal funds.[26] Third, because of the generality of the results on set of feasible debt structures, the model I developed in this paper can be used to study the dynamic implications of debt heterogeneity for firm investment and growth, as well as to trace out the aggregate effects of shocks on borrowing and investment

[26] Hackbarth, Hennessy, and Leland (2007), in the context of their model with fixed investment size, prove that bank debt seniority is generally preferable.

in an economy with different types of financial institutions.[27] I leave all three topics to future research.

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[27]See Reis (2010) for an example of such a model, focusing on the implications of securitization, rather than on the bank/market debt choice.

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A Appendix to section 2

A.1 Proof of proposition 1

Proof of proposition 1. Note first that the payoff to the entrepreneur under the optimal restructuring offer, given by equation (7), can be rewritten as:

$$\pi_R(\phi) = \pi(\phi) - min(\chi\pi(\phi), R_b) - R_m. \tag{10}$$

First, assume that the repayments promised to the bank and market lenders satisfy:

$$\frac{R_b}{\chi} > \frac{R_m}{1-\chi}. \tag{11}$$

Since:

$$R_m + R_b = \chi\frac{R_b}{\chi} + (1-\chi)\frac{R_m}{1-\chi},$$

condition (11) implies that:

$$\frac{R_b}{\chi} > R_m + R_b > \frac{R_m}{1-\chi}.$$

The threshold $\frac{R_b}{\chi}$ is the one where restructuring becomes profitable relative to repayment for the entrepreneur, while the threshold $R_m + R_b$ is the "natural" bankrupcy threshold – the one that would obtain absent the possibility of private workouts with the bank. Given condition (11), consider the three following possibilities:

- **if** $\pi(\phi) \geq \frac{R_b}{\chi}$**,** using equation (10), the payoff of the entrepreneur under restructuring is indentical to that under repayments; that is, the firm cannot lower its repayments in a private workout. Moreover, as $\pi(\phi) > \frac{R_b}{\chi} > R_m + R_b$, repayment is preferable to bankrupcy. Thus, the entrepreneur chooses full repayment.

- **if** $\frac{R_b}{\chi} > \pi(\phi) \geq \frac{R_m}{1-\chi}$**,** the entrepreneur prefers restructuring to repayment since:

$$\pi_R(\phi) = \pi(\phi) - \chi\pi(\phi) - R_m > \pi(\phi) - R_b - R_m.$$

Moreover, the entrepreneur also prefers restructuring to bankrupcy, since when $\pi(\phi) > \frac{R_m}{1-\chi}$,

$$\pi_R(\phi) > 0 = \pi_B(\phi).$$

Thus, the entrepreneur chooses to restructure her debt with the bank,

- **if** $\frac{R_m}{1-\chi} > \pi(\phi)$**,** the entrepreneur now prefers bankrupcy to restructuring, as the gains achieved under restructuring would still not be sufficient to repay market creditors in full and end up with a strictly positive amount of cash on hand. Furthermore, since under condition (11), $\pi(\phi) < \frac{R_m}{\chi} < R_m + R_b$, the entrepreneur also prefers bankrupcy to repayment. Thus, in this case, the entrepreneur chooses to file for bankrupcy.

This establishes the first part of the proposition on the structure of R-equilibria. Additionally, note that in this type of equilibrium, the payoff to market lenders under bankrupcy is always 0, because bankrupcy occurs only when $\pi(\phi) < \frac{R_m}{1-\chi} < \frac{R_b}{\chi}$.

Next, consider the case:

$$\frac{R_b}{\chi} \leq \frac{R_m}{1-\chi}, \tag{12}$$

which implies that:

$$\frac{R_b}{\chi} < R_m + R_b < \frac{R_m}{\chi}.$$

Under condition (12), consider the three following possibilities:

- **if** $\pi(\phi) \geq R_m + R_b$, repayment is preferable to bankrupcy. Moreover, because in this case $\pi(\phi) \geq R_m + R_b > \frac{R_b}{\chi}$, restructuring cannot lead to any gains for the entrepreneur. Thus, the entrepreneur chooses repayment.

- **if** $R_m + R_b > \pi(\phi) \geq \frac{R_b}{\chi}$, bankrupcy is preferable to repayment, and moreover, restructuring still cannot achieve any gains relative to repayment; thus, the entrepreneur chooses bankrupcy.

- **if** $\frac{R_b}{\chi} > \pi(\phi)$, restructuring can now achieve gains relative to repayment. However, because $\frac{R_m}{1-\chi} > \frac{R_b}{\chi} > \pi(\phi)$ the payoff under restructuring satisfies:

$$\pi_R(\phi) = (1-\chi)\pi(\phi) - R_m < 0 = \pi_B(\phi).$$

Thus, the entrepreneur still chooses bankrupcy.

This establishes the second part of the proposition. Additionally, note that unde the seniority structure assumed, bankrupcy payments to market lenders are $\chi\pi(\phi) - R_b$ if $R_m + R_b > \pi(\phi) \geq \frac{R_b}{\chi}$, and 0 otherwise.

In the latter case, this is obviously smaller than R_m. In the former case, since bankrucpy only occurs when $\pi(\phi) < R_m + R_b$, we have that

$$\pi(\phi) < R_m + R_b < \frac{R_m + R_b}{\chi},$$

so that:

$$\chi\pi(\phi) - R_b < R_m.$$

Therefore, under the assumed priority structure, market lenders never obtain a repayment under bankrupcy that exceeds the promised repayment R_m.

■

A.2 Debt pricing

In this section, I describe the payoff and the expected gross return functions of lenders that were omitted from the main text.

A.2.1 Payoff functions

Given the description of the equilibria in proposition 1, the payoffs to the lenders and the entrepreneur can be expressed as a function of ϕ. I denote them $\tilde{R}_b(\phi)$ and $\tilde{R}_b(\phi)$ for the bank and market lenders, respectively, and $\tilde{\pi}(\phi)$ for the entrepreneur.

In B-equilibria $\left(\frac{R_b}{\chi} < \frac{R_m}{1-\chi}\right)$, payoffs are given by:

$$\tilde{R}_b(\phi) = \begin{cases} R_b & \text{if} \quad \frac{R_b}{\chi} \leq \pi(\phi) \\ \chi\pi(\phi) & \text{if} \quad \pi(\phi) < \frac{R_b}{\chi} \end{cases}$$

$$\tilde{R}_m(\phi) = \begin{cases} R_m & \text{if} \quad R_m + R_b \leq \pi(\phi) \\ \chi\pi(\phi) - R_b & \text{if} \quad \frac{R_b}{\chi} \leq \pi(\phi) < R_m + R_b \\ 0 & \text{if} \quad \pi(\phi) < \frac{R_b}{\chi} \end{cases}$$

$$\tilde{\pi}(\phi) = \begin{cases} \pi(\phi) - R_b - R_m & \text{if} \quad R_m + R_b \leq \pi(\phi) \\ 0 & \text{if} \quad \pi(\phi) < R_m + R_b \end{cases}$$

In R-equilibria $\left(\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}\right)$, payoffs are given by:

$$\tilde{R}_b(\phi) = \begin{cases} R_b & \text{if} \quad \frac{R_b}{\chi} \leq \pi(\phi) \\ \chi\pi(\phi) & \text{if} \quad \pi(\phi) < \frac{R_b}{\chi} \end{cases}$$

$$\tilde{R}_m(\phi) = \begin{cases} R_m & \text{if} \quad \frac{R_m}{1-\chi} \leq \pi(\phi) \\ 0 & \text{if} \quad \pi(\phi) < \frac{R_m}{1-\chi} \end{cases}$$

$$\tilde{\pi}(\phi) = \begin{cases} \pi(\phi) - R_b - R_m & \text{if} \quad \frac{R_b}{\chi} \leq \pi(\phi) \\ (1-\chi)\pi(\phi) - R_m & \text{if} \quad \frac{R_m}{1-\chi} \leq \pi(\phi) < \frac{R_b}{\chi} \\ 0 & \text{if} \quad \pi(\phi) < \frac{R_m}{1-\chi} \end{cases}$$

A.2.2 Gross expected returns

I now turn to the gross expected return functions of lenders. Throughout, I use the same change of variables as in the text:

$$b = ds$$

$$m = d(1-s)$$

d thus denotes the total amount borrowed, s denotes the fraction borrowed from the bank, and $1-s$ denotes the fraction borrowed from market lenders; clearly $s \in [0,1]$ and $d \geq 0$. I use the notation:

$$\tilde{E}_i(R_b, R_m; d, s, e) = E\left[\Pi_i\left(e, ds, d(1-s), R_b, R_m\right)\right]$$

for the associated gross return of lenders.

Lemma 3 (Gross expected return functions) *The gross expect return of bank lenders is given by:*

$$\tilde{\mathbb{E}}_b(R_b; e+d) = \begin{cases} R_b & \textit{if} \quad \frac{R_b}{\chi} < (1-\delta)(e+d) \\ \chi\left((e+d)^{\zeta} G(\phi_b) + (1-\delta)(e+d)\right) & \textit{if} \quad \frac{R_b}{\chi} \geq (1-\delta)(e+d) \end{cases}$$

where:

$$\phi_b \equiv \frac{R_b - \chi(1-\delta)(e+d)}{\chi(e+d)^{\zeta}} \quad \textit{and} \quad G(x) \equiv x(1-F(x)) + \int_0^x \phi dF(\phi)$$

When $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, *the gross expected return of market lenders is given by:*

$$\tilde{\mathbb{E}}_{m,R}(R_m; e+d) = \begin{cases} R_m & \text{if} \quad \frac{R_m}{1-\chi} < (1-\delta)(e+d) \\ (1-\chi)\left((e+d)^\zeta I(\phi_{m,R}; e+d) + (1-\delta)(e+d)\right) & \text{if} \quad \frac{R_m}{1-\chi} \geq (1-\delta)(e+d) \end{cases}$$

where:

$$\phi_{m,R} \equiv \frac{R_m - (1-\chi)(1-\delta)(e+d)}{(1-\chi)(e+d)^\zeta} \quad \text{and} \quad I(x; e+d) \equiv x(1-F(x)) - F(x)(1-\delta)(e+d)^{1-\zeta}.$$

When $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, *the gross expected return of market lenders is given by:*

$$\tilde{\mathbb{E}}_{m,B}(R_b, R_m; e+d) = \begin{cases} R_m & \text{if} \quad R_m + R_b < (1-\delta)(e+d) \\ (e+d)^\zeta M(\phi_{m,B}; e+d) + (1-\delta)(e+d) & \text{if} \quad R_m + R_b \geq (1-\delta)(e+d) \\ \qquad -\tilde{\mathbb{E}}_b(R_b; e+d) & \end{cases}$$

where:

$$\phi_{m,B} \equiv \frac{R_b + R_m - (1-\delta)(e+d)}{(e+d)^\zeta} \quad \text{and} \quad M(x; e+d) \equiv (1-\chi)I(x; e+d) + \chi G(x).$$

Proof. In the case $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, the entrepreneur never defaults when $(1-\delta)(e+d) > R_m + R_b$, since in that case, the lower bound on her output is greater than the sum of her promised repayments. Expected repayments for the bank and market lenders are therefore equal to promised repayments R_b and R_m. When $\frac{R_b}{\chi} < (1-\delta)(e+d) \leq R_m + R_b$, there may be default on market debt, but even in liquidation bank debt will be repayed in full, so that $\tilde{\mathbb{E}}_b = R_b$. Similarly, in the case $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, there is no default when $(1-\delta)(e+d) > \frac{R_b}{\chi}$, and no default on market debt if $\frac{R_m}{1-\chi} < (1-\delta)(e+d) \leq \frac{R_b}{\chi}$. The rest of the expressions follow from the expressions of the payoff functions.

■

A.2.3 Zero profit conditions

Using the results of lemma 3, the zero profit condition (ZPC) of bank lenders can be written as:

$$\tilde{G}(R_b; e+d) = \frac{(1+r_b)ds}{\chi(e+d)^\zeta} \quad , \quad \text{with} \quad \tilde{G}(R_b; e+d) \equiv \frac{\tilde{\mathbb{E}}_b(R_b; e+d)}{\chi(e+d)^\zeta}. \tag{13}$$

Lemma 4 (The zero profit condition of bank lenders) *Let* (d, s, e) *be given. Then, there exists a* *unique solution $R_b(d,s,e)$ to the ZPC of bank lenders, equation (13), if an only if,*

$$0 \leq \frac{(1+r_b)ds}{\chi(e+d)^\zeta} \leq \mathbb{E}(\phi) + (1-\delta)(e+d)^{1-\zeta}. \tag{14}$$

Moreover, $R_b(d,s,e)$ is given by:

$$R_b(d,s,e) = \begin{cases} (1+r_b)ds & \text{if} \quad 0 \leq \frac{(1+r_b)ds}{\chi(e+d)^\zeta} < (1-\delta)(e+d)^{1-\zeta} \\ \chi(1-\delta)(e+d) & \text{if} \quad (1-\delta)(e+d)^{1-\zeta} \leq \frac{(1+r_b)ds}{\chi(e+d)^\zeta} \\ \qquad +\chi(e+d)^\zeta G^{-1}\left(y_G(d,s,e)\right) & \qquad\qquad \leq \mathbb{E}(\phi) + (1-\delta)(e+d)^{1-\zeta} \end{cases}$$

Here, $y_G(d,s,e) \equiv \frac{(1+r_b)ds - \chi(1-\delta)(e+d)}{\chi(e+d)^\zeta}$, and $G^{-1}(.)$ denotes the inverse of $G(.)$, defined on $[0, \mathbb{E}(\phi)]$ with the abuse of notation that $G^{-1}(\mathbb{E}(\phi)) = +\infty$.

Proof. Note that $\tilde{G}(.;e+d)$ is strictly increasing on $\mathbb{R}_+$, that $\tilde{G}(0;e+d) = 0$ and $\lim_{R_m \to +\infty} \tilde{G}(R_m;e+d) = \mathbb{E}(\phi) + (1-\delta)(e+d)^{1-\zeta}$. Similarly, $G(.)$ is strictly increasing on $\mathbb{R}_+$, $G(0) = 0$, and $\lim_{x \to +\infty} G(x) = \mathbb{E}(\phi)$. ■

Likewise, in the case $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, the zero profit condition of market lenders can be written as:

$$\tilde{I}(R_b;e+d) = \frac{(1+r_m)d(1-s)}{(1-\chi)(e+d)^\zeta} \quad , \quad \text{with} \quad \tilde{I}(R_b;e+d) \equiv \frac{\tilde{\mathbb{E}}_{m,R}(R_m;e+d)}{(1-\chi)(e+d)^\zeta}. \tag{15}$$

Lemma 5 (The zero profit condition of market lenders when $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$) *Let (d,s,e) be given. Then, there exist exactly two solutions, $R_{m,l}(d,s,e) \leq R_{m,L}(d,s,e)$, to equation (15), if and only if,*

$$0 \leq \frac{(1+r_m)d(1-s)}{(1-\chi)(e+d)^\zeta} \leq \hat{I}(e+d) + (1-\delta)(e+d)^{1-\zeta}, \tag{16}$$

where $\hat{I}(e+d) = I(\phi_I(e+d);e+d)$ is the maximum of $I(.;d+e)$, attained at $\phi_I(e+d)$, the unique (strictly positive) solution to:

$$1 - F(\phi_I(e+d)) - f(\phi_I(e+d))\left(\phi_I(e+d) + (1-\delta)(e+d)^{1-\zeta}\right) = 0.$$

Moreover, $R_{m,l}(d,s,e)$ is given by:

$$R_{m,l}(d,s,e) = \begin{cases} (1+r_m)d(1-s) & \text{if} \quad 0 \leq \frac{(1+r_m)d(1-s)}{(1-\chi)(e+d)^\zeta} < (1-\delta)(e+d)^{1-\zeta} \\ (1-\chi)(1-\delta)(e+d) & \text{if} \quad (1-\delta)(e+d)^{1-\zeta} \leq \frac{(1+r_m)d(1-s)}{(1-\chi)(e+d)^\zeta} \\ \qquad +(1-\chi)(e+d)^\zeta I^{-1}\left(y_I(d,s,e);e+d\right) & \qquad\qquad \leq \hat{I}(e+d) + (1-\delta)(e+d)^{1-\zeta} \end{cases}$$

Here, $y_I(d,s,e) \equiv \frac{(1+r_m)d(1-s)-(1-\chi)(1-\delta)(e+d)}{(1-\chi)(e+d)^\zeta}$, *and* $I^{-1}(.;e+d)$ *denotes the mapping from* $\left[0, \hat{I}(e+d)\right]$ *to* $[0, \phi_I(e+d)]$ *such that* $I(I^{-1}(y;e+d);e+d) = y$, $\forall y \in \left[0, \hat{I}(e+d)\right]$.

Proof. Note that $\tilde{I}(0;e+d) = 0$ and $\lim_{R\to+\infty} \tilde{I}(R;e+d) = 0$. Moreover, $\frac{\partial \tilde{I}}{\partial R_b} = \frac{1}{(1-\chi)(e+d)^\zeta} > 0$ when $R_b < (1-\chi)(1-\delta)(e+d)^{1-\zeta}$, so that $\tilde{I}(.,e+d)$ is increasing in R_b on that range. For $R_b \geq (1-\chi)(1-\delta)(e+d)^{1-\zeta}$, note that $\frac{\partial \tilde{I}}{\partial R_b} \geq 0$, if and only if, $h(\phi_{m,R}) = \frac{f(\phi_{m,R})}{1-F(\phi_{m,R})} \leq \frac{1}{\phi_{m,R}+(1-\delta)(e+d)^{1-\zeta}}$, with $\phi_{m,R}$ defined as in lemma 3. Fix e and d and let $\Delta(x) = h(x) - \frac{1}{x+(1-\delta)(e+d)^{1-\zeta}}$. h is strictly increasing by assumption 1. As the sum of two strictly increasing functions, Δ is therefore strictly increasing on $]0;+\infty[$. Moreover, $\Delta(0) = -\frac{1}{(1-\delta)(e+d)^{1-\zeta}}$ and $\lim_{x\to+\infty} \Delta(x) = \lim_{x\to+\infty} h(x)$. Since $h(0) = 0$ (as $f(0) = 0$ by assumption 1) and h is strictly increasing, $\lim_{x\to+\infty} h(x) > 0$. Thus $\Delta(x)$ has a unique strictly positive root at $\phi_{m,R} = \phi_I(e+d)$. In turn, because $\phi_{m,R}$ is a strictly increasing function of R_m, $\hat{R}_I(e+d) \equiv (1-\chi)\left(\phi_I(e+d)(e+d)^\zeta + (1-\delta)(e+d)\right)$ is the global maximum of $\tilde{I}(.;e+d)$. There are moreover no local maxima, so that $\tilde{I}(.;d,e)$ is strictly increasing to the left of $\hat{R}_I(e+d)$ and strictly decreasing to the right. (Note however that $I(.;e+d)$ and $\tilde{I}(.;e+d)$ need not be concave). This proves that there are exactly two solutions $\tilde{R}_{m,l}(d,s,e) \leq \hat{R}_I(e+d) \leq \tilde{R}_{m,L}(d,s,e)$ to equation (15) under the conditions given in the lemma. The expression for $R_{m,l}(d,s,e)$ follows from recognizing that, for the same reasons as $\tilde{I}(.;e+d)$, $I(.;e+d)$ is strictly increasing on $[0,\phi_I(e+d)]$ and strictly decreasing thereafter. ■

Finally, when $\frac{R_m}{1-\chi} > \frac{R_b}{\chi}$, when the zero profit condition of the bank holds, the zero profit condition of market lenders can be rewritten as:

$$\tilde{M}(R_b + R_m; e+d) = \frac{(1 + r_m(1-s) + r_b)d}{(e+d)^\zeta}, \tag{17}$$

where:

$$\tilde{M}(R;e+d) = \begin{cases} \frac{R}{(e+d)^\zeta} & \text{if} \quad R < (1-\delta)(e+d) \\ M(\phi_{m,B};e+d) + (1-\delta)(e+d)^{1-\zeta} & \text{if} \quad R \geq (1-\delta)(e+d) \end{cases}$$

and $\phi_{m,B} \equiv \frac{R-(1-\delta)(e+d)}{(e+d)^\zeta}$.

Lemma 6 (The zero profit condition of market lenders when $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$) *Let* (d,s,e) *be given. Then, there exist exactly two solutions,* $R_l(d,s,e) \leq R_L(d,s,e)$, *to equation (17), if and only if,*

$$\chi\mathbb{E}(\phi) + (1-\delta)(e+d)^{1-\zeta} \leq \frac{(1 + r_m(1-s) + r_b s)d}{(e+d)^\zeta} \leq \hat{M}(e+d) + (1-\delta)(e+d)^{1-\zeta}, \tag{18}$$

where $\hat{M}(e+d) = M(\phi_M(e+d), e+d)$ *is the maximum of* $M(.;e+d)$, *attained at* $\phi_M(e+d) > \phi_I(e+d)$, *which is the unique (strictly positive) solution to:*

$$1 - F(\phi_M(e+d) - (1-\chi) f(\phi_M(e+d)) \left(\phi_M(e+d) + (1-\delta)(e+d)^{1-\zeta}\right) = 0.$$

There exists exactly one solution $R_l(d,s,e)$ to equation (17), if and only if:

$$0 \leq \frac{(1 + r_m(1-s) + r_b s)d}{(e+d)^{\zeta}} < \chi \mathbb{E}(\phi) + (1-\delta)(e+d)^{1-\zeta},$$

Moreover, $R_l(d,s,e)$ is given by:

$$R_l(d,s,e) = \begin{cases} (1 + r_m(1-s) + r_b s)d & \text{if} \quad 0 \leq \frac{(1+r_m(1-s)+r_b s)d}{(e+d)^{1-\zeta}} < (1-\delta)(e+d)^{1-\zeta} \\ +(1-\delta)(e+d) & \text{if} \quad (1-\delta)(e+d)^{1-\zeta} \leq \frac{(1+r_m(1-s)+r_b s)d}{(e+d)^{1-\zeta}} \\ \quad (e+d)^{\zeta} M^{-1}\left(y_M(d,s,e); e+d\right) & \qquad \leq \hat{M}(e+d) + (1-\delta)(e+d)^{1-\zeta} \end{cases}$$

Here, $y_M(d,s,e) \equiv \frac{(1+r_m(1-s)+r_b s)d - (1-\delta)(e+d)}{(e+d)^{\zeta}}$, and $M^{-1}(.; e+d)$ denotes the mapping from $\left[0, \hat{M}(d+e)\right]$ to $[0, \phi_M(e+d)]$ such that $M(M^{-1}(y; e+d); e+d) = y \ \forall y \in \left[0, \hat{M}(e+d)\right]$.

A.3 The lending menu in the general case

This subsection contains the proof of proposition 2. This proposition states that the set $\mathcal{S}(e)$ can be separated into two subsets, $\tilde{S}_R(e)$ and $\tilde{S}_B(e)$. $\tilde{\mathcal{S}}_K(e)$ roughly correspond to debt structures associated with dominating contracts with $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, while $\tilde{\mathcal{S}}_K(e)$ corresponds to debt structures associated with dominating contracts such that $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$. The main difficulty of this proof is that when a debt structure (d,s) is such that $\mathcal{L}(d,s,e) \neq \emptyset$, $\mathcal{L}(d,s,e)$ will typically contain multiple pairs (R_b, R_m) (up to four pairs), each of which satisfy the zero profit conditions of lenders. The multiplicity of contracts comes from two sources. First, zero profit conditions of lenders generically have two solutions, as established in lemmas 5-6. Second, there are debt structures for which the zero profit conditions of lenders have solutions with $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$ as well as solutions with $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$ (that is, $\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e) \neq \emptyset$). All this multiplicity creates the potential for contracts that cannot be ordered using the product order. The main contribution of the lemma is to show that these contracts can in all instances be ordered, so that there is always a unique dominating contract, even when the contract menu contains multiple elements.

Proof. The proof draws heavily upon the results of lemmas 4-6 and proceeds in three steps:

Step 1 : Any $(d,s) \in \tilde{\mathcal{S}}(e)$ must be an element of either $\tilde{\mathcal{S}}_K(e)$ or $\tilde{\mathcal{S}}_R(e)$, so that $\tilde{\mathcal{S}}(e) \subset \left(\tilde{\mathcal{S}}_K(e) \cup \tilde{\mathcal{S}}_R(e)\right)$;

Step 2-a : $\forall (d,s) \in \tilde{\mathcal{S}}_R(e)$, $\mathcal{L}(d,s,e)$ has a least element (R_b, R_m) (for the product order $\geq_{\times}$), so that $\tilde{\mathcal{S}}_R(e) \subset$

$\tilde{\mathcal{S}}(e)$; moreover, this element satisfies $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$;

Step 2-b : $\forall (d,s) \in \tilde{\mathcal{S}}_K(e) \setminus \left(\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e)\right)$, $\mathcal{L}(d,s,e)$ has a least element (R_b, R_m) (for the product order $\geq_\times$), so that $\left(\tilde{\mathcal{S}}_K(e) \setminus \left(\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e)\right)\right) \subset \tilde{\mathcal{S}}(e)$; moreover, the least element satisfies $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$.

Step 1 Let $(d,s) \in \mathcal{S}(e)$ and let (R_b, R_m) be the associated dominating contract, that is, the least element of $\mathcal{L}(d,s,e)$ for $\geq_\times$. It must be the case that either $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$ or $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$.

Assume that $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$. Then $(R_b, R_m) \in \mathcal{L}(d,s,e)$, so that $\mathcal{L}(d,s,e) \neq \emptyset$. Moreover, (R_b, R_m) must solve the zero profit conditions (ZPC) of the lenders in the case $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, equations (13) and (17). Using the results of lemmas 4 and 6, conditions $(\mathrm{c}-\mathrm{bank})$ and $(\mathrm{c}-\mathrm{joint})$ are necessary for the existence of solutions to (13) and (17). Thus, (d,s,e) must satisfy these two conditions; moreover, by lemma 4, $R_b = R_b(d,s,e)$.

If $\frac{(1+r_m(1-s)+r_b s)d}{(e+d)^\varsigma} < \chi \mathbb{E}(\phi)$, the solution to equation (17) is unique, according to lemma 6. Therefore we must have $R_m = R_l(d,s,e) - R_b(d,s,e)$. Since, by assumption, $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, we also have $\frac{R_b}{\chi} < R_m + R_b$. Therefore, it must also be the case that $\frac{R_b(d,s,e)}{\chi} < R_l(d,s,e)$, so that condition $(\mathrm{frontier}-\mathrm{B})$ must hold.

If, on the other hand, $\frac{(1+r_m(1-s)+r_b s)d}{(e+d)^\varsigma} \geq \chi \mathbb{E}(\phi)$, there are exactly two solutions to equation (17), which satisfy $0 \leq R_l(d,s,e) \leq R_L(d,s,e)$. However, $(R_b(d,s,e), R_L(d,s,e) - R_b(d,s,e)) \geq_\times (R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e))$. Since (R_b, R_m) is the least element of $\mathcal{L}(d,s,e)$, it must be therefore be the case that $R_m = R_l(d,s,e) - R_b(d,s,e)$. The fact that condition $(\mathrm{frontier}-\mathrm{B})$ holds then follows from the same steps as above. This finishes the proof that if $\frac{R_m}{1-\chi} > \frac{R_b}{\chi}$, then $(d,s) \in \tilde{\mathcal{S}}_K(e)$.

Assume that $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$. Similar to the other case, this implies that $\mathcal{L}(d,s,e) \neq \emptyset$ and therefore that conditions $(\mathrm{c}-\mathrm{bank})$ and $(\mathrm{c}-\mathrm{market})$, which are necessary for the existence of a solution to the ZPC of lenders when $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, must hold. Moreover, by lemma 4, $R_b = R_b(d,s,e)$.

Under condition $(\mathrm{c}-\mathrm{market})$, there exist exactly two solutions $R_{m,l}(d,s,e) \leq R_{m,L}(d,s,e)$, once of which R_m must be equal to. Since (R_b, R_m) is the least element of $\mathcal{L}(d,s,e)$, it must be the case that $R_m = R_{m,l}(d,s,e)$. Since $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, this in turn imply that condition $(\mathrm{frontier}-\mathrm{B})$ holds. This finishes the proof that if $\frac{R_m}{1-\chi} \leq \frac{R_b}{\chi}$, then $(d,s) \in \tilde{\mathcal{S}}_R(e)$.

Thus, $\mathcal{S}(e) \in \left(\tilde{\mathcal{S}}_K(e) \cup \tilde{\mathcal{S}}_R(e)\right)$.

Step 2-a : Let $(d,s) \in \tilde{S}_R(e)$. We need to prove that $(d,s) \in \mathcal{S}_e$, that is, that $\mathcal{L}(d,s,e)$ is non-empty and has a least element. First, under conditions $(\mathrm{c}-\mathrm{bank})$ and $(\mathrm{c}-\mathrm{market})$, $(R_b(d,s,e), R_{m,l}(d,s,e)) \in \mathcal{L}(d,s,e)$, so that $\mathcal{L}(d,s,e) \neq \emptyset$. If $\mathcal{L}(d,s,e)$ contains only $(R_b(d,s,e), R_m(d,s,e))$, then it is the least element and we are done. Otherwise, let $(R_b, R_m) \in \mathcal{L}(d,s,e)$, $(R_b, R_m) \neq (R_b(d,s,e), R_{m,l}(d,s,e))$. If (R_b, R_m) is such that $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, then, since $(R_b, R_m) \neq (R_b(d,s,e), R_{m,l}(d,s,e))$, it must be the case that $R_b = R_b(d,s,e)$ and $R_m = R_{m,L}(d,s,e)$; therefore, $(R_b, R_m) \geq_\times (R_b(d,s,e), R_{m,l}(d,s,e))$. If, on the other hand, $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$, it must still be the case that $R_b = R_b(d,s,e)$; hence,

$$\frac{R_m}{1-\chi} > \frac{R_b}{\chi} = \frac{R_b(d,s,e)}{\chi} \geq \frac{R_{m,l}(d,s,e)}{1-\chi}.$$

Therefore, $(R_b, R_m) \geq_{\times} (R_b(d,s,e), R_{m,l}(d,s,e))$ in the case $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$ as well. Thus, $\mathcal{L}(d,s,e)$ has a least element, $(R_b(d,s,e), R_{m,l}(d,s,e))$, so that $(d,s) \in \mathcal{S}(e)$. Hence, $\tilde{S}_R(e) \subset \tilde{S}(e)$. Note that since the least element is always unique, I have also established the proposition's claim that the dominating contract (least element) $(R_b = R_b(d,s,e), R_m = R_{m,l}(d,s,e))$ associated to $(d,s) \in \tilde{S}_R(e)$ satisfies $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$.

Step 2-b : Let $(d,s) \in \tilde{S}_B(e) \setminus \left(\tilde{S}_B(e) \cap \tilde{S}_R(e)\right)$. First, under conditions (c − bank) and (c − joint), we have that $(R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e)) \in \mathcal{L}(d,s,e)$, so $\mathcal{L}(d,s,e) \neq \emptyset$. Consdier $(R_b, R_m) \in \mathcal{L}(d,s,e)$, $(R_b, R_m) \neq (R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e))$. If (R_b, R_m) are such that $\frac{R_b}{\chi} \geq \frac{R_m}{1-\chi}$, then, as in step 1, it must be that $(d,s) \in \tilde{S}_R(e)$; but we ruled this out by assumption. So, $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$. In this case, since $(R_b, R_m) \neq (R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e))$, it must be that $R_b = R_b(d,s,e)$ and $R_m = R_L(d,s,e) - R_b(d,s,e)$ (and additionally that $\frac{(1+r_m(1-s)+r_b s)d}{(e+d)^{\varsigma}} > \chi\mathbb{E}(\phi)$, since otherwise the solution to the ZPC of market lenders is unique). So, $(R_b, R_m) \geq_{\times} (R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e))$, and so $(R_b(d,s,e), R_l(d,s,e) - R_b(d,s,e))$ is the least element of $\mathcal{L}(d,s,e)$. This proves that $(d,s) \in \tilde{S}(e)$, and therefore that:

$$\left(\tilde{S}_B(e) \setminus \left(\tilde{S}_B(e) \cap \tilde{S}_R(e)\right)\right) \subset \tilde{S}(e).$$

Note that again because of unicity of the least element, I have also established the lemma's claim that the dominating contract $(R_b = R_b(d,s,e), R_m = R_l(d,s,e) - R_b(d,s,e))$ associated to $(d,s) \in \tilde{S}_B(e) \setminus \left(\tilde{S}_B(e) \cap \tilde{S}_R(e)\right)$ satisfies $\frac{R_b}{\chi} < \frac{R_m}{1-\chi}$.

■

A.4 The lending menu when $\delta = 1$

I next turn to characterizing the set $\mathcal{S}_R(e) = \tilde{\mathcal{S}}_R(e)$ in more detail, in the special case $\delta = 1$, where an analytical characterization can be obtained.

Proposition 9 (A parametrization of the set $\tilde{\mathcal{S}}_R(e)$) *The set $\tilde{\mathcal{S}}_R(e)$ can be described as:*

$$\tilde{\mathcal{S}}_R(e) = \left\{ (d,s) \in \mathbb{R}_+ \times [\underline{s}_R, 1] \;\middle|\; 0 \leq d \leq \overline{d}_R(s,e) \right\},$$

where:

$$\overline{d}_R(s,e) = \begin{cases} \overline{d}_{R,F}(s,e) & \textit{if} \quad \underline{s}_R \leq s < s_{R,1} \\ \overline{d}_{R,m}(s,e) & \textit{if} \quad s_{R,1} \leq s < s_{R,2} \\ \overline{d}_{R,b}(s,e) & \textit{if} \quad s_{R,2} \leq s \leq 1 \end{cases}$$

The thresholds $\underline{s}_R < s_{R,1} < s_{R,2} < 1$ *are given by:*

$$\begin{aligned} \underline{s}_R &= \frac{1}{1+\frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}} \\ s_{R,1} &= \frac{1}{1+\frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}\frac{I(\phi_I)}{G(\phi_I)}} \\ s_{R,2} &= \frac{1}{1+\frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}\frac{I(\phi_I)}{\mathbb{E}(\phi)}} \end{aligned}$$

while the functions $\overline{d}_{R,F}(s,e)$, $\overline{d}_{R,m}(s,e)$ *and* $\overline{d}_{R,b}(s,e)$ *are implicitly defined by:*

$$\begin{aligned} G^{-1}\left(\frac{(1+r_b)s\overline{d}_{R,F}(s,e)}{\chi(e+\overline{d}_{R,F}(s,e))^\zeta}\right) &= I^{-1}\left(\frac{(1+r_m)(1-s)\overline{d}_{R,F}(s,e)}{(1-\chi)(e+\overline{d}_{R,F}(s,e))^\zeta}\right) \\ \frac{(1+r_m)(1-s)\overline{d}_{R,m}(s,e)}{(1-\chi)(e+\overline{d}_R(s,e))^\zeta} &= I(\phi_I) \\ \frac{(1+r_b)s\overline{d}_{R,b}(s,e)}{\chi(e+\overline{d}_R(s,e))^\zeta} &= \mathbb{E}(\phi) \end{aligned}$$

Proof. For this proof, I proceed in four steps:

Step 1: If $(d,s) \in \tilde{\mathcal{S}}_R(e)$, then $\underline{s}_R \leq s \leq 1$;

Step 2: If $\underline{s}_R \leq s < s_{R,1}$, then, $(d,s) \in \tilde{\mathcal{S}}_R(e)$, if and only if, condition (frontier − R) is verified;

Step 3: If $s_{R,1} \leq s < s_{R,2}$, then, $(d,s) \in \tilde{\mathcal{S}}_R(e)$, if and only if, condition (c − market) is verified;

Step 4: If $s_{R,2} \leq s \leq 1$, then, $(d,s) \in \tilde{\mathcal{S}}_R(e)$, if and only if, condition (c − bank) is verified.

Step 1: First, I prove that when $s < \underline{s}_R$, condition (frontier − R) does not hold, so that $\tilde{\mathcal{S}}_R(e)$ cannot containt debt structures with $s < \underline{s}_R$. To see this, first note that $\forall 0 \leq x \leq \phi_I$, $G(x) \geq I(x)$, so that $\forall 0 \leq y \leq I(\phi_I)$, $G^{-1}(y) \leq I^{-1}(y)$ (with equality only at $y = 0$). Note moreover that:

$$s < \underline{s}_R \quad \Longrightarrow \quad \frac{(1+r_b)ds}{\chi(e+d)^\zeta} < \frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}$$

Thus,

$$\begin{aligned} s < \underline{s}_R \quad &\implies \quad G^{-1}\left(\frac{(1+r_b)ds}{\chi(e+d)^\zeta}\right) < G^{-1}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\right) \\ &\implies \quad G^{-1}\left(\frac{(1+r_b)ds}{\chi(e+d)^\zeta}\right) < I^{-1}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\right). \end{aligned}$$

Therefore, $(d,s) \in \tilde{\mathcal{S}}_R(e) \implies s \geq \underline{s}_R$.

Step 2: Next, I prove that when $(d,s) \in \tilde{\mathcal{S}}_R(e)$, $\underline{s}_R \leq s < s_{R,1}$, then only condition (frontier – R) is relevant to the definition of $\tilde{\mathcal{S}}_R(e)$; that is,

$$\Big((d,s) \in \tilde{\mathcal{S}}_R(e) \text{ and } \underline{s}_R \leq s < s_{R,1}\Big) \implies \left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta} < I(\phi_I) \text{ and } \frac{(1+r_b)ds}{\chi(e+d)^\zeta} < \mathbb{E}(\phi)\right). \tag{19}$$

I establish this by proving the contraposition. First, if $\frac{(1+r_b)ds}{\chi(e+d)^\zeta} > \mathbb{E}(\phi)$, then $(d,s) \notin \tilde{\mathcal{S}}_R(e)$. If $\frac{(1+r_b)ds}{\chi(e+d)^\zeta} = \mathbb{E}(\phi)$, then:

$$\begin{aligned} \mathbb{E}(\phi) = \frac{(1+r_b)ds}{\chi(e+d)^\zeta} \quad &< \quad \frac{(1+r_b)s_{R,1}d}{\chi(e+d)^\zeta} \\ &= \quad \frac{(1+r_m)(1-s_{R,1})d}{(1-\chi)(e+d)^\zeta}\frac{G(\phi_I)}{I(\phi_I)} \\ &< \quad \frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\frac{G(\phi_I)}{I(\phi_I)} \end{aligned}$$

Therefore:

$$\frac{(1+r_m)(1-s_{R,1})d}{(1-\chi)(e+d)^\zeta} > \frac{\mathbb{E}(\phi)}{G(\phi_I)}I(\phi_I) > I(\phi_I) \implies (d,s) \notin \tilde{\mathcal{S}}_R(e).$$

Second, if $\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta} > I(\phi_I)$, then $(d,s) \notin \tilde{\mathcal{S}}_R(e)$. If $\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta} = I(\phi_I)$, then:

$$\begin{aligned} I(\phi_I) = \frac{(1+r_b)(1-s)d}{(1-\chi)(e+d)^\zeta} \quad &> \quad \frac{(1+r_m)(1-s_{1,R})d}{(1-\chi)(e+d)^\zeta} \\ &= \quad \frac{(1+r_b)s_{R,1}d}{\chi(e+d)^\zeta}\frac{I(\phi_I)}{G(\phi_I)} \\ &> \quad \frac{(1+r_b)sd}{\chi(e+d)^\zeta}\frac{I(\phi_I)}{G(\phi_I)} \end{aligned}$$

Therefore,

$$\frac{(1+r_b)ds}{\chi(e+d)^\zeta} < G(\phi_I) \implies G^{-1}\left(\frac{(1+r_b)ds}{\chi(e+d)^\zeta}\right) < \phi_I = I^{-1}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\right) \implies (d,s) \notin \tilde{\mathcal{S}}_R(e),$$

which finishes proving statement (19). It is then straightforward to show that condition (frontier $-$ R) is equivalent to $0 \leq d \leq \overline{d}_{R,F}(s,e)$.

Step 3: Next, I prove that when $(d,s) \in \tilde{\mathcal{S}}_R(e)$, $s_{R,1} < s < s_{R,2}$, then only condition (c $-$ market) is relevant to the definition of $\tilde{\mathcal{S}}_R(e)$; that is,

$$\begin{gathered}\Big((d,s) \in \tilde{\mathcal{S}}_R(e) \text{ and } s_{R,1} < s < s_{R,2}\Big) \implies \\ \left(\frac{(1+r_b)ds}{\chi(e+d)^\zeta} < \mathbb{E}(\phi) \text{ and } G^{-1}\left(\frac{(1+r_b)ds}{\chi(e+d)^\zeta}\right) > I^{-1}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\right)\right)\end{gathered} \tag{20}$$

Assume that $s_{R,1} < s < s_{R,2}$ and $(d,s) \in \tilde{\mathcal{S}}_R(e)$, then:

$$\begin{aligned}\frac{(1+r_b)sd}{\chi(e+d)^\zeta} &= \frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}\frac{s}{1-s}\frac{(1+r_m)(1-s)d}{(e+d)^\zeta} \\ &\leq \frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}\frac{s}{1-s}I(\phi_I) \\ &< \frac{1-\chi}{\chi}\frac{1+r_b}{1+r_m}\frac{s_{R,2}}{1-s_{R,2}}I(\phi_I) = \mathbb{E}(\phi)\end{aligned}$$

so that condition (c $-$ bank) is satisfied with strict inequality. I next prove that when $(d,s) \in \tilde{\mathcal{S}}_R(e)$ and $s_{R,1} < s$, then condition (frontier $-$ R) holds with strict inequality. As a first step, define $\Delta(x) = \frac{G(\phi_I)}{I(\phi_I)}I(x) - G(x)$. Note that $\Delta(0) = \Delta(\phi_I) = 0$. Moreover, $\Delta'(x) = \left(\frac{G(\phi_I)}{I(\phi_I)} - 1\right)(1-F(x)) - \frac{G(\phi_I)}{I(\phi_I)}xf(x)$, so that, following steps similar to the proof of lemma 3, $\Delta(.)$ is increasing on $\left[0, \underline{\phi}\right]$ and decreasing on $\left[\underline{\phi}, \phi_I\right]$, where $\underline{\phi} < \phi_I$ is the unique solution to $\Delta'(x) = 0$. Thus,

$$0 \leq x \leq \phi_I \implies \frac{G(\phi_I)}{I(\phi_I)}I(x) \geq G(x),$$

so that:

$$0 \leq y \leq I(\phi_I) \implies G^{-1}\left(\frac{G(\phi_I)}{I(\phi_I)}y\right) \geq I^{-1}(y)$$

Moreover, using the definition of $s_{R,1}$, note that:

$$s > s_{R,1} \quad \implies \quad \frac{(1+r_b)sd}{\chi(e+d)^\zeta} > \frac{G(\phi_I)}{I(\phi_I)}\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}$$

Thus,

$$s > s_{R,1} \implies G^{-1}\left(\frac{(1+r_b)sd}{\chi(e+d)^\zeta}\right) > G^{-1}\left(\frac{G(\phi_I)}{I(\phi_I)}\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\right) \geq I^{-1}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\right),$$

This finishes establishing statement (20). It is then straightforward to show that condition (c − market) is equivalent to $0 \leq d \leq \overline{d}_{R,m}(s,e)$.

Step 4: Finally, I prove that when $(d,s) \in \tilde{\mathcal{S}}_R(e)$, $s_{R,2} < s \leq 1$, then only condition (c − bank) is relevant to the definition of $\tilde{\mathcal{S}}_R(e)$; that is,

$$\begin{gathered}\Big((d,s) \in \tilde{\mathcal{S}}_R(e) \text{ and } s_{R,2} < s < 1\Big) \implies \\ \left(\frac{(1+r_m)d(1-s)}{(1-\chi)(e+d)^\zeta} < I(\phi_I) \text{ and } G^{-1}\left(\frac{(1+r_b)ds}{\chi(e+d)^\zeta}\right) > I^{-1}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\right)\right)\end{gathered} \tag{21}$$

This is straightforward: note that $s > s_{R,2}$ immediately implies the first part of statement (21), while the second part obtains because $s > s_{R,2} > s_{R,1}$. It is then straightforward to show that condition (c − bank) is equivalent to $0 \leq d \leq \overline{d}_{R,b}(s,e)$.

For the thresholds $s_{R,1}$ and $s_{R,2}$, it is simple to establish that when $s = s_{R,1}$, conditions (c − market) and (frontier − R) coincide, and both imply condition (c − bank); while, when $s = s_{R,2}$, conditions (c − market) and (c − bank) coincide, and imply condition (frontier − R). This concludes the proof of the proposition.

■

I next turn to the structure of the set $\tilde{\mathcal{S}}_K(e)$.

Proposition 10 (A parametrization of the set $\tilde{\mathcal{S}}_K(e)$) *The set $\tilde{\mathcal{S}}_K(e)$ can be parametrized as:*

$$\tilde{\mathcal{S}}_K(e) = \left\{ (d,s) \in \mathbb{R}_+ \times [0, \overline{s}_K] \;\middle|\; \underline{d}_K(s,e) < d \leq \overline{d}_K(s,e) \right\},$$

where:

$$\underline{d}_K(s,e) = \begin{cases} 0 & \text{if} \quad 0 \leq s < \underline{s}_R \\ \underline{d}_{K,F}(s,e) & \text{if} \quad \underline{s}_R \leq s \leq \overline{s}_K \end{cases}$$

The threshold $\overline{s}_K$ is given by:

$$\overline{s}_K = \frac{1}{1 + \left(\frac{1}{\chi}\frac{M(\phi_M)}{G(\phi_M)} - 1\right)\frac{1+r_b}{1+r_m}}$$

while the functions $\underline{d}_{K,F}(s,e)$ and $\overline{d}_K(s,e)$ are implicitly defined by:

$$\begin{aligned} G^{-1}\left(\frac{(1+r_b)s\underline{d}_{K,F}(s,e)}{\chi(e+\underline{d}_{K,F}(s,e))^\zeta}\right) &= M^{-1}\left(\frac{(1+(1-s)r_m+sr_b)\,\underline{d}_{K,F}(s,e)}{(e+\underline{d}_{K,F}(s,e))^\zeta}\right) \\ \frac{(1+(1-s)r_m+sr_b)\,\overline{d}_K(s,e)}{(e+\overline{d}_K(s,e))^\zeta} &= M(\phi_M) \end{aligned}$$

Proof. For this proof, I proceed in three steps:

Step 1: If $(d, s) \in \tilde{S}_B(e)$, then $0 \leq s \leq \overline{s}_K$;

Step 2: Let $(d, s) \in \mathbb{R}_+ \times [0, \overline{s}_K]$; then if condition $(\mathrm{c} - \mathrm{joint})$ holds, condition $(\mathrm{c} - \mathrm{bank})$ also holds;

Step 3: Let $(d, s) \in \mathbb{R}_+ \times [0, \overline{s}_K]$; then conditions $(\mathrm{frontier} - \mathrm{R})$ and $(\mathrm{c} - \mathrm{joint})$ are equivalent to, respectively, lower and upper bounds on d.

Step 1: I prove the contraposition. Assume that $s > \overline{s}_K$. Note that in this case,

$$\frac{(1+r_b)s}{\chi} \frac{d}{(e+d)^\zeta} > \frac{G(\phi_M)}{M(\phi_M)} \left[(1+r_m)(1-s) + (1+r_b)s\right] \frac{d}{(e+d)^\zeta},$$

for any $d, e > 0$. Moreover, using steps analogous to the proof of proposition 7, one can establish that, $\forall 0 \leq y \leq M(\phi_M)$, $G^{-1}\left(\frac{G(\phi_M)}{M(\phi_M)} y\right) \geq M^{-1}(y)$. Thus,

$$\begin{aligned} G^{-1}\left(\frac{(1+r_b)sd}{\chi(e+d)^\zeta}\right) &> G^{-1}\left(\frac{G(\phi_M)}{M(\phi_M)} \left[(1+r_m)(1-s) + (1+r_b)s\right] \frac{d}{(e+d)^\zeta}\right) \\ &\geq M^{-1}\left(\left[(1+r_m)(1-s) + (1+r_b)s\right] \frac{d}{(e+d)^\zeta}\right), \end{aligned}$$

where the first line uses the second result above, and the second line uses the first result. This is a violation of $(\mathrm{frontier} - \mathrm{R})$, so $(d, s) \notin \tilde{S}_K(e)$. This proves the result of announced.

Step 2: This result follows from using the fact that, when $s \leq \overline{s}_K$,

$$\frac{(1+r_b)s}{\chi} \frac{d}{(e+d)^\zeta} \leq \frac{G(\phi_M)}{M(\phi_M)} \left[(1+r_m)(1-s) + (1+r_b)s\right] \frac{d}{(e+d)^\zeta}.$$

When condition $(\mathrm{c} - \mathrm{joint})$ holds, we therefore have:

$$\frac{(1+r_b)s}{\chi} \frac{d}{(e+d)^\zeta} \leq G(\phi_M) < \mathbb{E}(\phi)$$

where the last inequality follows from the facts that $G(.)$ is strictly increasing. This shows that condition $(\mathrm{c} - \mathrm{bank})$ holds. Condition $(\mathrm{c} - \mathrm{bank})$ is therefore irrelevant to the definition of $\tilde{S}_B(e)$; only conditions $(\mathrm{frontier} - \mathrm{R})$ and $(\mathrm{c} - \mathrm{joint})$ matter.

Step 3: I first show that the condition (frontier − R) is tantamount to a lower bound on d. First, note that if $0 \leq \underline{s}_B$, then for any $d, e > 0$:

$$\frac{(1+r_b)sd}{\chi(e+d)^\zeta} \quad \leq \quad \frac{(1+r_m(1-s)+r_b s)d}{\chi(e+d)^\zeta}$$

This implies that, when $\frac{(1+r_m(1-s)+r_b s)d}{\chi(e+d)^\zeta} \leq M(\phi_M)$, ie condition (c − joint), then:

$$\begin{aligned} G^{-1}\left(\frac{(1+r_b)sd}{\chi(e+d)^\zeta}\right) \quad &\leq \quad G^{-1}\left(\frac{(1+r_m(1-s)+r_b s)d}{(e+d)^\zeta}\right) \\ &< \quad M^{-1}\left(\frac{(1+r_m(1-s)+r_b s)d}{(e+d)^\zeta}\right) \end{aligned}$$

Thus, condition (frontier − R) is automatically verified. When $\underline{s}_B \leq s \leq \overline{s}_K$, it is straightforward to check that condition (frontier − R) is equivalent to $d \geq \underline{d}_{K,F}(s,e)$, where $\underline{d}_{K,F}(s,e)$ is defined in the statement of the proposition. Finally, it is also straightforward to show that condition (c − joint) corresponds to is equivalent to the upper bound $\overline{d}_K(s,e)$ given in the proposition.

■

The two previous propositions are useful to characterize the two sets that we are actually interested in, that is, the sets $\mathcal{S}_R(e)$ and $\mathcal{S}_K(e)$. Recall that $\mathcal{S}_R(e) = \tilde{\mathcal{S}}_R(e)$, so that the parametrization established in proposition 7 is also characterizes $\tilde{\mathcal{S}}_R(e)$. We are left with the task of describing the intersection $\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e)$. This is the object of the following proposition.

Proposition 11 (A parametrization of the intersection $\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e)$) *The intersection $\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e)$ can be parametrized as:*

$$\tilde{\mathcal{S}}_R(e) \cap \tilde{\mathcal{S}}_K(e) = \left\{ \, (d,s) \in \mathbb{R}_+ \times]\underline{s}_R, \overline{s}_K[\; \middle| \; \underline{d}_K(s,e) \leq d \leq \overline{d}_{R\cap K}(s,e) \, \right\},$$

where:

$$\overline{d}_{R\cap K}(s,e) = \begin{cases} \overline{d}_R(s,e) & \text{if} \quad \underline{s}_R \leq s < s_{R\cap K} \\ \overline{d}_K(s,e) & \text{if} \quad s_{R\cap K} \leq s \leq \overline{s}_K \end{cases}$$

where the threshold $s_{R\cup K}$ is given by:

$$s_{R\cup K} = \frac{1}{1 + \frac{1}{\frac{M(\phi_M)}{(1-\chi)I(\phi_I)} - 1}\frac{1+r_b}{1+r_m}},$$

and satisfies $\underline{s}_R < s_{R\cup K} < s_{R,1}$.

Proof. Note first that the intersection $\tilde{\mathcal{S}}_K(e) \cap \tilde{\mathcal{S}}_R(e)$ may only contain debt structures with $\underline{s}_R \leq s \leq \overline{s}_K$. Given this, the proof of this proposition proceeds in three steps:

Step 1: If $\underline{s}_R \leq s \leq s_{R,1}$, then $\overline{d}_R(s,e) = \underline{d}_K(s,e)$, so that $\tilde{\mathcal{S}}_K(e) \cap \tilde{\mathcal{S}}_R(e)$ contains no elements such that $\underline{s}_R \leq s \leq s_{R,1}$;

Step 2: If $s_{R,1} \leq s \leq \overline{s}_K$, then $\overline{d}_R(s,e) > \underline{d}_K(s,e)$, so that $\tilde{\mathcal{S}}_K(e) \cap \tilde{\mathcal{S}}_R(e)$ contains debt structures (d,s) for each $s_{R,1} \leq s \leq \overline{s}_K$;

Step 3: There is a unique $s_{R,1} < s_{R\cup K} < \overline{s}_K$ such that $\overline{d}_K(s,e) \geq \overline{d}_R(s,e)$, if and only if, $s \leq s_{R\cup K}$.

Step 1: When $\underline{s}_R \leq s \leq s_{R,1}$, using proposition 7, $\overline{d}_R(s,e)$ solves:

$$I^{-1}\left(\frac{(1+r_m)(1-s)\overline{d}_R(s,e)}{(1-\chi)(e+\overline{d}_R(s,e))^{\zeta}}\right) = G^{-1}\left(\frac{(1+r_b)s\overline{d}_R(s,e)}{\chi(e+\overline{d}_R(s,e))^{\zeta}}\right).$$

This implies that:

$$\begin{aligned} x_1 \equiv \frac{(1+r_m)(1-s)\overline{d}_R(s,e)}{(1-\chi)(e+\overline{d}_R(s,e))^{\zeta}} \quad &\leq \quad I(\phi_I) \\ x_2 \equiv \frac{(1+r_b)s\overline{d}_R(s,e)}{\chi(e+\overline{d}_R(s,e))^{\zeta}} \quad &\leq \quad G(\phi_I) \end{aligned}$$

Therefore,

$$\begin{aligned} (1-\chi)x_1 + \chi x_3 = \frac{(1+r_m(1-s)+r_b s)\,\overline{d}_R(s,e)}{\left(e+\overline{d}_R(s,e)\right)^{\zeta}} \quad &\leq \quad (1-\chi)I(\phi_I) + \chi G(\phi_I) \\ &= \quad M(\phi_I) \\ &< \quad M(\phi_M) \end{aligned}$$

Thus, $M^{-1}\left((1-\chi)+x_1+\chi x_2\right)$ is well-defined. Furthermore:

$$\begin{aligned} I^{-1}(x_1) = G^{-1}(x_2) \quad &\Longrightarrow \quad x_2 = G\left(I^{-1}(x_1)\right) \\ &\Longrightarrow \quad x_2 = x_1 + \int_0^{I^{-1}(x_1)} \phi dF(\phi) \\ &\Longrightarrow \quad \chi x_2 + (1-\chi)x_1 = M\left(I^{-1}(x_1)\right) \\ &\Longrightarrow \quad M^{-1}\left(\chi x_2 + (1-\chi)x_1\right) = I^{-1}(x_1) = G^{-1}(x_2) \end{aligned}$$

Therefore, $\overline{d}_R(s,e)$ also solves:

$$M^{-1}\left(\frac{(1+r_m(1-s)+r_b s)\,\overline{d}_R(s,e)}{\left(e+\overline{d}_R(s,e)\right)^{\zeta}}\right)=G^{-1}\left(\frac{(1+r_b)s\overline{d}_R(s,e)}{\chi(e+\overline{d}_R(s,e))^{\zeta}}\right),$$

so that $\overline{d}_R(s,e)=\underline{d}_K(s,e)$. This concludes the proof of step 1.

Step 2: For $s_{R,1}\le s\le \overline{s}_K$, $\underline{d}_K(s,e)$ solves:

$$M^{-1}\left(\frac{(1+r_m(1-s)+r_b s)\,\underline{d}_K(s,e)}{\left(e+\underline{d}_K(s,e)\right)^{\zeta}}\right)=G^{-1}\left(\frac{(1+r_b)s\underline{d}_K(s,e)}{\chi(e+\underline{d}_K(s,e))^{\zeta}}\right),$$

while $\overline{d}_R(s,e)$ solves one of the two conditions:

$$\begin{aligned}\frac{(1+r_m)(1-s)\overline{d}_R(s,e)}{(1-\chi)(e+\overline{d}_R(s,e))^{\zeta}} &= I(\phi_I)\\ &\text{or}\\ \frac{(1+r_b)s\overline{d}_R(s,e)}{\chi(e+\overline{d}_R(s,e))^{\zeta}} &= \mathbb{E}(\phi).\end{aligned}$$

In order to show that $\overline{d}_R(s,e)>\underline{d}_K(s,e)$, we need to prove that at $d=\underline{d}_K(s,e)$, we have:

$$\begin{aligned}\frac{(1+r_m)(1-s)\underline{d}_K(s,e)}{(1-\chi)(e+\underline{d}_K(s,e))^{\zeta}} &< I(\phi_I)\\ &\text{and}\\ \frac{(1+r_b)s\underline{d}_K(s,e)}{\chi(e+\underline{d}_K(s,e))^{\zeta}} &< \mathbb{E}(\phi).\end{aligned}$$

The second condition follows from the fact that because of the definition of $\underline{d}_K(s,e)$, it must be the case that:

$$x_2\equiv\frac{(1+r_b)s\underline{d}_K(s,e)}{\chi(e+\underline{d}_K(s,e))^{\zeta}}\le G\left(\phi_M\right)<\mathbb{E}(\phi).$$

Furthermore, let:

$$x_3=\frac{(1+r_m(1-s)+r_b s)\underline{d}_K(s,e)}{\left(e+\underline{d}_K(s,e)\right)^{\zeta}}=M\left(G^{-1}(x_2)\right)$$

Then:

$$\begin{aligned}x_1\equiv\frac{(1+r_m)(1-s)\underline{d}_K(s,e)}{(1-\chi)(\underline{d}_K(s,e)+e)^{\zeta}} &= \frac{1}{1-\chi}\left(x_3-\chi x_2\right)\\ &= \frac{1}{1-\chi}\left(M\left(G^{-1}(x_2)\right)-\chi x_2\right)\end{aligned}$$

Note that $\forall 0 \leq x \leq \phi_M$, $M(x) = \chi G(x) + (1-\chi)x(1-F(x))$. Therefore:

$$\begin{aligned}
M\left(G^{-1}(x_2)\right) &= \chi x_2 + (1-\chi)G^{-1}(x_2)\left(1 - F\left(G^{-1}(x_2)\right)\right) \\
M\left(G^{-1}(x_2)\right) - \chi x_2 &= (1-\chi)G^{-1}(x_2)\left(1 - F\left(G^{-1}(x_2)\right)\right) \\
x_1 &= G^{-1}(x_2)\left(1 - F\left(G^{-1}(x_2)\right)\right) < I(\phi_I)
\end{aligned}$$

where the last inequality follows because $I(x) = x(1-F(x))$ is maximized at ϕ_I. This concludes the proof of step 2.

Step 3: Define $s_{R\cap K}$ as in the text of the proposition. I first prove that $\hat{s}_{R,1} < s_{R\cap K} < \hat{s}_{R,2}$. Note that $s_{R,1} < s_{R\cap K}$, if and only if:

$$M(\phi_M) > \chi G(\phi_I) + (1-\chi)I(\phi_I).$$

This statement is true because $M(\phi_M) > M(\phi_I) = \chi G(\phi_I) + (1-\chi)I(\phi_I)$. Next, $s_{R\cap K} < s_{R,2}$, if and only if,

$$M(\phi_M) > \chi \mathbb{E}(\phi) + (1-\chi)I(\phi_I).$$

To prove this inequality, define $\Delta(\chi) = M(\phi_M) - \chi\mathbb{E}(\phi) - (1-\chi)I(\phi_I)$ for $\chi \in [0,1]$. Note that:

$$\begin{aligned}
\Delta(0) = \Delta(1) &= 0 \\
\frac{\partial \Delta}{\partial \chi} &= \int_0^{\phi_M} x dF(x) - \mathbb{E}(\phi) + I(\phi_I) \\
\frac{\partial \Delta}{\partial \chi}|_{\chi=0} &= G(\phi_I) - \mathbb{E}(\phi) < 0 \\
\frac{\partial \Delta}{\partial \chi}|_{\chi=1} &= I(\phi_I) > 0 \\
\frac{\partial^2 \Delta}{\partial \chi^2} &= \frac{\partial \phi_M}{\partial \chi}\phi_M f(\phi_M) > 0
\end{aligned}$$

Taken together, these properties imply that $\Delta(\chi)$ is strictly convex on $[0,1]$, and that $\forall \chi \in]0,1]$, $\Delta(\chi) < 0$, so that the initial inequality holds, and $s_{R\cap K} < s_{R,2}$.

Next, I prove that $\overline{d}_K(s,e) \geq \overline{d}_R(s,e)$, if and only if, $s \leq s_{R\cup K}$. First note that by definition of $s_{R\cap K}$, $\overline{d}_B(s_{R\cap K}) = \overline{d}_R(s_{R\cap K})$. To prove the statement, it is therefore sufficient to show that $\overline{d}_K(s,e) - \overline{d}_R(s,e)$ is strictly decreasing in s. To show this, I establish that $\overline{d}_K(s,e)$ is strictly decreasing in s, and that $\overline{d}_R(s,e)$ is strictly increasing in s. The implicit definition of $\overline{d}_K(s,e)$ implies that:

$$\frac{\partial \overline{d}_K(s,e)}{\partial s} = \frac{-\overline{d}_K(s,e)(r_b - r_m)}{1 + (1-s)r_m + sr_b - \zeta M(\phi_M)(e + \overline{d}_K(s,e))^{\zeta}}$$

$$= \frac{-\overline{d}_K(s,e)(e+\overline{d}_K(s,e))(r_b - r_m)}{(1+(1-s)r_m + sr_b)e + (1-\zeta)M(\phi_M)(e+\overline{d}_K(s,e))^\zeta} < 0,$$

where the last inequality follows from the fact that $r_b > r_m$. Likewise, using the implicit definition of $\overline{d}_R(s,e)$, one obtains that:

$$\frac{\partial \overline{d}_R(s,e)}{\partial s} = \frac{\overline{d}_R(s,e)(e+\overline{d}_R(1+r_m)}{\frac{(1+r_m)(1-s)}{1-\chi}e + (1-\zeta)I(\phi_I)(e+\overline{d}_R(s,e))^\zeta} > 0$$

This concludes the proof of step 3.

Step 3 shows that elements of $\tilde{\mathcal{S}}_K(e) \cap \tilde{\mathcal{S}}_R(e)$ with $s < s_{R\cup K}$ are therefore exactly those with $\underline{d}_K(s,e) \leq s \leq \overline{d}_R(s,e)$; while elements of $\tilde{\mathcal{S}}_K(e) \cap \tilde{\mathcal{S}}_R(e)$ with $s \geq s_{R\cup K}$ are *all elements* of $\tilde{S}_B(e)$, that is, those with $\underline{d}_K(s,e) \leq s \leq \overline{d}_K(s,e)$. This is the main statement of proposition 10.

■

B Appendix to section 3

Proof of proposition 4. Assume that $(b = sd, m = (1-s)d) \in \mathcal{S}_K(e)$ and that $((1+r_m)(1-s) + (1+r_b)s)d \geq (1-\delta)(e+d)$; in that case, using the expression of the return function of the entrepreneur under a K-contract reported in appendix A:

$$\mathbb{E}\left[\tilde{\pi}(\phi; e, sd, (1-s)d)\right] = \left(\int_{\phi_K}^{+\infty} \phi dF(\phi)\right)(e+d)^\zeta - \phi_K(1-F(\phi_K))(e+d)^\zeta$$

where:

$$\phi_K = \frac{R_K(d,s,e) - (1-\delta)(e+d)}{(e+d)^\zeta}.$$

Using the results of lemmas 4-6, we then have that:

$$\begin{aligned}
& \mathbb{E}\left[\tilde{\pi}(\phi; e, sd, (1-s)d)\right] + \tilde{E}_{m,K}(R_b, R_m; e+d) + \tilde{E}_b(R_b; e+d) \\
= \quad & \left(\int_{\phi_K}^{+\infty} \phi dF(\phi)\right)(e+d)^\zeta - \phi_K(1-F(\phi_K))(e+d)^\zeta \\
& +\chi\left(\int_0^{\phi_K} \phi dF(\phi)\right)(e+d)^\zeta - (1-\chi)F(\phi_K)(1-\delta)(e+d) \\
= \quad & \mathbb{E}(\pi(\phi)) - (1-\chi)\int_0^{\phi_K} \pi(\phi) dF(\phi).
\end{aligned}$$

Since $\tilde{\mathbb{E}}_{m,K}(R_b, R_m; e+d) = (1+r_m)(1-s)d$ and $\tilde{\mathbb{E}}_b(R_b; e+d) = (1+r_b)sd$, this in turn implies that:

$$\mathbb{E}\left[\tilde{\pi}(\phi; e, sd, (1-s)d)\right] = \mathbb{E}(\pi(\phi)) - (1-\chi)\int_0^{\phi_K} \pi(\phi)dF(\phi) - ((1+r_m)(1-s) + (1+r_b)s)d.$$

When instead $((1+r_m)(1-s) + (1+r_b)s)d < (1-\delta)(e+d)$, the entrepreneur never defaults, and moreover $R_b = (1+r_b)sd$ and $R_m = (1+r_m)(1-s)d$. Thus:

$$\mathbb{E}\left[\tilde{\pi}(\phi; e, sd, (1-s)d)\right] = \mathbb{E}(\pi(\phi)) - (1+r_m)(1-s)d - (1+r_b)sd.$$

This proves the lemma's claim in the case $(b = sd, m = (1-s)d) \in S_K(e)$, with:

$$\overline{\phi}(e, d, s) = \frac{R_K(d, s, e) - (1-\delta)(e+d)}{(e+d)^{\zeta}},$$

where note that $\overline{\phi}(e, b, m) = 0$, if and only if, $(1-\delta)(e+b+m) \geq (1+r_b)b + (1+r_m)m$.

The proof is similar when $(b = sd, m = (1-s)d) \in \mathcal{S}_R(e)$, with in that case:

$$\overline{\phi}(e, d, s) = \frac{R_{m,l}(d, s, e) - (1-\chi)(1-\delta)(e+d)}{(1-\chi)(e+d)^{\zeta}}.$$

■

Proof of proposition 5.

Consider, first, the sub-problem of a firm with internal finance e restricted to use contracts in $\mathcal{S}_R(e)$. Abusing somewhat the notation of propositions 6 and 4, this problem can be written as:

$$\max_{(d,s)} \quad O_R(e, d, s) = \mathbb{E}(\pi(\phi)) - (1+r_b)sd - (1+r_m)(1-s)d - (1-\chi)\int_0^{\underline{\phi}(e,d,s)} \pi(\phi)dF(\phi)$$

$$\text{s.t. } \underline{s}_R \leq s \leq 1 \text{ and } 0 \leq d \leq \overline{d}_R(s, e)$$

The Lagrangian associated with this problem is:

$$\mathcal{L} = O_R(e, d, s) + \overline{\lambda}(\bar{d}(s, e) - d) + \underline{\lambda} d + \overline{\mu}(1-s) + \underline{\mu}(s - \underline{s}_R).$$

A first-order necessary conditions for optimality is:

$$\frac{\partial O_R}{\partial s} + \frac{\partial \bar{d}_R}{\partial s}\overline{\lambda} = (\overline{\mu} - \underline{\mu}),$$

where the derivative of the objective function with respect to s is given by:

$$\frac{\partial O_R}{\partial s} = (r_m - r_b)d + \underline{\phi}(e,d,s)f(\underline{\phi}(e,d,s))(1+r_m)d(1-s)\frac{\partial I^{-1}}{\partial y}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^{\zeta}}\right).$$

Since $\frac{\partial I^{-1}}{\partial y} > 0$, we have that $\frac{\partial O_R}{\partial s} > 0$ if $r_b - r_m < 0$. Note, then, that if $\underline{s} \leq s \leq s_{R,M}$, then by theorem 6, $\frac{\partial \bar{d}_R}{\partial s} \geq 0$, so that $\overline{\mu} > 0$ and necessarily $s = 1$. This is a contradiction because $s_{R,M} < 1$. So, when $r_b - r_m < 0$, any solution to the restricted problem above must satisfy $s_{R,M} \leq s \leq 1$.

Then, there are two possibilities:

- Either the bank borrowing constraint is binding, that is, $\overline{\lambda} > 0$; in that case, it need not be that $\bar{\mu} > 0$, that is, the firm may still be borrowing from both sources although it is exhausting its bank borrowing capacity;

- The bank borrowing constraint is loose, that is $\overline{\lambda} = 0$. In that case, it must be that $\overline{\mu} > 0$, so that $s = 1$ and the firm is borrowing only from banks.

This establishes that, provided that the solution of the firm's problem is an element of $\mathcal{S}_R(e)$, then the firm is either entirely bank-financed, or its bank borrowing constraint is binding.

I next show that the solution to the firm's problem is necessarily an element of $\mathcal{S}_R(e)$. To establish this, let $\tilde{d}$ be a scale of borrowing that is feasible in $\mathcal{S}_K(e)$, that is, such that there exists $\tilde{s} \in [0, \bar{s}_K]$ such that $(\tilde{d}, \tilde{s}) \in \mathcal{S}_K(e)$. First, note that, for any s such that $\tilde{d}$ is feasible in $\mathcal{S}_K(e)$:

$$\frac{\partial O_K}{\partial s}(e,\tilde{d},s) = (r_m - r_b)\tilde{d}\left(1 + (1-\chi)\underline{\phi}(\tilde{d},\tilde{s},e)f\left(\underline{\phi}(\tilde{d},\tilde{s},e)\right)\frac{\partial M^{-1}}{\partial y}\left(\frac{(1+r_m(1-s)+r_b s)\tilde{d}}{(e+\tilde{d})^{\zeta}}\right)\right) > 0, \quad (22)$$

where the latter inequality holds because $r_b < r_m$.

Furthermore, using the results of proposition 2, the joint borrowing constraint defining $\mathcal{S}_K(e)$ can be written as:

$$0 \leq (1 + r_b s + r_m(1-s))d \leq \hat{M}(e+d)^{\zeta}$$

When $r_b < r_m$, if $(\tilde{d}, \tilde{s})$ satisfy this constraint, then $(\tilde{d}, s)$ also satisfy it for any $s \geq \tilde{s}$. Therefore, all the points on the line defined, in (d,s) space, by $d = \tilde{d}$ (that is, an anti-diagonal in (b,m) where $b+m$ is a constant) are in the set $S_K(e)$ so long as they satisfy condition (frontier-K) from proposition 2. On this line, the objective function O_K is increasing, as noted above. Finally, it is straightforward to show that, when the condition (frontier-K) is binding:

$$\underline{\phi}_K(e,\tilde{d},s) = M^{-1}\left(\frac{(1+r_b s + r_m(1-s))\tilde{d}}{(e+\tilde{d})^{\zeta}}\right) = I^{-1}\left(\frac{(1+r_m(1-s))\tilde{d}}{(1-\chi)(e+\tilde{d})^{\zeta}}\right) = \underline{\phi}_R(e,\tilde{d},s),$$

so that at the frontier between the two sets, $O_K(\tilde{d}, s, e) = O_R(\tilde{d}, s, e)$. Thus, the value of the firm at the point $(\tilde{d}, s) \in \mathcal{S}_R(e)$ at the intersection of the line $d = \tilde{d}$ and the frontier between the sets $S_K(e)$ and $S_R(e)$, is weakly larger than at any point on that line within the set $S_K(e)$. Since I established this for any $\tilde{d}$ feasible in $S_K(e)$, this shows that the global maximum to the firms' problem is in $S_R(e)$, and concludes the proof. ■

Proofs of propositions 6 and 7

In order to establish the different claims of propositions 6 and 7, I first characterize the solution to the two subproblems:

$$\begin{aligned}
\max_{(d,s)} \quad O_K(e,d,s) \quad &= \quad \mathbb{E}(\pi(\phi)) - (1+r_b)sd - (1+r_m)(1-s)d - (1-\chi)\int_0^{\underline{\phi}_K(e,d,s)} \pi(\phi)dF(\phi) \\
\text{s.t.} \quad & \quad 0 \leq (1+r_b)sd + (1+r_m)(1-s)d \leq \hat{M}(e+d)^\zeta \\
& \quad G^{-1}\left(\frac{(1+r_b)sd}{\chi(e+d)^\zeta}\right) < M^{-1}\left(\frac{(1+r_b s + r_m(1-s))d}{(e+d)^\zeta}\right)
\end{aligned} \tag{23}$$

$$\begin{aligned}
\max_{(d,s)} \quad O_R(e,d,s) \quad &= \quad \mathbb{E}(\pi(\phi)) - (1+r_b)sd - (1+r_m)(1-s)d - (1-\chi)\int_0^{\underline{\phi}_R(e,d,s)} \pi(\phi)dF(\phi) \\
\text{s.t.} \quad & \quad 0 \leq (1+r_b)sd \leq \chi\mathbb{E}(\phi)(e+d)^\zeta \\
& \quad 0 \leq (1+r_m)(1-s)d \leq (1-\chi)\hat{I}(e+d)^\zeta \\
& \quad I^{-1}\left(\frac{(1+r_m)(1-s)d}{(1-\chi)(e+d)^\zeta}\right) < G^{-1}\left(\frac{(1+r_b)sd}{\chi(e+d)^\zeta}\right)
\end{aligned} \tag{24}$$

Characterization of the solution to subproblem (23) Let (d, s) be a feasible debt structure in subproblem (23) with $s > 0$. Then, note that $G(0) < M^{-1}\left(\frac{(1+r_m)d}{(e+d)^\zeta}\right)$, so that $(d, 0)$ is also feasible. Furthermore, $O_K(e, d, 0) > O_K(e, d, s)$ because $\frac{\partial O_K}{\partial s} > 0$ when $r_b > r_m$, as indicated by equation (22). Thus, the maximum of subproblem (23) necessarily satisfies $\hat{s} = 0$, that is, is entirely market-financed. Moreover, the optimal amount of borrowing $\hat{d}$ is necessarily an interior solution, i.e. satisfies $(1+r_m)\hat{d} < \hat{M}(e+\hat{d})^\zeta$, so long as $e < \hat{k}_m$, where $\hat{k}_m$ is the unconstrained optimal size of the firm when liquidation is costless, as defined in lemma 2. To establish this, note that when $e < \hat{k}_m$:

$$\frac{\partial O_K}{\partial d}(e, 0, 0) = \zeta E(\phi)(e+d)^{\zeta-1} - (1+r_m) > 0.$$

Furthermore, the derivative of the objective function is give by:

$$\frac{\partial O_K}{\partial d}(e,d,0) \quad = \quad \zeta E(\phi)(e+d)^{\zeta-1} - (1+r_m) - (1-\chi)\zeta(e+d)^{\zeta-1}\int_0^{\underline{\phi}_K(e,d,s)} \phi dF(\phi) \tag{25}$$

$$-(1+r_m)(1-\chi)\underline{\phi}_K(e,d,s)f\left(\underline{\phi}_K(e,d,s)\right)\frac{\partial M^{-1}}{\partial y}\left(\frac{(1+r_m)d}{(e+d)^{\zeta}}\right)\left[1-\zeta\frac{d}{e+d}\right]$$

In this last expression, all terms are bounded as $d \to \overline{d}_K(e)$, except the derivative $\frac{\partial M^{-1}}{\partial y}$, since $\lim_{y\to\hat{M}} \frac{\partial M^{-1}}{\partial y} = +\infty$. Thus, $\lim_{d\to\overline{d}_K(e)} \frac{\partial O_K}{\partial d}(e,d,0) = -\infty$. This establishes that the maximum in subproblem (23) is always such that $0 < \hat{d} < \overline{d}_K(e)$, and furthermore that it is attained when the derivative in equation (25) is equal to 0.

Characterization of the solution to subproblem (24) I start the analysis of subproblem (24) by characterizing its unconstrained (”interior”) solution (d_{int}, s_{int}). Manipulating the necessary first-order condition $\frac{\partial O_R}{\partial s} = 0$, the unconstrained solution must statisfy:

$$I^{-1}\left(\frac{(1+r_m)s_{int}d_{int}}{(1-\chi)(e+d_{int})^{\zeta}}\right) = \phi_{int}$$

where ϕ_{int} is the unique solution to:

$$\frac{\phi_{int}f\left(\phi_{int}\right)}{1-F\left(\phi_{int}\right)} = \frac{r_b-r_m}{1+r_m}.$$

Using this expression along with the necessary first-order condition $\frac{\partial O_R}{\partial d} = 0$, the interior solution is given by:

$$\begin{aligned}
\hat{s}_{int}(e) &= 1-\frac{\Gamma}{1+r_m}\frac{\hat{k}_{int}^{\zeta}}{\hat{k}_{int}-e} \\
\hat{d}_{int}(e) &= \hat{k}_{int}-e \\
\hat{O}_{R,int}(e) &= \Xi(1-\zeta)\hat{k}_{int}^{\zeta}+(1+r_b)e \\
\hat{k}_{int} &= \left(\frac{\zeta\Xi}{1+r_b}\right)^{\frac{1}{1-\zeta}} \\
\Gamma &= (1-\chi)I\left(\phi_{int}\right) \\
\Xi &= \mathbb{E}(\phi)+(1-\chi)\left[\frac{r_b-r_m}{1+r_m}I\left(\phi_{int}\right)-\int_0^{\phi_{int}}\phi dF(\phi)\right]
\end{aligned}$$

Provided that the interior solution does not violate any of the constraints in subproblem (24), it is the maximum of this problem. Note that, since $\phi_{int} < \hat{\phi}_I$, the second inequality in this problem (the market borrowing constraint) always holds strictly, so we need only study the two other inequalities, the bank borrowing constraint and the frontier with the set $S_K(e)$.[28]

By evaluating the bank borrowing constraint at the interior solution, one can establish that the bank

[28] The inequality $\phi_{int} < \hat{\phi}_I$ is true so long as $r_b < 1 + 2r_m$, which it will under any reasonable calibration of the model.

borrowing constraint is binding at the interior solution, if and only if, $e \leq \tilde{e}$, where:

$$\tilde{e} = \hat{k}_{int} \left(1 - \frac{1}{\zeta} \frac{1}{1 + \frac{(1-\chi)(\mathbb{E}(\phi) - G(\phi_{int}))}{\chi \mathbb{E}(\phi) + (1-\chi)\frac{1+r_b}{1+r_m} I(\phi_{int})}} \right) \tag{26}$$

Along similar lines, the third constraint in subproblem (24), the frontier condition, is binding at the interior solution, if and only if $e > \tilde{e}^F$, where:

$$\tilde{e}^F = \hat{k}_{int} \left(1 - \frac{1}{\zeta} \frac{\frac{\chi G(\phi_{int}) + (1-\chi)\frac{1+r_b}{1+r_m} I(\phi_{int})}{\chi \mathbb{E}(\phi) + (1-\chi)\frac{1+r_b}{1+r_m} I(\phi_{int})}}{1 + \frac{(1-\chi)(\mathbb{E}(\phi) - G(\phi_{int}))}{\chi \mathbb{E}(\phi) + (1-\chi)\frac{1+r_b}{1+r_m} I(\phi_{int})}} \right) > \tilde{e} \tag{27}$$

Then, there are three possible cases:

- If $0 \leq e \leq \tilde{e}$, then, at the optimum of subproblem (24), the bank borrowing constraint is binding. (Indeed, note that, as long as a point satisfies the bank borrowing constraint, it also satisfies the third constraint in subproblem (24), the frontier condition);
- If $\tilde{e} \leq e \leq \tilde{e}^F$, then the optimum of the subproblem is the interior solution described above;
- If $e > \tilde{e}^F$, then at the optimum of the subproblem (24), the third constraint in subproblem (24), the frontier condition, is binding.

Note that, as in the proof of proposition 4, it is straightforward to show that when $r_b > r_m$, $\frac{\partial O_r}{\partial s} < 0$ on the frontier defined by the third condition in subproblem (24). Thus, when $e > \tilde{e}^F$, the solution to subproblem (24) is dominated, locally, by a feasible point in subproblem (23), and therefore cannot be a the global solution to the firm's problem. This indicates that the threshold for switching from mixed to market-finance e^* must satisfy $e^* < \tilde{e}^F$.

The switching threshold e^* In what follows, I prove that $\hat{O}_R(0) > \hat{O}_K(0)$. Since, as established before, $\hat{O}_R\left(\tilde{e}^F\right) < \hat{O}_K\left(\tilde{e}^F\right)$, and since both maximized functions are continuous in e, this establishes that there exists a switching threshold e^* such that: $\hat{O}_R(e^*) = \hat{O}_K(e^*)$.

Note first that the objective function in the subproblem (23) can be written as:

$$O_K(d, 0, 0) = \left(E(\phi) - G\left(M^{-1}\left((1+r_m)d^{1-\zeta}\right)\right)\right) d^{\zeta}.$$

This objective function is defined for $0 \leq d \leq \overline{d}_K(0)$, or:

$$0 \leq d \leq \left(\frac{\hat{M}}{1+r_m}\right)^{\frac{1}{1-\zeta}}.$$

Alternatively, using the change of variable $\phi_K = M^{-1}\left((1+r_m)d^{1-\zeta}\right)$, this problem can reformulated as:

$$\max_{\phi_K \in [0,\phi_M]} \left[\int_{\phi_K}^{+\infty} (1-F(\phi))\, d\phi\right]\left[\frac{1}{1+r_m} M(\phi_K)\right]^{\frac{\zeta}{1-\zeta}}.$$

The first order necessary condition for this problem is:

$$\frac{\zeta}{1-\zeta}\left[\int_{\phi_K}^{+\infty} (1-F(\phi))\, d\phi\right]\left[1-(1-\chi)\frac{\phi_K f(\phi_K)}{1-F(\phi_K)}\right] = M(\phi_K). \tag{28}$$

In a similar manner, the objective function in the subproblem (24), when the bank borrowing constraint binds, can be written as:

$$O_R(d,0,0) = (1-\chi)\left(E(\phi) - G\left(I^{-1}\left(\frac{1+r_m}{1-\chi}d^{1-\zeta} - \frac{1+r_m}{1+r_b}\frac{\chi}{1-\chi}\mathbb{E}(\phi)\right)\right)\right)d^{\zeta}.$$

Given the results of proposition 9, this objective function is defined for $1 \geq s \geq s_{R,2}$, or (given that the borrowing constraint binds):

$$\left(\frac{\chi\mathbb{E}(\phi)}{1+r_b}\right)^{\frac{1}{1-\zeta}} \leq d \leq \left(\frac{\chi\mathbb{E}(\phi)}{1+r_b} + \frac{(1-\chi)I(\phi_I)}{1+r_m}\right)^{\frac{1}{1-\zeta}}.$$

Using the change of variable $\phi_R = I^{-1}\left(\frac{1+r_m}{1-\chi}d^{1-\zeta} - \frac{1+r_m}{1+r_b}\frac{\chi}{1-\chi}\mathbb{E}(\phi)\right)$, this problem can reformulated as:

$$\max_{\phi_R \in [0,\phi_I]} (1-\chi)\left[\int_{\phi_R}^{+\infty} (1-F(\phi))\, d\phi\right]\left[\frac{\chi\mathbb{E}(\phi)}{1+r_b} + \frac{(1-\chi)I(\phi_R)}{1+r_m}\right]^{\frac{\zeta}{1-\zeta}}.$$

The first order necessary condition for this problem is:

$$\frac{\zeta}{1-\zeta}(1-\chi)\left[\int_{\phi_R}^{+\infty} (1-F(\phi))\, d\phi\right]\left[1-\frac{\phi_R f(\phi_R)}{1-F(\phi_R)}\right] = (1-\chi)I(\phi_R) + \frac{1+r_m}{1+r_b}\chi\mathbb{E}(\phi). \tag{29}$$

Given the assumption that $F(.)$ has a strictrly increasing hazard rate, the left hand side of 29 is strictly decreasing, and moreover it always strictly smaller than the left hand side of 28 (which is itself strictly decreasing). Since $M(\phi) = (1-\chi)I(\phi) + \chi G(\phi) \leq (1-\chi)I(\phi) + \chi\mathbb{E}(\phi)$, the right hand side of 29 is always strictly larger than the right hand side of 28; both are moreover strictly increasing. This implies that:

$\hat{\phi}_K > \hat{\phi}_R$ (ie, the optimal K-debt structure results in more frequent liquidation). Moreover, it also implies that: $(1-\chi)I(\hat{\phi}_R) + \frac{1+r_m}{1+r_b}\chi\mathbb{E}(\phi) > M(\hat{\phi}_K)$ (the optimal R-debt structure involves firms operating at larger scales). Combining the two facts and using the expressions of the objective functions then establishes that $\hat{O}_R(0) > \hat{O}_K(0)$.

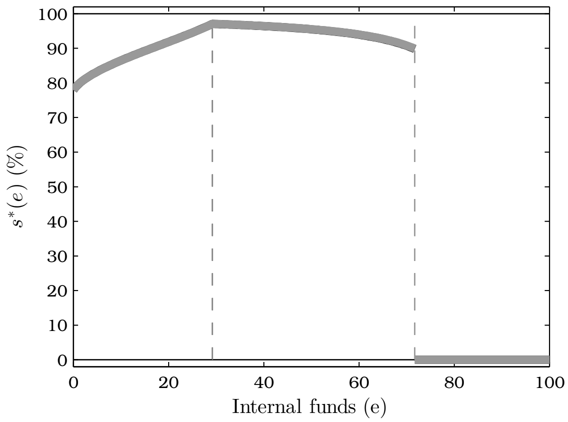
C Appendix to section 4

C.1 Calibrations

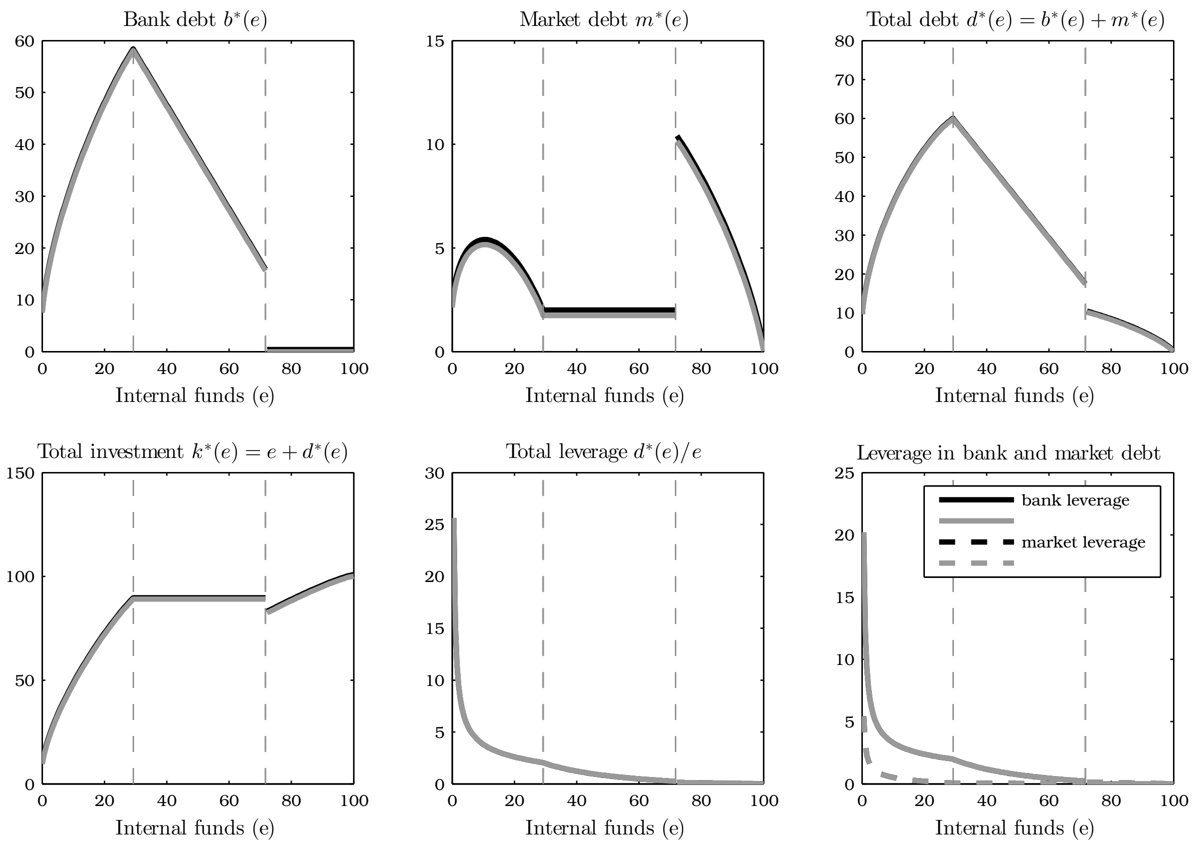
	Baseline	**High cost spread**	**High cost levels (1)**	**High cost levels (2)**	**Low average productivity**	**High productivity dispersion**
Parameters:						
r_m	0.05	0.05	0.06	0.06	0.05	0.05
r_b	0.06	0.07	0.07	0.07	0.06	0.06
χ	0.60	0.60	0.60	0.60	0.60	0.60
ξ	1.60	1.60	1.60	1.60	1.60	1.37
λ	1.84	1.84	1.86	1.84	1.82	1.82
ζ	0.92	0.92	0.92	0.92	0.92	0.92
Moments of $F(.)$:						
$\mathbb{E}(\phi)$	1.65	1.65	1.67	1.65	1.63	1.65
$\sigma(\phi)$	1.06	1.06	1.07	1.06	1.05	1.22
$\frac{\sigma(\phi)}{\mathbb{E}(\phi))}$	0.64	0.64	0.64	0.64	0.64	0.74
Shape of solution:						
$s^*(0)$	0.78	0.78	0.78	0.78	0.78	0.79
$\tilde{e}$	29.19	25.20	29.24	25.97	25.81	29.81
$s^*(\tilde{e})$	0.97	0.96	0.97	0.97	0.97	0.98
e^*	71.90	55.87	72.08	64.18	64.03	74.85
$s^*(e^*)$	0.90	0.90	0.90	0.90	0.90	0.92
k^*	88.94	79.31	89.04	79.28	79.09	88.90
$\overline{e}$	100.00	100.00	100.00	88.83	88.40	100.00

Table 1: Calibrations of the model discussed in section 4. The parameters ξ and λ are the location and scale parameters of the Weibull distribution. e^* refers to the level of equity such that the firm will choose not to borrow from either sources.

C.2 Comparative statics

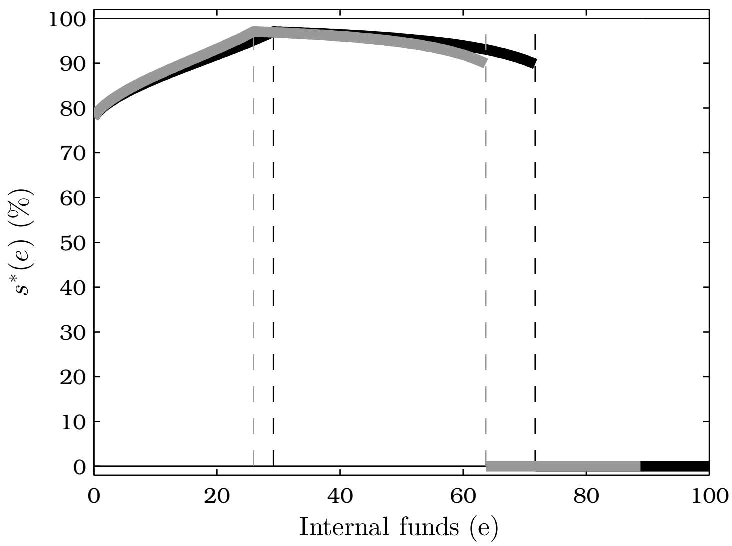


(a) Change in the bank share

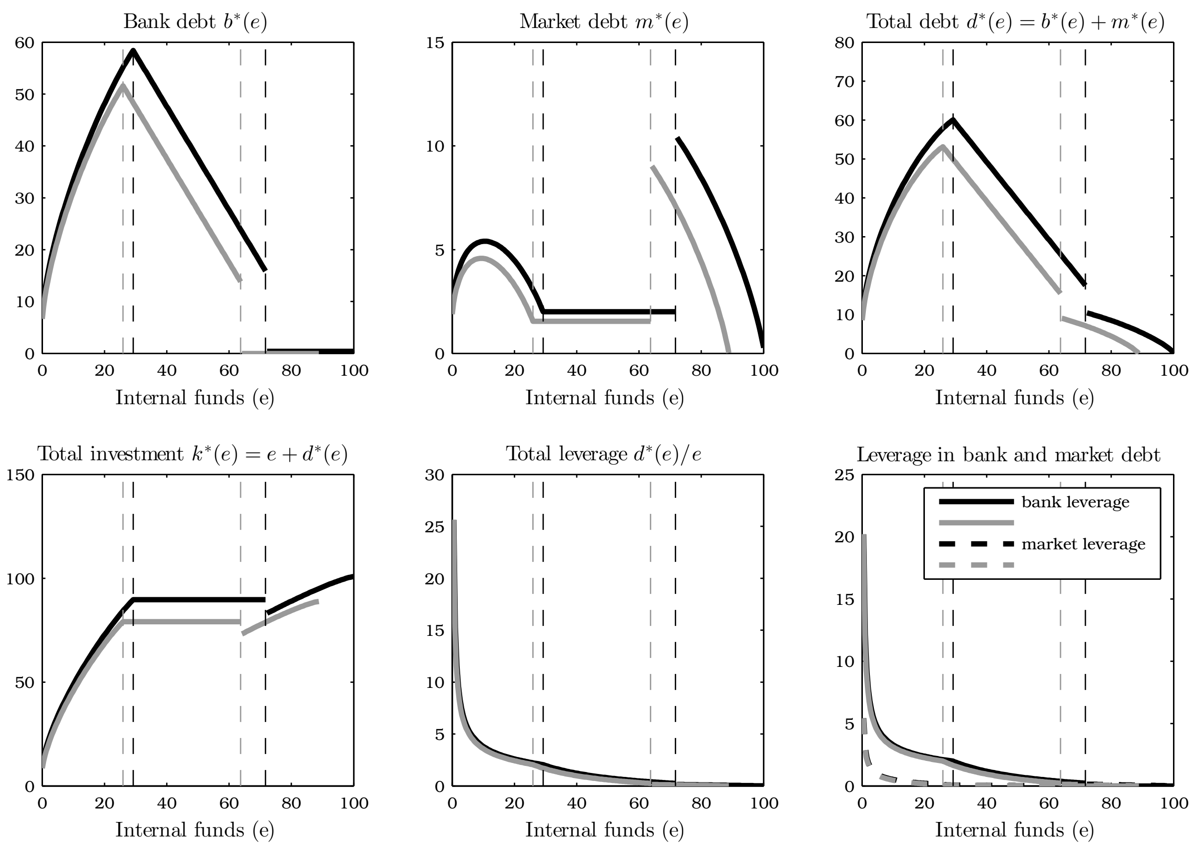


(b) Changes in borrowing

Figure 13: The effect of an increase in lending costs: first case. Graphs 13(a) and 13(b) report the characteristics of the debt structure in the "Baseline" calibration (black line), and in the "High cost levels (1)" calibration (grey line); see table 1 for details.

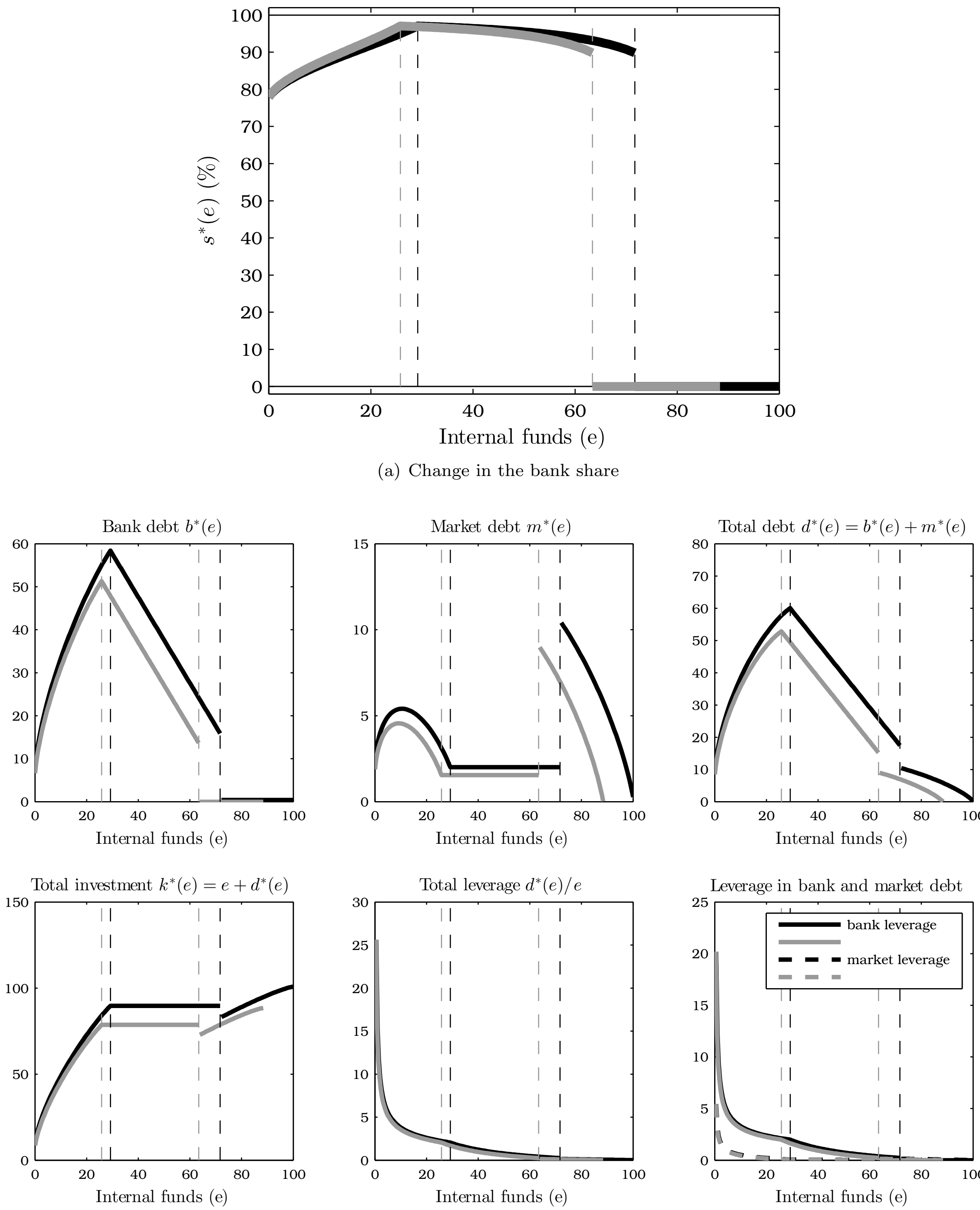


(a) Change in the bank share



(b) Changes in borrowing

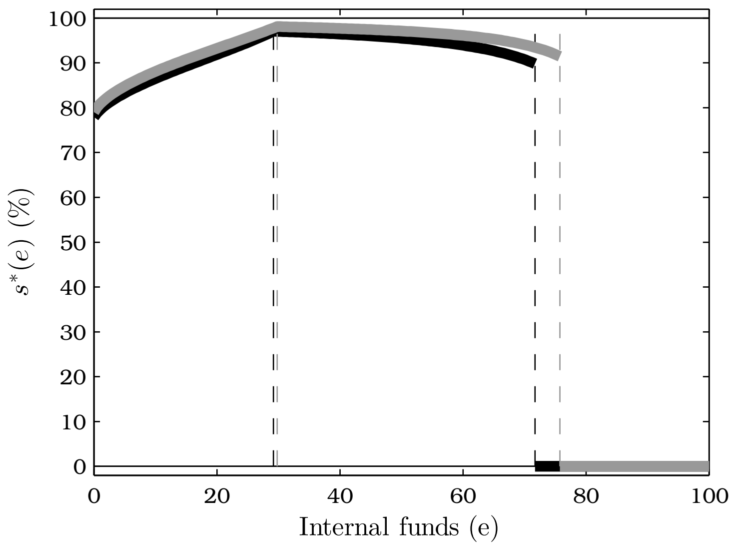
Figure 14: The effect of an increase in lending costs: second case. Graphs 14(a) and 14(b) report the characteristics of the debt structure in the "Baseline" calibration (black line), and in the "High cost levels (2)" calibration; see table 1 for details.



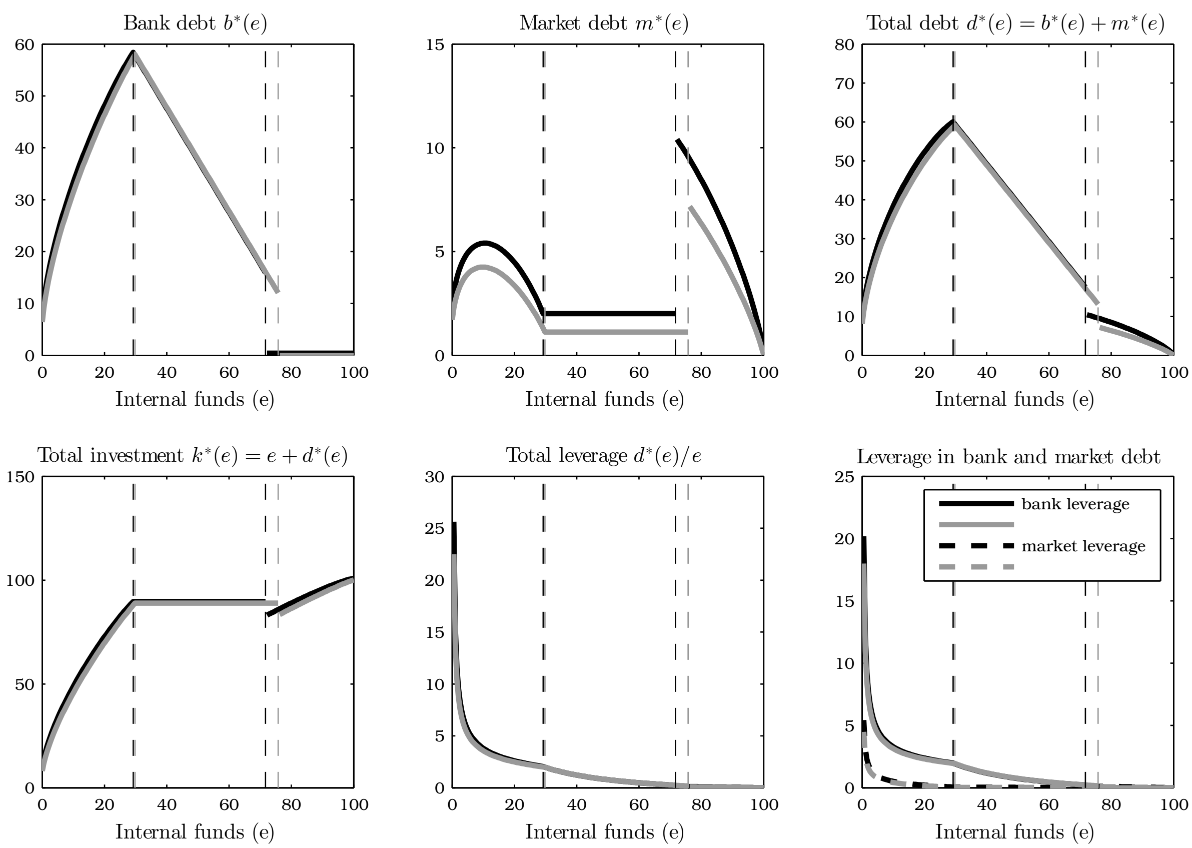
(a) Change in the bank share

(b) Changes in borrowing

Figure 15: The effect of a fall in productivity. Graphs 15(a) and 15(b) report the characteristics of the debt structure in the "Baseline" calibration (black line), and in the "Low average productivity" calibration; see table 1 for details.



(a) Change in the bank share



(b) Changes in borrowing

Figure 16: The effect of an increase in productivity dispersion. Graphs 16(a) and 16(b) report the characteristics of the debt structure in the "Baseline" calibration (black line), and in the "High productivity dispersion" calibration; see table 1 for details.

D Appendix to section 5

D.1 Measures of the debt structure in the QFR

Breaking down the liability side of firms' balance sheets Along with financial liabilities, the QFR balance sheets contain information on non-financial items, such as accounts payables and various forms of tax liabilities. I restrict my attention to financial liabilities, and more specifically to debt. This excludes, in particular, stockholders' equity and trade credit, an important component of liabilities, especially for smaller firms. Within the category of financial liabilities, some are current (due in more than one year) and some are non-current (due in one year or less). I construct measures of the composition of debt for both maturity categories, as well as measures combining short with long debt maturities. For large firms, the behavior of current debt during recessions differs substantially from that of non-current debt. The focus of this paper is on the behavior of total liabilities of firms; I do not adress changes in the maturity structure.

Bank and non-bank liabilities The QFR sample contains two subsamples of firms, one of small and medium-sized firms and one of large firms. Smaller firms report their liabilities with less detail than large firms (appendix D.2 contains more information on the differences between short and long form samples). In particular, small firms report as a group all their non-bank financial liabilities, whereas the large firm samples reports separately commercial paper and long-term debt. Table 3 below provides a breakdown of current and non-current financial liabilities in the short and long sample. To maintain comparability across asset size classes, I focus on "non-bank" liabilities as whole. This includes, for largest firms, both commercial paper and long-term debt. As reported below, the facts described in section 5 are robust to the exclusion of commercial paper from measures of non-bank liabilities for large firms.

Aggregates of financial liabilities In the following discussion, I define two debt aggregates for each firm category: bank liabilities, denoted by CB and TB; and non-bank financial liabilities, including commercial paper and bonds (denoted by CNB and TNB). Variables beginning with a C denote aggregates computed for current liabilities only, while variables beginning with a T denote aggregates computed for total liabilities. I also contruct measures of total financial liabilities ($CFIN$ and $TFIN$), eliminating non-financial liabilities from the aggregate balance sheet measures. Table 4 in appendix D.2 summarizes the construction of these variables.

D.2 Variable definitions in the QFR

The QFR survey is constructed on a quarterly basis from two separate samples. The first sample (the "short-form" sample) is a representative sample from the universe of manufacturing firms with assets less than or equal to 25m\$. The second sample (the "long-form" sample) contains manufacturing firms with at least 25m\$. Firms with between 25m\$ and 250m\$ are sampled from the universe of manufacturing firms, while all existing manufacturing firms with more than 250m\$ in assets are included. For both samples, the QFR contains a variety of information on firms' real and financial variables. The samples differ in the detail with which firms report their balance sheets, as detailed below.

Asset size class	1	2	3	4	5	6	7	8
Assets ($m\$$)	< 5	$[5, 10[$	$[10, 25[$	$[25, 50[$	$[50, 100[$	$[100, 250[$	$[250, 1000[$	≥ 1000

Table 2: Definition of asset size brackets in the QFR.

	Short form variables		Long form variables	
	Name	*Description*	*Name*	*Description*
Current liabilities	STBANK	Short-term bank debt	STBANK	Short-term bank debt
	INSTBANK	Long-term bank debt with maturity $\leq$ 1 year	INSTBANK	Long-term bank debt with maturity $\leq$ 1 year
	STDEBTOTH	Other short-term debt (incl. commercial paper)	STDEBTOTH	Other short-term debt (excl. commercial paper)
			COMPAPER	Commercial paper
	INSTOTH	Other long-term debt with maturity $\leq$ 1 year (incl. bonds)	INSTOTH	Other long-term debt with maturity $\leq$ 1 year (excl. bonds)
			INSTBONDS	Bonds maturing in $\leq$ 1 year
Non-Current liabilities	LTBNKDEBT	Long-term bank debt with maturity $>$ 1 year	LTBNKDEBT	Long-term bank debt with maturity $>$ 1 year
	LTOTHDEBT	Other debt with maturity $>$ 1 year (incl. bonds)	LTOTHDEBT	Other debt with maturity $>$ 1 year (excl. bonds)
			LTBNDDEBT	Bonds maturing in $>$ 1 year

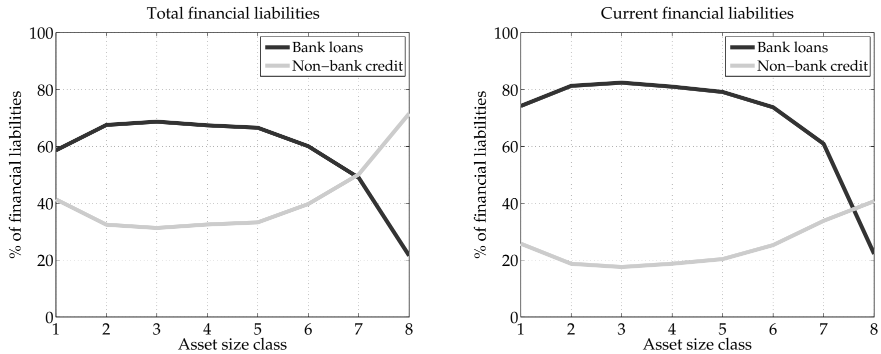
Table 3: Financial liabilities reported in the QFR, short and long form samples.

		Definition for short form sample	Definition for long form sample
Current liabilities	CB	STBANK + INSTBANK	STBANK + INSTBANK
	CNB	STDEBTOTH + INSTOTH	COMPAPER + INSTBONDS+STDEBTOTH+INSOTH
	CNBX	n.a.	CNB - COMPAPER
	CFIN	CB + CNB	CB + CNB
Total liabilities	TB	CB + LTBNKDEBT	CB + LTBNKDEBT
	TNB	CNB + LTOTHDEBT	CNB + LTBNDDEBT + LTOTHDEBT
	TNBX	n.a.	CNBX + LTBNDDEBT + LTOTHDEBT
	TFIN	TB + TNB	TB + TNB

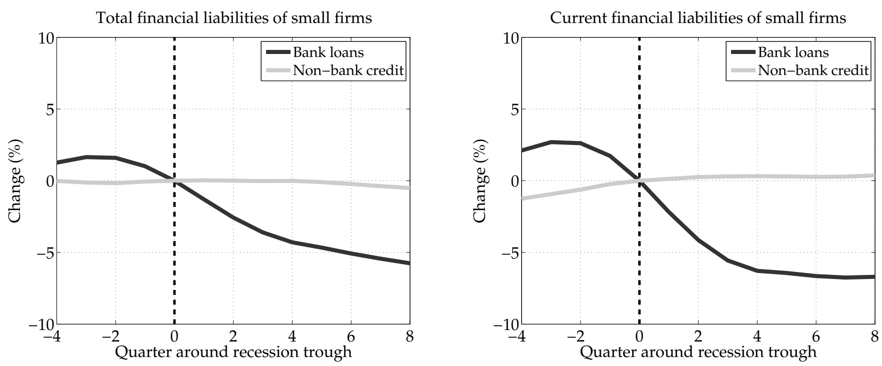
Table 4: Definitions of aggregates of financial liabilities.

D.3 Robustness

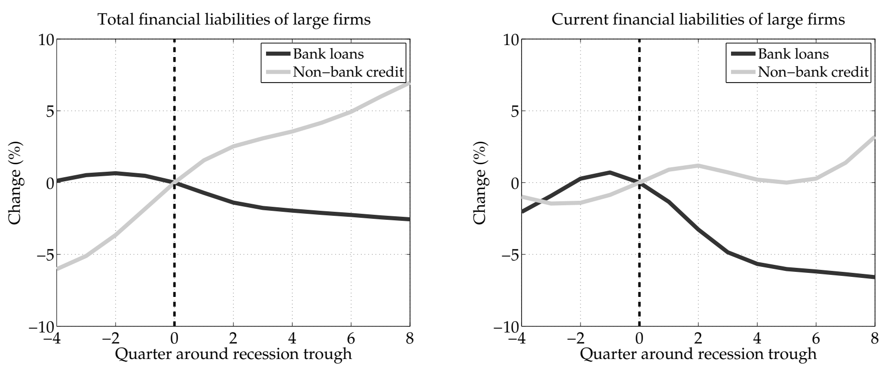
D.3.1 Excluding commercial paper from non-bank liabilities



(a) Composition of debt in the cross-section



(b) Debt structure of small firms during recessions



(c) Debt structure of large firms during recessions

Figure 17: Robustness check 1: no commercial paper. Graphs 17(a), 17(b) and 17(c) report the same series as graphs 12(a), 12(b) and 12(c), but excluding commercial paper from non-bank debt in the case of long-form sample firms.

D.3.2 An alternative definition of small and large firms

To examine whether the facts discussed in section 5 are robust to other groupings of "small" and "large" firms, I use the definitions of Gertler and Gilchrist (1994). They use information on total sales of different asset size brackets in order to determine a cutoff between small and large firms (in terms of asset size). The (gross) growth rate of sales of small firms for a quarter is computed as a weighted average of the growth rate of total sales in the two categories that straddle the thirtieth percentiles of cumulative sales. A series for total sales in each size category is then constructed by taking the cumulative sum of the log of the quarterly gross growth rates obtained in this fashion.

I describe this method more formally. Fix t. Let $\left\{x^{(1)},...,x^{(n)}\right\}$ denote the asset size brackets for quarter t, and let $s_{i,t}$ and $x_{i,t}$ denote sales and total assets of firm i at time t. First, let $S_t = \sum_i s_{i,t}$ and define a cutoff in terms of assets, $\underline{x}_t$, by:

$$\underline{x}_t = \max\left\{x \in \left\{x^{(1)},...,x^{(n)}\right\} \Big/ \frac{\sum_{x_{i,t}\leq x} s_{i,t}}{S_t} \leq 0.3\right\}.$$

Furthermore, let $\underline{x}_t^+$ be the cutoff immediately above $\underline{x}_t$ in the list $\left\{x^{(1)},...,x^{(n)}\right\}$. The cumulative sales of all firms with at most $\underline{x}_t^+$ are less than 30 % of the total of sales of manufacturing firms. Second, compute weights w_t such that:

$$w_t\frac{\sum_{x_{i,t}\leq \underline{x}_t} s_{i,t}}{S_t} + (1-w_t)\frac{\sum_{x_{i,t}\leq \underline{x}_t^+} s_{i,t}}{S_t} = 0.3.$$

Third, for any series $y_{i,t}$, compute "weighted" growth rates according to:

$$G^y_{S,t-1,t} = w_t\frac{\sum_{\left\{i/x_{i,t}\leq \underline{x}_t\right\}} y_{i,t}}{\sum_{\left\{i/x_{i,t-1}\leq \underline{x}_t\right\}} y_{i,t-1}} + (1-w_t)\frac{\sum_{\left\{i/x_{i,t}\leq \underline{x}_t^+\right\}} y_{i,t}}{\sum_{\left\{i/x_{i,t-1}\leq \underline{x}_t^+\right\}} y_{i,t-1}}.$$

Roughly speaking, the goal of this approximation is to compute the growth rate of the variable y for a synthetic size class which represents, on average, 30% of total sales. These weighted growth rates form the basis of a synthetic series for small firms for variable y, $y_{S,t}$, through:

$$y_{S,t} = \sum_{j=0}^{t} log(G^Y_{S,j-1,j}).$$

For large firms, Gertler and Gilchrist (1994) use[29]:

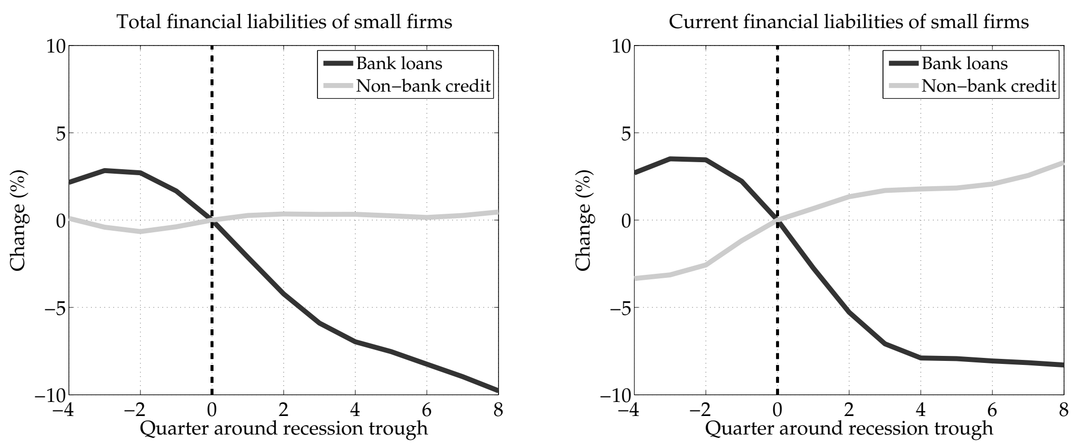
$$G_{L,t-1,t} = w_t\frac{\sum_{\left\{i/x_{i,t}\geq \underline{x}_t\right\}} y_{i,t}}{\sum_{\left\{i/x_{i,t-1}\geq \underline{x}_t\right\}} y_{i,t-1}} + (1-w_t)\frac{\sum_{\left\{i/x_{i,t}\geq \underline{x}_t^+\right\}} y_{i,t}}{\sum_{\left\{i/x_{i,t-1}\geq \underline{x}_t^+\right\}} y_{i,t-1}},$$

[29] In the later part of the sample, the cutoff $\underline{x}_t$ turns out to be $1b\$$, the largest asset class. For these years, I set the weight w_t in $G_{L,t-1,t}$ to be equal to 1.

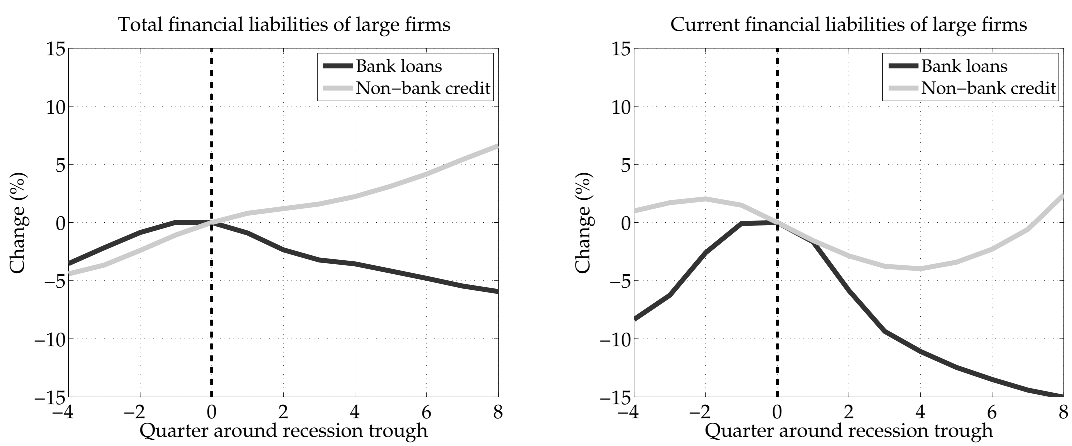
where w_t is the same value as above, and likewise define a synthetic series for large firms, for the variable y, $y_{L,t}$, as:

$$y_{L,t} = \sum_{j=0}^{t} log(G_{L,j-1,j}).$$

The results are reported in figures 19(a) and 19(b), when y corresponds to the series CB, CNB, TB and TNB defined above. Consistently with the facts discussed in section 5, these figures suggest that, while the "synthetic" small firm experiences a contraction in her liabilities driven entirely by a reduction in bank loan, the large firm experiences a substitution of market credit for bank credit. Notably, the upward trend in non-bank liabilities is less noticeable in definition of small and large firms; in turn, over the first year after the recession, the substitution is less than one for one.



(a) The debt structure of small firms during recessions

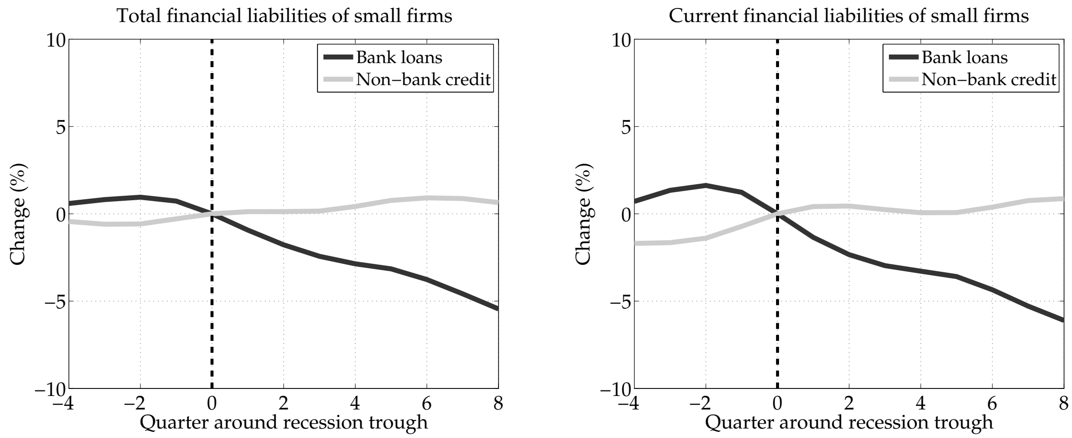


(b) The debt structure of large firms during recessions

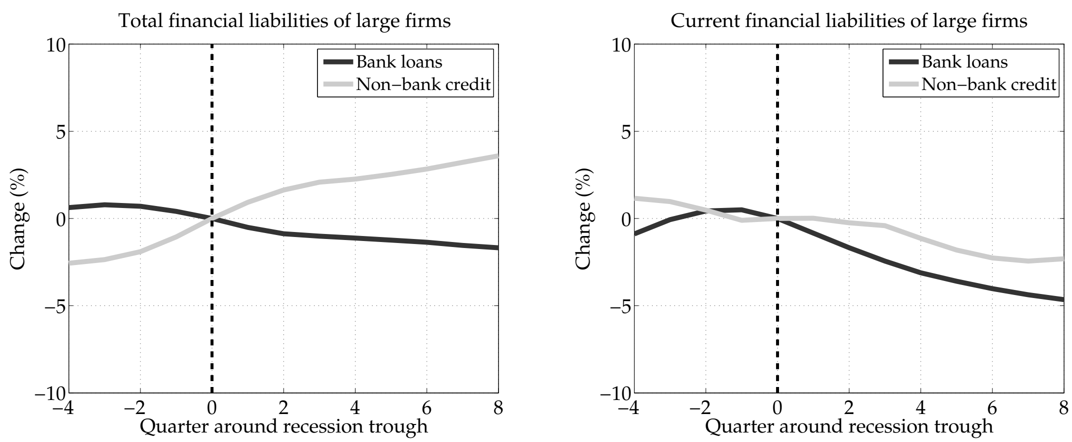
Figure 18: Robustness check 2: broader definition of large firms. Graphs 19(a) and 19(b) report the same series as graphs 12(b) and 12(c), but for the definition of "large" and "small" developed by Gertler and Gilchrist (1994); see text for details.

D.3.3 Excluding the Great Recession

The following graph reports changes in the composition of liabilities of small and large firms in recessions, excluding the 2007-2009 recession. The series are thus the average of the change in liabilities for the 1990-1991 and 2000 recession.



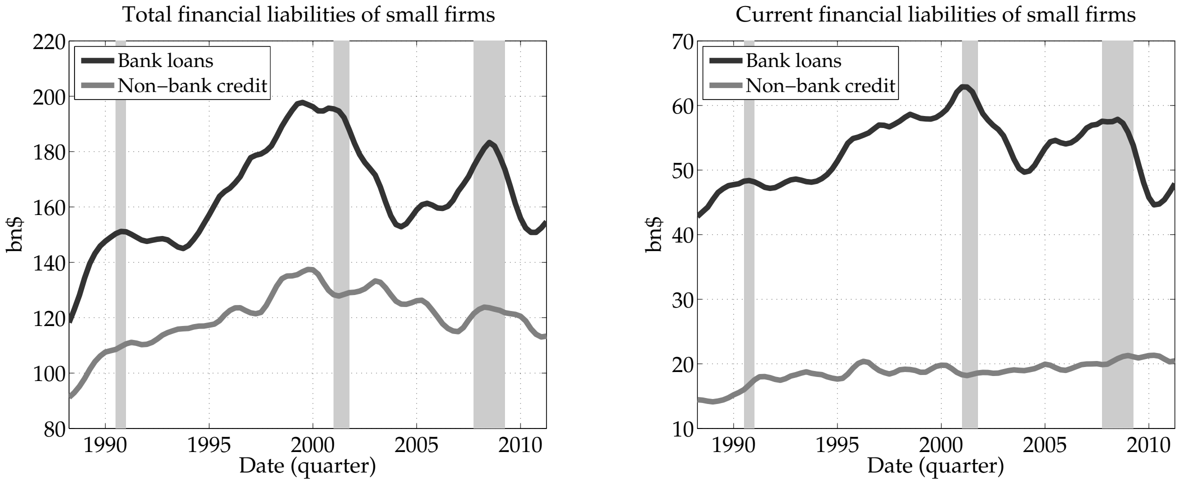
(a) Debt structure of small firms during recessions



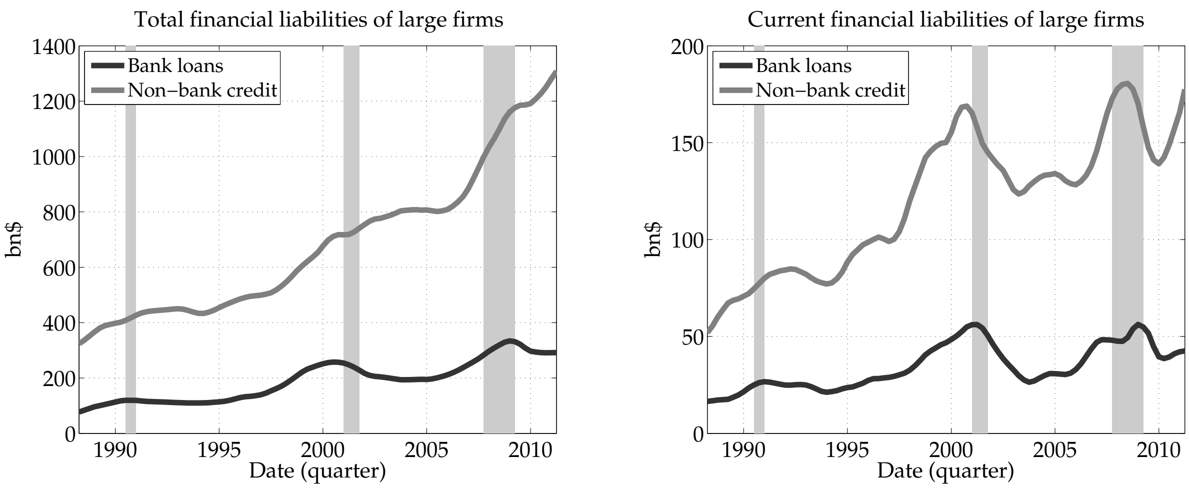
(b) Debt structure of large firms during recessions

Figure 19: Robustness check 3: excluding the 2007 recession. Graphs 19(a) and 19(b) report the same series as graphs 12(b) and 12(c), excluding the 2007-2009 recession.

D.4 Aggregate financial liabilities in levels

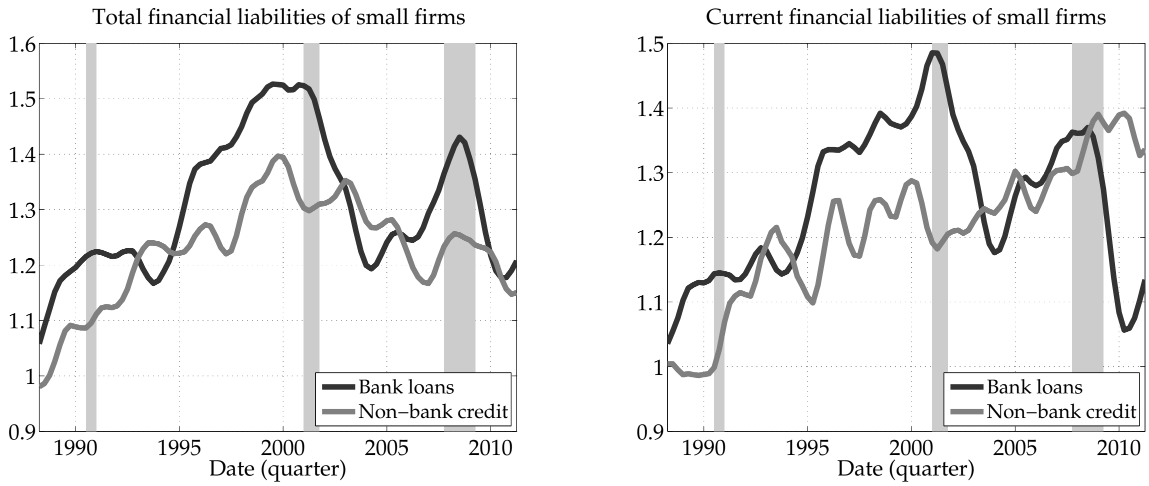


(a) **Small firms**

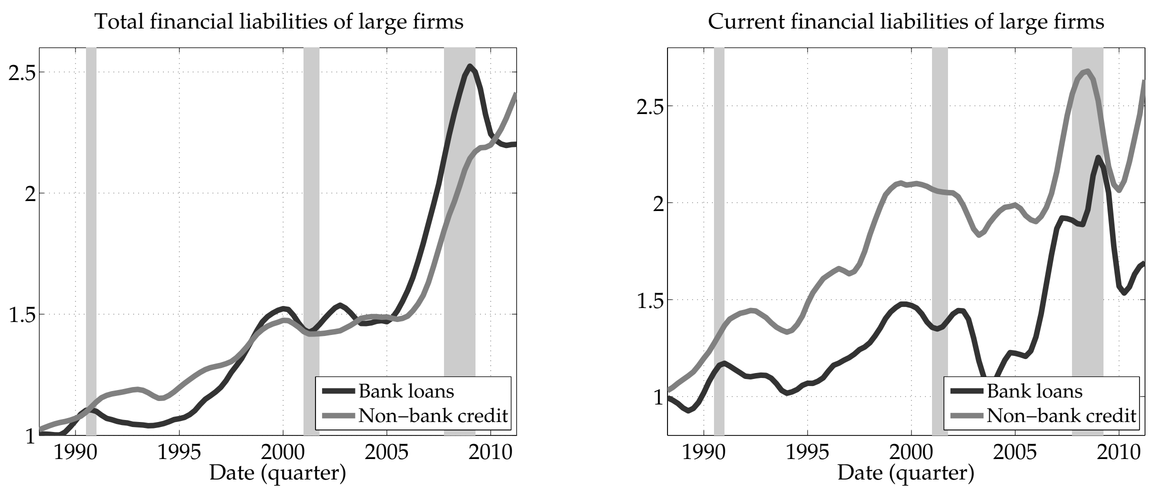


(b) **Large firms**

Figure 20: Financial liabilities in levels, baseline small/large classification. All series are smoothed with a 2 by 4 MA smoother to remove seasonal variation. Shaded areas indicate NBER recessions.



(a) **Small firms**



(b) **Large firms**

Figure 21: Financial liabilities in levels, using the Gertler and Gilchrist (1994) definition of small and large firms. All series are smoothed with a 2 by 4 MA smoother to remove seasonal variation. Shaded areas indicate NBER recessions. Units on the y-axis are arbitrary, since the series are computed from synthetic growth rates; see text ofr details.