Information Sharing in Supply Chains: An Empirical and Theoretical Valuation

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April 10, 2013

We provide an empirical and theoretical assessment of the value of information sharing in a two-stage supply chain. The value of downstream sales information to the upstream firm stems from improving upstream order fulfillment forecast accuracy. Such improvement can lead to lower safety stock and better service. According to recent theoretical work, the value of information sharing is zero under a large spectrum of parameters. Based on the data collected from a CPG company, however, we empirically show that if the company includes the downstream demand data to forecast orders, the mean squared error percentage improvement ranges from 7.1% to 81.1% in out-of-sample tests. Thus, there is a discrepancy between the empirical results and existing literature: the empirical value of information sharing is positive even when the literature predicts zero value. While the literature assumes that the decision maker strictly adheres to a given inventory policy, our model allows him to deviate, accounting for private information held by the decision maker, yet unobservable to the econometrician. This turns out to reconcile our empirical findings with the literature. These "decision deviations" lead to information losses in the order process, resulting in strictly positive value of downstream information sharing. We prove that this result holds for any forecast lead time and for more general policies. We also systematically map the product characteristics to the value of information sharing.

Key words: supply chain, information sharing, information distortion, decision deviation, time series, forecast accuracy, empirical forecasting, ARIMA process.

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1. Introduction

The abundance of information technology has had a massive impact on supply chain coordination. Sharing downstream demand information with upstream suppliers has improved supply chain performance in practice. Costco and 7-Eleven share warehouse-specific, daily, item level point of sale data with their suppliers via SymphonyIRI platform, a company offering business advice to retailers (see Costco collaboration 2006). In addition to this uni-directional information sharing, Collaborative Planning, Forecasting and Replenishment (CPFR) programs advocate joint visibility and joint replenishment. According to Terwiesch et al. (2005), the benefit of CPFR programs

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can be significant: the GlobalNetXchange, a consortium consisting of more than 30 trade partners including Sears, Kroger etc, have reported a 5% to 20% reduction in inventory costs and an increase in off-the-shelf availability of 2% to 12% following the launch of their CPFR programs.

In sharp contrast, however, the academic literature shows that the value of sharing downstream customer sales to improve upstream forecasting is limited. (We will also refer to customer sales as *customer demand* or *demand*.) For example, Gaur et al. (2005) model the demand process as an autoregressive moving average ARMA$(1,1)$ and show that the value of information sharing is zero under 75% of the demand parameters. Therefore, a direct conclusion of this literature is that the value of information sharing is zero under a large spectrum of parameters.

Companies spend billions of dollar on demand forecasting software and other supply chain solutions (Ledesma 2004). Given the implementation cost of collaboration technology and the limited theoretical benefits, it is not clear in practice whether a firm should invest in information sharing systems. The decision to implement an information sharing system thus hinges on the following question: how much would sharing downstream information improve the supplier's order forecast accuracy? We were approached with this question by the statistical forecasting team of a leading global consumer packaged goods (CPG) company that manufactures and sells beverages and snack foods to wholesalers and retail chains. Forecasting is necessary for the company due to the lead time to adjust manufacturing runs and deploy inventory. In the absence of downstream demand information, the upstream supplier uses its own demand history (i.e., its retailer's order history), to forecast how much to manufacture. Not satisfied with its current forecasting performance, the firm sought solutions in information sharing by collecting downstream operations data (e.g., point of sale and inventory) from both its customers and a third party organization (e.g., RSI). Using this data set, we directly measure the supplier's forecast accuracy improvement.

Surprisingly, our empirical results indicate a substantial value of information sharing: (1) incorporating order and demand correlation yields a statistically significant improvement for some products even without accounting for the applied inventory policies; and (2) applying the underlying replenishment policies, we find that the value of information sharing is strictly positive (7.1% to 81.1% MSE percentage improvement) across *all* products with stationary demand while the literature suggests positive value for only 30% (4 out of 14) of products. To put this in perspective, the company views the forecast accuracy improvement opportunities of 10% as important and 30% as very significant. These empirical findings suggest that we need a better theoretical understanding of how demand propagation impacts forecasting.

The works of Gaur et al. (2005) and Giloni et al. (2012) are important antecedents of our paper. In their setting, the decision maker strictly follows an order-up-to policy, via which the demand process propagates upstream and becomes the order process. If, for example, the retailer follows a demand replacement policy (the retailer orders the demand in the current week), the order process then equals the demand process. It is as if the demand process propagates fully upstream and the order process carries full demand information. In such setting, the value of including the downstream demand process is zero. The downstream demand information might be lost in this demand-to-order transformation. The authors derive the following insights: the value of information sharing is zero when the order process effectively carries all demand information, or equivalently, when the order is *invertible* with respect to demand shocks. The authors show that the order is invertible (thus the value of information sharing is zero) under 75% of demand parameters for an ARMA$(1,1)$ demand process.

The key underlying assumption in the theoretical literature is that the decision maker consistently and strictly follows a given replenishment policy. In practice, however, we learned that the decision makers deviate from their target inventory policy based on private information that we cannot observe. The planner might round the order quantity due to truck load constraints and delivery multiple brands of products in batches. From an econometric perspective, we model the agent's deviation from the exact policy, in the spirit of Rust (1997), by an "error term" that accounts for a state variable which is observed by the agent but not by the statistician. We demonstrate that including these idiosyncratic shocks in the model significantly increases the theoretical value of information sharing, in agreement with the empirical findings.

In the presence of decision deviations, we provide a different and important insight into the value of information sharing. Unlike before, the demand process now propagates together with the decision deviation. The value of information sharing is zero if the order process carries both all demand information and all decision deviation information. The demand and decision deviation processes follow distinct evolution patterns to produce the order process: the evolution of inventory governs the translation of decision deviations into replenishment decisions while the evolution of inventory and current demand together dictates the translation of demand. This difference causes the order process to no longer carry full downstream information. Information sharing then becomes valuable to recover the order's elaborate information structure and to forecast more accurately. At first glance, the decision uncertainty seems to diminish the attractiveness of analyzing a retailer's replenishment process due to the unpredictability of the order decision. Such uncertainty, however, opens the door to information loss in the upstream orders, because the decision deviation distorts the normal demand propagation. We prove that as long as the variability in decision deviation and demand are both nonzero, the value of information sharing is strictly positive for any forecast lead time (regardless of the demand and policy). Our extended model induces qualitatively different results than the literature and reconciles our empirical findings.

We also conduct comparative statics and detailed numerical studies to examine the impact of product demand characteristics on the value of information sharing. These insights can help managers rank the potential gains from information sharing depending on the demand characteristics for different products such as sport drinks and orange juice.

Our study is grounded in both empirical evidence and theory and attempts to understand the cause of strictly positive value of information sharing. We analyze a data set containing weekly downstream demand, upstream order fulfillment, and the price plan over a period of two and a half years. This allows us to make the following four main contributions: first, this paper complements the emerging area of research in information sharing with empirical evidence. Specifically, we directly measure the value of information sharing at a leading CPG company and demonstrate a positive value of information sharing in all the settings that we study. Second, we allow for decision deviations in our theoretical model to explicitly capture the decision maker's private information that is unobservable to us. This model extends the existing literature and recovers the results from the literature as a special case where decision deviation is zero. Third, we prove that if both demands and order decisions are subject to random shocks, the value of information sharing is always positive. We demonstrate that the decision deviation distorts the normal demand propagation in a way that obscures the detailed information of the two processes. The resulting less informative order process induces larger forecast uncertainty, which indicates a positive value from using the downstream demand to recover the original elaborate information structure. Finally, we provide guidelines for the magnitude of the value of information sharing depending on the demand characteristics.

2. Literature Review

Motivated by practice and theory, we study the value of a retailer to share its retailer's downstream demand information with its supplier to help the supplier forecast the retailer's order. We find that the results in the literature point towards the inconsistencies between the empirical evaluation of the benefit and the theoretical predictions. This paper reconciles these findings by extending the established theory. Therefore, our paper is related to two streams of literature: (1) theoretical work on information sharing and demand propagation through supply chains and (2) empirical work that bridges the above theory and operational data.

There is a vast theoretical literature on the subject of demand propagation and information sharing in supply chains. A company's demand propagates through the supply chain and becomes its order to the supplier. The properties of orders can help answer important questions in supply chains, e.g. is sharing retailer's demand information beneficial for the supplier to forecast its own order and manage its inventory; is there incentive for the agents to share their own information; is there a bullwhip effect and what is the driver? The demand propagation relies on two basic characteristics of the supply chain: demand structure and replenishment policy. We focus on the work that assumes truthful and complete information disclosure. We begin by introducing the various demand and policy structures studied in the literature. Next, we discuss our paper's contribution relative to the two most related work: Gaur et al. (2005) and Giloni et al. (2012).

Theoretical Work. The literature studies demand propagation under various demand structures. Lee et al. (2000) and Raghunathan (2001) adopt an autoregressive AR(p) process, Miyaoka and Hausman (2004) and Graves (1999) assume an integrated moving average IMA(d, q) process, Zhang (2004), Gaur et al. (2005), Kovtun et al. (2012) and Giloni et al. (2012) consider an autoregressive and moving average ARMA(p, q) process, Gilbert (2005) applies an ARMA with integration model called ARIMA(p, d, q) process, and Aviv (2003) uses the linear state space framework. Another body of literature applies the Martingale Model of Forecast Evolution (MMFE) structure. It uses the incremental signal, generated from the minimum mean squared error, to model the evolution of a process. Heath and Jackson (1994), Graves et al. (1998), Aviv (2001a) and Chen and Lee (2009) apply such demand structure to study production and forecasting. The general expression in the optimal forecast revision drives MMFE's theoretical advantage: most time-series models can be interpreted as a special case of the MMFE model (Chen and Lee 2009). We will show our main conclusion holds under the MMFE structure (see Online Companion). Gaur et al. (2005) point out the ARMA model closely resembles the real-life demand structure and finds it valuable from the manager perspective to study such demand process. For our studies, we use an ARIMA structure to model and empirically fit the demand process.

In the above literatures, the most commonly studied replenishment decision is the myopic order-up-to policy. The following papers investigate information sharing and the bullwhip effect under other policies. Caplin (1985) proves the existence of the bullwhip effect under periodically reviewed (s,S) policy. Cachon and Fisher (2000) quantify the value of information sharing with a batching allocation rule between one supplier and multiple retailers. These two papers model batching in replenishment, which is not amenable to exact and mathematically-tractable analysis. The following papers adopt a "linear replenishment rule," in which orders are linear in past observed variables. Balakrishnan et al. (2004) propose an "order smoothing" inventory policy where the order is a convex combination of historical demands. Miyaoka and Hausman (2004) use the old demand forecasts to set the base stock level and show this can reduce the bullwhip effect. Graves et al. (1998) and Aviv (2001b) study the production smoothing policy and Chen and Lee (2009) extend it to a more general order-up-to policy (GOUTP), which bears an affine and time-invariant structure of the forecast revisions. We prove that our main result still holds under GOUTP (see Online Companion). According to the replenishment policy we observed at the firm that provided us with the data, our paper introduces an order rule that keeps the days of inventory constant and uses some order smoothing. Such policy specifies the order as a linear combination of past demands and inventory. To summarize, the ARIMA model determines our input demand structure and the linear policy dictates how demand propagates through the supply chain.

As reviewed in Section 1, Gaur et al. (2005) and Giloni et al. (2012) are the two most closely related works to our paper. They assume that the decision maker strictly and consistently follows the replenishment policy. They conclude that under certain demand and policy parameters, the value of sharing the demand information is zero. Under such strict policy adherence, orders only depend on the observed demand or demand signals. In practice, however, decision makers adjust their purchases by other factors. To account for these factors, we introduce an "error term" into the empirical model with the interpretation of a state variable observed by the agent but not by the statistician, in the spirit of Rust (1997). Therefore, our theoretical model differs from the existing literature in relaxing the strict adherence to the replenishment policy. Such an extension explains our substantial empirically evaluated value of information sharing, thus fills the gap between the literature and the empirical observation.

Empirical Work. A growing body of empirical literature analyzes the bullwhip effect and information sharing. Cachon et al. (2007) investigate a wide range of industries and show insignificant variance amplification for some industries. Bray and Mendelson (2012b) further decompose the bullwhip by short, middle and long lead time signals. In the gaming environment, agents have incentives to partially rely on the data or share untruthful information. Cohen et al. (2003) model the supplier's optimal production starting time after receiving forecasts by the retailer. The estimated high cost of starting the production too early indicates the supplier's tendency to ignore the retailer's early forecast, thus inferring the low efficiency of forecast sharing. Terwiesch et al. (2005) also conclude the low efficient forecast sharing by finding the agent's forecast behavior falls in the noncooperative scenario. Bray and Mendelson (2012a) characterize the demand propagation under the MMFE structure and GOUTP rule. The authors suggest the positive value for the upstream supplier if the retailer better forecasts the demand.

Using an econometric model, Dong et al. (2011) find that the inventory decision-making transfer between firms, which means the supplier manages the retailer's inventory, benefit both upstream and downstream firms. They show a negative relation between the decision transfer and distributor's average inventory. Route (2003) captures the order demand correlation including the retailer's point of sale data, and evaluates the forecast accuracy improvement. Our study also applies the method to include the downstream demand information. In our paper, the retailer's demand information is an additional indicator included to help forecasting supplier orders. Similarly, one can use other potential indicators to predict customer demand, e.g. financial market index or accounting variables (see Gaur et al. 2009 and Kesavan et al. 2009, among others).

3. Including Downstream Demand Improves Order Forecasting

The goal of this section is to provide an empirical evidence that incorporating downstream sales data improves order forecast accuracy compared to the benchmark where the sales information is not shared. In this section, we proceed as follows: we first explain the supply chain structure; we then describe the data set; we next illustrate the forecasting procedure and finally show the empirical results.

Supply chain setting. We consider a two-echelon supply chain with a supplier and a retailer. The retailer places an order O_t to the supplier in each period t. In each period, the supplier predicts the future order, e.g. the 1-step prediction for period t given the history through period $t-1$, which we denote as $\hat{O}_{t-1,t}$ (throughout the paper, hat denotes forecasted quantities). The prediction error is the difference between the actual order and its predicted value, $O_t - \hat{O}_{t-1,t}$. We will measure the forecast accuracy as a function of the prediction error using two metrics for our empirical study. The supplier aims to improve the forecast accuracy of future orders. We will compare the forecast accuracy under two settings: *NoInfoSharing* and *InfoSharing*. The NoInfoSharing denotes the setting when the supplier only has access to the retailer's order history. Under the InfoSharing setting, in addition to the order data, the retailer also shares her sales history with the supplier.

Data. We obtain the data from a CPG company, which is a leading manufacturer and supplier in the US beverage and snack food industry. We utilize a specific retail customer's (1) sales from the retailer distribution center and (2) replenishment fulfillment from the supplier, over 126 weeks between 2009 and 2011. The sales data corresponds to the actual demand due to the few stockout. We study two brands of products: a sports drink and orange juice. We choose 14 low-promotional products for the study because of their stationary nature[1].

Forecasting procedure. For the purpose of our study, we choose the last 26 weeks in our data as the out-of-sample test period. This out-of-sample comparison is made in two stages. First, we forecast the 1-period-ahead order over the out-of-sample test period. To be specific, the forecast begins 26 weeks before the end of the data. Given information history through the end of period $t-1$, we predict the order for period t. Then we update the information history from the beginning of the data through the end of week t to predict for period $t+1$. We periodically update the available information history to obtain the order forecast and calculate the forecast error by comparing the actual observation and predicted value. Second, we conduct tests of equal forecast accuracy on the two sequences of forecast errors generated from two candidate forecasting methods.

Next we explain the NoInfoSharing and InfoSharing forecasting methods. For the NoInfoSharing benchmark, we use order history to predict future orders. We fit the autoregressive integrated moving average (ARIMA) model to the order history to obtain the best estimator with the

[1] The stationary nature means the sales process has constant mean and covariance over time.

lower Bayesian information criterion (BIC)[2]. We then predict the order by applying the estimated ARIMA model.

The ARIMA model is generally referred to as an ARIMA(p,d,q) model where p, d and q are non-negative integers that refer to the degree of the autoregressive, integrated and moving average parts of the model respectively. In the ARIMA structure, the order is a linear combination of past observations and shocks. The "first order differenced" process $O_t - O_{t-1}$ will be denoted by O_t^1. We assume O_t^1 is an ARMA(p,q) process

$$O_t^1 = \mu + \rho_1 O_{t-1}^1 + \rho_2 O_{t-2}^1 + \cdots + \rho_p O_{t-p}^1 + \eta_t + \lambda_1 \eta_{t-1} + \lambda_2 \eta_{t-2} + \cdots + \lambda_q \eta_{t-q}. \quad (1)$$

where μ is the process mean, η_t is the order shock, ρ_i is the autoregressive parameter and λ_i is the moving average coefficient. Suppose the available information history is through the end of period $t-1$, the differenced order forecast for period t is $\hat{O}_{t-1,t}^1 = \mu + \rho_1 O_{t-1}^1 + \rho_2 O_{t-2}^1 + \cdots + \rho_p O_{t-p}^1 + \lambda_1 \eta_{t-1} + \lambda_2 \eta_{t-2} + \cdots + \lambda_q \eta_{t-q}$ or the order forecast for period t is $\hat{O}_{t-1,t} = O_{t-1} + \mu + \rho_1 O_{t-1}^1 + \rho_2 O_{t-2}^1 + \cdots + \rho_p O_{t-p}^1 + \lambda_1 \eta_{t-1} + \lambda_2 \eta_{t-2} + \cdots + \lambda_q \eta_{t-q}$.

To analyze the impact of including retail demand data, we consider four InfoSharing forecasting methods: three "naive" methods and our "advanced" method. The "naive" methods capture the correlation between order and demand by specifying a linear model. The "advanced" method considers a specific replenishment policy, which we will discuss in detail in Section 4 and Section 6. The "advanced" method is used to evaluate the additional value of carefully considering the underlying order policy structure over the naive ones. We refer to such forecast scheme as the policy structure method.

The first two naive methods capture the order demand correlation by regressing order on demands, or on both orders and demands, within the past five periods. If regressing order only on demands, we refer to this method as Reg D method where the order is expressed as

$$O_t = c_0 D_t + c_1 D_{t-1} + \cdots + c_5 D_{t-5} + \varepsilon_t. \quad (2)$$

We fit the ARIMA model to the demand process to forecast $\hat{D}_{t-1,t}$. With the parameters estimated from equation (2), the order prediction in period t becomes $\hat{O}_{t-1,t} = c_0 \hat{D}_{t-1,t} + c_1 D_{t-1} + \cdots + c_5 D_{t-5}$.

In the second naive method refered to as "Reg D and O" method, we regress order on historical demands and orders. This method expresses the order as

$$O_t = c_0 D_t + c_1 D_{t-1} + \cdots + c_5 D_{t-5} + b_1 O_{t-1} + \cdots + b_5 O_{t-5} + \varepsilon_t. \quad (3)$$

As the demand D_t is not known at time $t-1$, we fit the ARIMA model to the demand process to forecast $\hat{D}_{t-1,t}$. With the parameters estimated from equation (3), the order prediction in period t becomes $\hat{O}_{t-1,t} = c_0 \hat{D}_{t-1,t} + c_1 D_{t-1} + \cdots + c_5 D_{t-5} + b_1 O_{t-1} + \cdots + b_5 O_{t-5}$.

[2] BIC is a criterion for model section for time series analysis and model regression. It selects the set of parameters that maximizes the likelihood function with the least number of parameters in the model.

The third naive method adds the observed demands to equation (1), i.e. it assumes

$$O_t^1 = \mu + \rho_1 O_{t-1}^1 + \rho_2 O_{t-2}^1 + \cdots + \rho_p O_{t-p}^1 + \eta_t + \lambda_1 \eta_{t-1} + \lambda_2 \eta_{t-2} + \cdots + \lambda_q \eta_{t-q} \qquad (4)$$
$$+ a_0 D_t + a_1 D_{t-1} + \cdots + a_p D_{t-p}.$$

A forecast for O_t^1 then can be achieved in two steps: estimate the parameters in equation (4) and obtain the demand forecast $\hat{D}_{t-1,t}$. The parameters in equation (4) can be estimated by fitting O_t and D_{t+1} series in a two dimensional vector ARIMA model. Note that equation (4) serves as a more general method than equation (1). We specify a vector ARIMA$(3,1,1)$[3] model with $\mu = 0$ for O_t. As the demand D_t is not known at time $t-1$, the ARIMA model is fitted to the demand process to forecast $\hat{D}_{t-1,t}$. The differenced order forecast for period t then becomes $\hat{O}_{t-1,t}^1 = \mu + \rho_1 O_{t-1}^1 + \cdots + \rho_p O_{t-p}^1 + \lambda_1 \eta_{t-1} + \cdots + \lambda_q \eta_{t-q} + a_0 \hat{D}_t + a_1 D_{t-1} + \cdots + a_p D_{t-p}$. We adopt the vector ARIMA model to estimate and forecast the process and thus we refer to this as the Vector ARIMA method.

To measure the accuracy of various methods, we introduce two forecast error metrics used in the literature: mean absolute percentage error (MAPE) and mean squared zero-mean error (MSE). Let N be the number of weeks in the test period. The forecast metrics over the test period are:

$$MAPE = \frac{1}{N} \sum_{i=1}^{N} \left| O_{t+i} - \hat{O}_{t+i-1,t+i} \right| / O_{t+i}, \qquad (5)$$
$$MSE = \frac{1}{N} \sum_{i=1}^{N} (O_{t+i} - \hat{O}_{t+i-1,t+i} - \frac{1}{N} \sum_{i=1}^{N} (O_{t+i} - \hat{O}_{t+i-1,t+i}))^2.$$

MAPE is a widely used accuracy metrics in the literature (cf. Gaur et al. 2009, Kesavan et al. 2009). This metric is also closely related to the metric used by the company from which we received the data. MSE is a frequently adopted accuracy metric in the theoretical literature because of its mathematical tractability. We will also use this metric for our theoretical analysis. Note that in the MSE definition, $\sum_{i=1}^{N} (O_{t+i} - \hat{O}_{t+i-1,t+i})/N$ is the mean of the forecast error and as the sample size N goes to infinity, $\sum_{i=1}^{N} (O_{t+i} - \hat{O}_{t+i-1,t+i})/N \to 0$ under unbiased estimates. Therefore, as the sample size is large enough, the mean squared zero-mean error coincides with the mean squared error used in the theory literature.

With the alternative forecasting models, we empirically assess the value of incorporating the downstream information. To this end, we perform a product-by-product forecast accuracy comparison. The disaggregated analysis enables a detailed detection for each product. We conduct the

[3] VARIMA$(3,1,1)$ model is $\begin{bmatrix} O_t^d \\ D_{t+1}^d \end{bmatrix} = \begin{bmatrix} c_{11}^1 & c_{12}^1 \\ c_{21}^1 & c_{22}^1 \end{bmatrix} \begin{bmatrix} O_{t-1}^d \\ D_t^d \end{bmatrix} + \cdots + \begin{bmatrix} c_{11}^3 & c_{12}^3 \\ c_{21}^3 & c_{22}^3 \end{bmatrix} \begin{bmatrix} O_{t-3}^d \\ D_{t-2}^d \end{bmatrix} + \begin{bmatrix} \eta_t \\ \epsilon_{t+1} \end{bmatrix} + \begin{bmatrix} e_{11}^1 & e_{12}^1 \\ e_{21}^1 & e_{22}^1 \end{bmatrix} \begin{bmatrix} \eta_{t-1} \\ \epsilon_t \end{bmatrix}$, where c_{21}^i and c_{22}^i are restricted to zero for $i = 1,2,3$. e_{12}^1, e_{21}^1 and e_{22}^1 are restricted to zero, η_t is order shock and ϵ_t is demand shock. The larger the degree of AR and MA, the broader order pattern relative to the parameters found in equation (1). We choose $(3,1,1)$ due to the computational constraints. Such parameter can represent the majority of parameters found in equation (1).

Table 1 MAPE and MSE percentage improvement for the four methods that incorporate the downstream demand data. Significant accuracy improvement over the no sharing method is marked by star. Significant ($p = 0.1$) accuracy improvement of the policy structure method over the unbold others is marked with bold value.

		MAPE percentage improvement				MSE percentage improvement			
Brand	Product	Vector ARIMA	Reg D	Reg D and O	Policy Structure	Vector ARIMA	Reg D	Reg D and O	Policy Structure
Orange Juice	128 OR	11.1%	12.2%*	-14.6%	**45.0%****	8.7%	14.0%**	0.4%	**18.1%****
	128 ORCA	-18.3%	8.1%	1.9%	**30.3%***	-0.5%	7.8%	**18.8%***	**26.5%****
	12 OR	31.6%	15.7%	**50.2%***	**58.6%***	32.4%**	33.8%**	35.1%**	**53.4%****
	12 ORCA	**40.8%****	**40.0%****	**38.0%****	**50.2%****	30.5%	36.3%	**57.3%****	**53.1%****
	59 ORST	**16.1%***	4.1%	5.0%	**18.8%***	13.2%*	10.9%	10.7%	7.1%
	59 ORPC	12.8%**	**29.1%****	**23.8%****	**27.7%****	**16.2%****	**31.0%****	11.4%	**29.4%****
Sports Drink	500 BR	21.2%	26.2%	25.5%	**39.8%****	54.1%**	48.7%**	41.7%*	**62.5%****
	500 GP	**30.9%***	25.7%	26.5%	**36.0%****	**53.1%****	42.9%*	38.7%*	**68.4%****
	PD LL	2.8%	-15.5%	-18.4%	4.7%	5.6%	30.9%	31.3%	51.3%
	PD OR	26.8%**	26.2%**	26.2%**	**44.2%****	43.3%*	**81.0%***	**81.0%***	**81.1%***
	PD FRZ	22.1%	8.2%	11.4%	**39.5%***	**44.5%****	8.2%	9.2%	**56.9%****
	1GAL GLC	**23.7%****	**30.3%****	**26.4%****	**38.0%****	50.1%**	42.9%*	40.1%*	**54.2%****
	1GAL FRT	**24.3%****	**21.4%***	17.2%	**29.9%****	46.4%**	40.3%**	31.3%*	**54.0%****
	1GAL OR	**16.9%***	**18.3%***	14.0%	**30.4%***	30.2%	21.2%	18.3%	**44.8%****

** At level $p < 0.05$, the accuracy improvement over no information sharing method is significant.

* At level $p < 0.1$, the accuracy improvement over no information sharing method is significant.

pairwise t-test to determine the statistical significance of forecast performance improvement. Table 1 presents the MAPE and MSE percentage improvement of the four InfoSharing methods over the NoInfoSharing method for each product. The MAPE percentage improvement of method 1 over method 2 is given by $(MAPE_1 - MAPE_2)/MAPE_1$. Similarly, the MSE percentage improvement is $(MSE_1 - MSE_2)/MSE_1$. The larger the percentage improvement, the more accurate the forecast with information sharing. We carry out two sets of comparisons: the improvement with respect to the NoInfoSharing forecast and the improvement of the policy structure forecast over other forecasts. The star mark means that the forecast improvement with respect to the NoInfoSharing method is statistically significant. The policy structure forecast in bold induces a statistically significant improvement over the unbold forecasts.

Table 1 delivers two key messages. First, for all products, at least one of the InfoSharing methods generates statistically significant improvement over the NoInfoSharing method for one error metric[4]. On average, the NoInfoSharing forecasts have the lowest accuracy with MAPE around 56%, the number of which is representative of the typical number we observe at the CPG company. From these, we infer that for each product, the improvement of including the downstream demand

[4] The mean absolute error (MAE) is defined as $\sum_{i=1}^{N} \left| O_{t+i} - \hat{O}_{t+i-1,t+i} \right| / \sum_{i=1}^{N} O_{t+i}$. For the product PD LL, the MAE metric shows that the policy structure method is statistically significantly ($p < 0.1$) better than the NoInfoSharing method, although both MAPE and MSE metrics indicate insignificant improvement.

information is statistically significant. Furthermore, we test whether considering the replenishment policy further strengthens the InfoSharing forecasts. The second message is that incorporating the *policy structure* yields the greatest or one of the greatest improvements. For the MAPE metric, the policy structure method has the highest improvement for all products and statistically higher improvement than all other forecast methods at $p < 0.1$ for 5 out 14 products. For the MSE metric, the policy structure method has statistically significantly higher improvement than all other forecast methods at $p < 0.1$ for 6 out of 14 products. The forecasts generated from the naive methods can be statistically indistinguishable from the policy structure method for some products. This means the naive methods can correctly capture the correlation between orders and demands for those products. On average, however, the policy structure method yields 40% MAPE percentage improvement, which is statistically significantly ($p = 0.05$) higher than the three naive methods[5]. To summarize, (1) the downstream demand information adds positive value to the order forecast even if it is incorporated in a simple way but (2) incorporating the policy structure shows the largest improvement. Moreover, we will later develop theory to predict for which product characteristics we expect high forecast accuracy improvement.

4. Model Setup

In this section, we describe the model setup and some preliminary results on the value of sharing information. Recall that we introduced a two-echelon supply chain in section 3. There are two key ingredients in our model: customer demand and the firm's replenishment policy. Notice that the policy we will illustrate coincides with the policy structure method we discussed in section 3. In this section, we introduce the actual policy followed by the company that we studied, which we will call the ConDI policy with order smoothing. We then show that the main result from Giloni et al. (2012) and Gaur et al. (2005) still holds if the retailer follows such a replenishment policy. The contradicting empirical evidence, however, suggests the theoretical model fails to capture a key element which is the decision deviation. The decision deviation relaxes the assumption, commonly made in the literature, that the decision maker perfectly adheres to the inventory replenishment policy. And finally, we develop the order process under such relaxation.

Recall that we consider a supply chain with two stages. The retailer is faced with demand D_t and places order O_t to the supplier during week t. There is a transportation lead time L_R from the supplier to the retailer. The supplier is the retailer's only source. Backlogging is allowed for the retailer. The retailer and supplier review their inventory periodically. Within each period, the following sequence of events occur: (1) the retailer's demand is realized and then the retailer places an order to the supplier, (2) after receiving the order, the supplier releases the shipment, (3) then

[5] We also assess the overall prediction improvement for these four InfoSharing methods (see Online Companion).

the supplier collects the latest information and predicts the future h-step ahead orders, (4) based on the updated prediction, the supplier makes production and replenishment decisions.

4.1. Demand Process

During each week t, the retailer faces demand D_t. We assume that D_t follows an autoregressive integrated moving average (ARIMA) process. The model is generally referred to as an ARIMA(p,d,q) model, where p, d and q represent the degree of the autoregressive, integrated and moving average parts of the model, respectively. The ARIMA model assumes that demand is a linear combination of historical observations and demand shocks. We first illustrate the demand process under $d=0$ and then derive the abbreviated expression for $d \geq 0$. When $d=0$, the ARIMA$(p,0,q)$ process is reduced to an ARMA(p,q) process

$$D_t = \mu + \rho_1 D_{t-1} + \rho_2 D_{t-2} + \cdots + \rho_p D_{t-p} + \epsilon_t - \lambda_1 \epsilon_{t-1} - \lambda_2 \epsilon_{t-2} - \cdots - \lambda_q \epsilon_{t-q}. \tag{6}$$

where μ is the process mean, ϵ_t is an i.i.d. normal demand shock with zero mean and variance σ_ϵ^2, ρ_i is the autoregressive coefficient and λ_i is the moving average coefficient.

The backward shift operator B shifts variables backward in time; e.g. $B^d D_t$ shifts demand back by d times $B^d D_t = D_{t-d}$, and $(1-B)D_t$ differences demand once $(1-B)D_t = D_t - D_{t-1}$. Differencing the demand twice means differencing $D_t - D_{t-1}$ one more time, $(1-B)^2 D_t = D_t - 2D_{t-1} + D_{t-2}$. Similarly, $(1-B)^d D_t$ differences the demand d times, which we refer to as the dth-order differenced demand. We assume the mean of demand is constant. Under this assumption, $E[(1-B)^d D_t] = 0$ for $d > 0$, which means the differenced demand $(1-B)^d D_t$ is a zero-mean ARMA process. Therefore, the differenced demand has process mean $\mu = 0$ for $d > 0$.

Let the AR coefficient be denoted as $\phi_{AR}(B) = 1 - \rho_1 B - \rho_2 B^2 - \cdots - \rho_p B^p$, the integration coefficient as $\pi(B) = (1-B)^d$, the ARI coefficient as $\phi_{ARI}(B) = \phi_{AR}(B)\pi(B)$ and the MA coefficient as $\varphi_{MA}(B) = 1 - \lambda_1 B - \lambda_2 B^2 - \cdots - \lambda_q B^q$. If we assume $\pi(B)D_t$ is an ARMA(p,q) process, then D_t is an ARIMA(p,d,q) process with $d \geq 0$. Then we replace D_t by $(1-B)^d D_t$ in equation (6), and rewrite equation (6) as

$$\phi_{ARI}(B) D_t = \mu + \varphi_{MA}(B)\epsilon_t. \tag{7}$$

We can rewrite the dth differenced demand $\pi(B)D_t$ in equation(7) as an MA representation

$$\pi(B) D_t = \mu + \varphi(B)\epsilon_t. \tag{8}$$

where μ is the process mean, $\varphi(B) = \phi_{AR}^{-1}(B)\varphi_{MA}(B)$ is the coefficient. We work with the MA representation because it provides the same intuition as an ARMA representation with a more concise analysis.

In the rest of this section, we will review two basic, yet important, properties of an MA process from the time series literature: covariance stationarity and invertibility. For details, we refer readers to Hamilton (1994) and Brockwell and Davis (2002). We assume the *dth* differenced demand is covariance stationary; that is, the differenced demand has a finite and constant mean, finite variance and time invariant covariance of $\pi(B)D_t$ and $\pi(B)D_{t+h}$ for any t and h. One might think that the MA model is restricted to a convenient class of models. However, representation (7) is fundamental for any covariance stationary time series. Any covariance stationary process is equivalent to an MA process in terms of the same covariance matrix (Wold 1938). Therefore, assuming the ARIMA model is not restrictive. We adopt Hamilton (1994, p. 109)'s description of the equivalence between the stationarity and MA representation, which is known as the Wold Decomposition property.

Property 1 (Wold Decomposition) *Any zero-mean covariance stationary process X_t can be represented in the MA form $X_t = \sum_{i=0}^{\infty} \alpha_i \epsilon_{t-i}$, where $\alpha_o = 1$ and $\sum_{i=0i}^{\infty} \alpha^2 < \infty$. The term ϵ_t is white noise and represents the error in forecasting: $\epsilon_t \equiv X_t - \hat{E}(X_t | X_{t-1}, X_{t-2}, \ldots)$.*

An MA process is determined by a unique covariance matrix. A covariance stationary process may have multiple MA representations in terms of different sets of coefficients α_i relative to their corresponding white noise series. Among the alternative representations, we are only interested in one that leads to the second property: invertibility. An MA process $X_t = \mu + \varphi(B)\epsilon_t$ is invertible relative to $\{\epsilon_t\}$ if the shock can be written as an absolutely summable sequence of past demands. An infinite sequence $\{\alpha_t\}$ is said to be absolutely summable if $\lim_{n\to\infty} \sum_{i=0}^{n} |\alpha_i|$ is finite.

Property 2 (Invertibility) *Define $\varphi(z) = 1 - \lambda_1 z^1 - \lambda_2 z^2 - \cdots - \lambda_q z^q$. Then ϵ_t can be written as an absolutely summable series of $\{X_s\}$ with $s \le t$, if and only if all roots of $\varphi(z) = 0$ lie outside of the unit circle, $\{z \in \mathbb{C}, |z| > 1\}$. We say that X_t is invertible relative to $\{\epsilon_t\}$.*

The invertibility guarantees future-independence: X_t is only correlated with past value of ϵ_t. Noninvertibility would allow for correlation with future values, which is undesirable. Invertibility is a property of the MA coefficients relative to the corresponding white noise series. According to Brockwell and Davis (2002, p. 54), for any noninvertible process $X_t = \varphi(B)\epsilon_t$, we can find a new white noise sequence $\{w_t\}$ such that $X_t = \varphi'(B)w_t$ and X_t is invertible relative to $\{w_t\}$. We say that the coefficient $\varphi'(B)$ is in the invertible representation. Therefore, when estimating the parameters of a time series process, estimators are restricted in the invertible set. That is, the empirically identified parameters have invertible representations. Henceforth, we assume the differenced demand process, $(1-B)^d D_t$, satisfies invertibility. This assumption has both intuitive appeal and technical consequences (for Proposition 1 and 3).

As Hamilton (1994, p. 68) points out, an MA process has at most one invertible representation, which has larger white noise variance than any other noninvertible representations. Later, we will illustrate that the enlarged white noise caused by converting from the noninvertible to invertible representation, is one trigger to the positive value of information sharing.

4.2. The Theoretical Model

To understand the policy used in practice, we interviewed the planner that placed orders. According to the planner, the retailer aims at keeping the DOI (days of inventory) level of the total on-hand inventory and in transit inventory constant. The decision maker also admits that the end inventory might not reach the target days of inventory level because the actual replenishment is not fast enough, i.e. retailer's capacity restriction. The smoothed order can explain such phenomenon both theoretically and empirically. We refer to the policy with smoothed order as the "ConDI policy with order smoothing", where "Con" represents constant, "DI" represents days of inventory and "order smoothing" captures a linear control rule that smoothes orders to produce a desirable order-up-to level. We first define the ConDI policy and then extend it with order smoothing.

Under a ConDI policy, the retailer places an order at the end of week t so that the inventory level reaches the week of inventory level ($7^{-1}\times$ target DOI level) multiplies the retailer's total future demand forecast within transportation lead time L_R. For example, if the target DOI level equals 14 and L_R equals 2, the retailer orders up to $2\times$ the demand forecast of next two weeks. If the demand is i.i.d distributed, then the optimal order up to level is constant, which coincides with the ConDI policy. When demands are correlated, the optimal order up to level changes every period. A fluctuating inventory target level is not convenient from the management perspective. Therefore, the ConDI policy becomes an attractive policy in practice.

We assume the DOI level Γ is positive and constant. We assume that the retailer's demand forecast for week $t+k$ made in week t is $\hat{D}^R_{t,t+k}$. Then the retailer's order-up-to level at the end of week t is $\Gamma\sum_{k=1}^{L_R}\hat{D}^R_{t,t+k}$, where L_R is the transportation lead time from the supplier to the retailer.

According to the planner, their forecast of future demands is a linear combination of past demands. Therefore, we assume the retailer's forecast of future L_R period demands given $D_t, D_{t-1}, ...$ is a linear combination of past H demands and we denote it as $\hat{m}_t$

$$\hat{m}_t \equiv \sum_{k=1}^{L_R}\hat{D}^R_{t,t+k} = \sum_{j=0}^{H}\beta_j D_{t-j}. \tag{9}$$

where β_j is the coefficient of demand in past jth period. The sum of the demand coefficients is the retailer's lead time, $\sum_{j=0}^{H}\beta_j = L_R$. When $L_R = 1$, the forecast is the weighted sum of the current and past H periods' demands.

In order to capture order smoothing, we extend the ConDI policy by allowing a fixed proportion of last week's inventory to become the current week's inventory. Irvine (1981) introduces a similar notion and empirically confirms that firms attempt a partial adjustment towards the optimum level. Balancing the product inflow and outflow, the sum of the proportion of last week's inventory and target inventory under the ConDI policy should equal 1. Therefore, the inventory becomes

$$I_t = \gamma\Gamma\hat{m}_t + (1-\gamma)I_{t-1}. \tag{10}$$

where γ is the order smoothing level, which takes values in $[0,1]$.

Given the fundamental law of material conservation, $O_t = D_t + I_t - I_{t-1}$, equation (10) becomes

$$O_t = D_t + \gamma(\Gamma\hat{m}_t - I_{t-1}). \tag{11}$$

The order in week t is the current week's demand plus γ fraction of the net inventory under the ConDI policy. If $\gamma = 1$, it is reduced to the strict ConDI policy. If $\gamma = 0$, it becomes the demand replenishment policy. The larger γ, the faster the order adjusts to the target ConDI inventory level. The order smoothing component enables the extension of the ConDI policy to a rich family of linear policies.

We can iteratively replace I_{t-i} with $\gamma\Gamma\hat{m}_{t-i} + (1-\gamma)I_{t-i-1}$ for any $i \geq 0$ in equation (11). We define $a_i \equiv \Gamma\beta_i$ for $0 \leq i \leq H$, where a_i is the policy coefficient of the past ith demand. Then $\Gamma\hat{m}_t = \sum_{j=0}^{H} a_j D_{t-j}$ and the order becomes

$$O_t = D_t + \gamma \sum_{i=0}^{H} a_i D_{t-i} - \gamma^2 \sum_{i=1}^{\infty} (1-\gamma)^{i-1} \sum_{j=0}^{H} a_j D_{t-i-j}. \tag{12}$$

We define $\psi(B) \equiv 1 + \gamma\sum_{i=0}^{H} a_i B^i - \gamma^2 \sum_{i=1}^{\infty}\sum_{j=0}^{H}(1-\gamma)^{i-1} a_j B^{i+j}$ as the policy parameter. Applying the backshift operator, we abbreviate equation (12) as $O_t = \psi(B)D_t$. Thus we have $\pi(B)O_t = \pi(B)\psi(B)D_t$. Since demand satisfies $\pi(B)D_t = \mu + \varphi(B)\epsilon_t$, the demand process can be written as $\pi(B)\psi(B)D_t = \mu + \varphi(B)\psi(B)\epsilon_t$. Therefore, the order process follows an ARIMA process with white noise $\{\epsilon_t\}$:

$$\pi(B)O_t = \mu + \varphi(B)\psi(B)\epsilon_t. \tag{13}$$

Equation (13) has the same expression as equation (7) in Gaur et al. (2005): order is linear in demand shocks. It is worth noting that our policy parameters $\psi(B)$ capture a broader linear policy than the myopic order up to policy considered in Gaur et al. (2005). The myopic order up to policy corresponds to a special case when $\gamma = 1$ in equation (12) (this is equivalent to equation (4) in Gaur et al. 2005).

The coefficients of ϵ_t in equation (13) are obtained by multiplying the demand coefficient $\varphi(B)$ of ϵ_t in D_t with the policy coefficient $\psi(B)$ of D_t in O_t. Therefore, the first coefficient in equation (13) is $C \equiv 1 + \gamma a_0$. We normalize the first coefficient to be 1. Then the centered order follows an MA process with white noise $\{C\epsilon_t\}$

$$\pi(B)O_t - \mu = C^{-1}\varphi(B)\psi(B)C\epsilon_t. \tag{14}$$

The analysis of the value of information sharing. As introduced in section 3, the supplier aims to forecast the future order. In the rest of this section, we will focus on the 1-step ahead forecast as it provides the insight to the positive value of information sharing and serves as the theoretical foundation that we can compare with the empirical results. Section 6 will discuss the h-step ahead forecast in detail in a more general setting.

In our theoretical analysis, we adopt a mathematically tractable forecast error metric: the mean squared forecast error. We denote the space that contains the linear combination of the order history from period 1 to period t and all its limit points as Ω_t^O. By definition, Ω_t^O is the Hilbert space generated by the order history. Therefore, $\Omega_t^O \cup \Omega_t^D$ includes both the order and demand history. According to the Projection Theorem, the unique optimal estimator to minimize the mean squared error can be found conditional on either Ω_t^O or $\Omega_t^O \cup \Omega_t^D$. As before, we denote it as $\hat{O}_{t,t+1}$. The 1-step mean squared forecast error without information sharing is $\text{Var}(O_{t+1} - \hat{O}_{t,t+1}|\Omega_t^O)$ and with sharing is $\text{Var}(O_{t+1} - \hat{O}_{t,t+1}|\Omega_t^O \cup \Omega_t^D)$. The value of information sharing is positive if

$$\text{Var}(O_{t+1} - \hat{O}_{t,t+1}|\Omega_t^O) > \text{Var}(O_{t+1} - \hat{O}_{t,t+1}|\Omega_t^O \cup \Omega_t^D). \tag{15}$$

With downstream demand information, the demand and policy parameters can be estimated. We assume the parameters can be correctly estimated and are known to the supplier. The only uncertainty in $\hat{O}_{t,t+1} - O_{t+1}$ stems from the demand shock occurring in $t+1$. Therefore, $\text{Var}(O_{t+1} - \hat{O}_{t,t+1}|\Omega_t^O \cup \Omega_t^D) = \text{Var}(C\epsilon_t)$.

Without information sharing, the supplier analyzes the order history as an MA process. The MA process in equation (14) may not be invertible with respect to $\{C\epsilon_t\}$. If not, we can find an invertible representation relative to a new white noise series $\{w_t\}$, which has a larger variance than $\text{Var}(C\epsilon_t)$. Then inequality (15) holds and thus the value is positive. Gaur et al. (2005) and Giloni et al. (2012) show similar intuitions for the positive value of information sharing. The following proposition states the sufficient and necessary condition that sharing demand benefit the supplier's order forecast.

Proposition 1 *If the decision maker strictly adheres to the replenishment policy, the value of information sharing under the one step forecast lead time is positive if and only if at least one root of $\psi(z) = 0$ lies inside the unit circle.*

The value of information sharing is positive if and only if $\varphi(B)\psi(B)$ is in the noninvertible representation, which in turn is equivalent to the existence of at least one root of $\varphi(z)\psi(z) = 0$ that lies inside the unit circle. Since all roots of $\varphi(z) = 0$ lie outside the unit circle due to the invertible assumption, the order is noninvertible relative to ϵ_t if and only if there exists at least one root of $\psi(z) = 0$ that lies inside the unit circle.

Remark. We next illustrate two extreme setting of $\gamma = 0$ and $\gamma = 1$ and show that in both setting there is no value of information sharing. When $\gamma = 0$, it becomes the demand replacement policy $O_t = D_t$ and $\psi(z) = 1$. Since no root of $\psi(z) = 0$ lies inside the unit circle, the value of information sharing is zero. When $\gamma = 1$ and $H = 0$, it becomes the ConDI policy with policy parameter $\psi(z) = 1 + a_0 - a_0 z$. Since $\sum_{j=0}^{H} a_j = \Gamma L_R$, coefficient a_0 is positive. Since the unique root of $\psi(z) = 0$ is larger than 1, $z = (1 + a_0)/a_0 > 1$, the value of sharing is zero. From these two examples, we can see that the value of information sharing under a strict replenishment policy can be zero.

4.3. The Empirical Model: Decision Deviations

The empirical results in section 3 indicate incorporating the downstream demand properly yields statistically significantly positive value of information sharing for all low-promotional products. The above theoretical results, however, suggest zero value for 10 out of 14 products based on the estimated parameters that we will show in Section 6. These empirical deviations call for a better theoretical understanding of the model in previous literature.

The key underlying assumption in the theoretical model described above and in the literature is that the decision maker strictly and consistently follows a family of linear decision rules. Our discussion with the replenishment decision maker suggests this is rarely the case because the decision makers can implement their own adjustment based on the additional signals that we do not observe. The empirically observed idiosyncratic shocks in the order decisions also indicate that the decision makers may not replenish as the theory requires, or that the theoretical model does not capture all elements of reality.

From the interview with the planner, we understand that the deviation from the theoretical model stems from several operational causes. The order quantity is rounded due to transportation and truck load constraints. To increase transportation efficiency, the retailer tries to fill up a full truck when placing an order. Products with inventory above the target DOI level might still be replenished because delivering in batches can decrease set up cost. Such a phenomenon is common as the week approaches Friday, because the decision maker needs to guarantee enough inventory. Orders might be moved from peak to nonpeak periods if planners anticipate a spike in future demands (Donselaar et al. 2010 also points out such advancing orders as an important consideration of the decision maker). In practice, the retailer might place orders daily. However, for this study, we have access only to the weekly aggregate level instead of daily information. Looking through the lens of the aggregate data, we lose the detail on the replenishment decision, which is reflected by the actual order's departure from the theory.

Among the above different operational drivers, a common characteristic is that the decision maker adjusts replenishment according to those drivers while statisticians cannot observe them. We rationalize the agent's departure from the exact policy following the same spirit as Rust (1997): it is a state variable which is observed by the agent but not by the statistician. Since the actual observations always contain randomness from the observational perspective of the analyst, the empirical model should successfully capture it and might yield qualitatively different results than the literature.

We extend the theoretical framework by including the idiosyncratic shocks in decision making, and thus relax the strict adherence to the ConDI with order smoothing policy. We refer to such idiosyncratic shocks as decision deviation. The decision deviation is observable to the retailer, but not to statistician. We assume the decision deviation δ_t is normally distributed with zero mean and variance σ_δ^2, and independent with historical demand shock $\epsilon_s, s < t$. However, contemporaneous demand signals and decision deviation signals can be correlated. A common approach in the empirical literature is to model this error term as additively separable, in the decision. Using this approach, we obtain

$$O_t = D_t + \gamma(\Gamma \hat{m}_t - I_{t-1}) + \delta_t. \tag{16}$$

As before, we iteratively replace I_{t-i} with $\gamma\Gamma\hat{m}_{t-i} + (1-\gamma)I_{t-i-1} + \delta_{t-i}$ in equation (16) and obtain

$$O_t = D_t + \gamma \sum_{i=0}^{H} a_i D_{t-i} - \gamma^2 \sum_{i=1}^{\infty} (1-\gamma)^{i-1} \sum_{j=0}^{H} a_j D_{t-i-j} + \delta_t - \sum_{i=1}^{\infty} \gamma(1-\gamma)^{i-1} \delta_{t-i}. \tag{17}$$

We define $\kappa(B) = 1 - \gamma \sum_{i=1}^{\infty} (1-\gamma)^{i-1} B^i$ as the order smoothing parameter. Applying the backshift operator, equation (17) can be abbreviated as $O_t = \psi(B)D_t + \kappa(B)\delta_t$. Applying $\pi(B)$ to both sides, order process can be abbreviated as:

$$\pi(B)O_t = \mu + \varphi(B)\psi(B)\epsilon_t + \pi(B)\kappa(B)\delta_t. \tag{18}$$

where μ is the process mean, $\varphi(B)\psi(B)$ is the demand shock coefficient and $\pi(B)\kappa(B)$ is the decision deviation coefficient.

ARMA-in-ARMA-out property. The order with decision deviation has a stationary covariance. According to property 1, the order process in equation (18) follows an ARIMA model. This is consistent with the "ARMA-in-ARMA-out" (AIAO) property discussed in the literature (Zhang 2004, Gilbert 2005, Gaur et al. 2005 and Giloni et al. 2012), where AIAO means that the retailer's order process is also an ARMA process with respect to the demand shock. If the replenishment policy is an affine and time invariant function of the historical demand, inventory, demand shock and decision deviation, the order process has a stationary covariance. Therefore, the AIAO property holds for such policies.

5. Strictly Positive Value of Information Sharing

In this section, we study the impact of decision deviation on the value of information sharing and prove that the value of information sharing is always positive if there is uncertainty in both decision deviation and demand processes.

We rewrite the order in equation (18) as a centered process

$$O_t - \pi^{-1}(B)\mu = C^{-1}\pi^{-1}(B)\varphi(B)\psi(B)C\epsilon_t + \kappa(B)\delta_t. \tag{19}$$

where the constant C normalizes the first coefficient to one as defined before. Let q_ϵ denote the degree of $\pi^{-1}(B)\varphi(B)\psi(B)$ and q_δ denote the degree of $\kappa(B)$. The centered order is the summation of two MA processes with demand shock and decision deviation as their corresponding white noise series. We study the value of information sharing with the existence of decision deviations.

Preliminary results. Our key question is closely related to the general goal of forecasting the aggregation of multiple MA processes.

Consider N MA processes (N can be infinite) where the process i is $X_t^i = \chi_i(B)\epsilon_t^i$ with i.i.d. random shock ϵ_t^i. The coefficient is $\chi_i(B) = 1 + \lambda_1^i B + \lambda_2^i B^2 + \cdots + \lambda_{q_i}^i B^{q_i}$ with degree q_i. When predicting future value beyond q_i periods, the forecast is constant and uncertainty cannot be resolved. Thus q_i denotes the effective forecasting range for process X_t^i. We allow contemporaneous signals to be correlated, but require signals to be independent across periods. That is, ϵ_t^i is independent of ϵ_s^j for any $s < t$. The summation of these N processes is

$$S_t = \sum_{i=1}^{N} X_t^i. \tag{20}$$

According to Property 1, S_t can be rewritten as an MA process with degree $q_S \geq 0$, where q_S is the largest k that guarantees nonzero covariance $\mathrm{Cov}(S_t, S_{t+k}) \neq 0$.

With full information (or with information sharing), we have access to each process's history and parameters. With aggregate information (or without information sharing), we only have access to the aggregate process S_t. As before, $\Omega_t^{X^i}$ and Ω_t^S denotes the Hilbert space generated by X_t^i and S_t history through period t. Let $\hat{X}_{t,t+h}^i$ and $\hat{S}_{t+h}$ denote the best estimator to minimize the mean squared forecast error for X_{t+h}^i and S_{t+h}. With information sharing, the h-step ahead mean squared error is $\mathrm{Var}(S_{t+h} - \hat{S}_{t,t+h} | \cup_i \Omega_t^{X^i})$. Without information sharing, the h-step ahead mean squared error is $\mathrm{Var}(S_{t+h} - \hat{S}_{t,t+h} | \Omega_t^S)$. The value is positive for forecast lead time h if $\mathrm{Var}(S_{t+h} - \hat{S}_{t,t+h} | \cup_i \Omega_t^{X^i}) < \mathrm{Var}(S_{t+h} - \hat{S}_{t,t+h} | \Omega_t^S)$. The following theorem states the sufficient and necessary condition for the zero value of information sharing.

Theorem 2 *The* 1*-step mean squared forecast error is the same with and without sharing,* $\mathrm{Var}(S_{t+1} - \hat{S}_{t,t+1} | \cup_i \Omega_t^{X^i}) = \mathrm{Var}(S_{t+1} - \hat{S}_{t,t+1} | \Omega_t^S)$ *if and only if the MA processes satisfy* $\chi_i(B) = \chi_j(B)$ *for any* i, j*. If there exists* $i \neq j$ *such that* $\chi_i(B) \neq \chi_j(B)$*, then* $\mathrm{Var}(S_{t+h} - \hat{S}_{t,t+h} | \cup_i \Omega_t^{X^i}) < \mathrm{Var}(S_{t+h} - \hat{S}_{t,t+h} | \Omega_t^S)$ *for any finite forecast lead time* $h \leq \max_i\{q_i\}$*.*

Among N processes, if coefficients of any two processes differ, the aggregate process has strictly larger mean squared forecast error as long as the forecast is within the effective forecast range of one process, $h \leq \max_i\{q_i\}$. If $q_i = 0$, X_t^i becomes an i.i.d. normal model with the coefficient $\chi_i(B) = 1$. If $q_i = 0$ for all i, the processes have the same coefficients, and thus the value of information sharing is zero. If $\max_i\{q_i\} = \infty$, the value of information sharing is strictly positive for any finite forecast lead time if $\varphi_i(B) \neq \varphi_j(B)$.

Analysis of our model. Let us apply this general result to the order process in our setting in equation (19). The centered order has the same structure as equation (20), where the two processes are with respect to demand shocks and decision deviations

$$X_t^1 = C^{-1}\pi^{-1}(B)\varphi(B)\psi(B)C\epsilon_t, \tag{21}$$
$$X_t^2 = \kappa(B)\delta_t.$$

We can apply Theorem 2 to determine whether the value of information sharing is positive. Similar as before, if $\text{Var}(O_{t+h} - \hat{O}_{t,t+h}|\Omega_t^O) > \text{Var}(O_{t+h} - \hat{O}_{t,t+h}|\Omega_t^O \cup \Omega_t^D)$, then the value of information sharing is positive. The following proposition illustrates the result.

Proposition 3 *If the demand shock and decision deviation are nonzero, the value of information sharing is strictly positive for any finite forecast lead time* $h \leq \max\{q_\epsilon, q_\delta\}$.

When $q_\epsilon = q_\delta = 0$, both X_t^1 and X_t^2 are i.i.d. processes and S_t is also an i.i.d. process. Any forecast is a constant, and thus there is no value from sharing the downstream sales information. This situation can only occur when $\varphi(B) = \pi(B) = \psi(B) = \kappa(B) = 1$, which means the retailer faces an i.i.d. demand processes and adopts a demand replacement policy, which we refer to as "i.i.d demand replacement". In the rest of the paper, we will exclude the discussion on this situation, because using historical observations cannot resolve any uncertainty of the future forecast.

If not both processes are i.i.d. models, or equivalently $q_\epsilon = q_\delta = 0$ is not true, then the two sets of parameters $C^{-1}\pi^{-1}(B)\varphi(B)\psi(B)$ and $\kappa(B)$ can never be the same. The key ingredient in the proof is to show that the polynomial $(1-B)$ is a factor in $\kappa(B)$ but not a factor in $C^{-1}\pi^{-1}(B)\varphi(B)\psi(B)$. Therefore, the value of information is strictly positive for any forecast lead time.

Compared with the conditions on the policy parameter $\psi(B)$ that induces positive value of information sharing under the strict adherence to a linear policy, Proposition 3 establishes a qualitatively different conclusion: the benefit of information sharing is strictly positive within any forecasting period. Under the strict adherence to the inventory policy, the planner makes replenishment decisions based only on information that statisticians also have access to, which leads to a pure demand propagation. The interview with the planner and our data suggests that this is rarely the case and the decision departures from the ideal policy due to private information that statisticians can not observe. Thus, unlike before, the demand now propagates together with the decision deviation. The value of information sharing is zero if the order process carries all demand information and all decision deviation information. The different propagation patterns of the demand process and decision deviation process, however, drive the loss of information as demand and decision deviation propagate upstream. To be specific, the ending inventory level carries current period's decision deviation and rolls it over to next period's replenishment decision which further determines the next period's ending inventory. Thus the evolution of inventory governs the translation of exogenous decision deviation signals into orders. Demand signals, on the other hand, are governed by the evolution of both inventory and current demand. As both signals propagate together to become orders in such innately different patterns, the detailed information of two processes is lost and is replaced with the less informative (larger uncertainty) order signals. Consequently, the value of information sharing becomes positive regardless of the policy parameter $\psi(B)$.

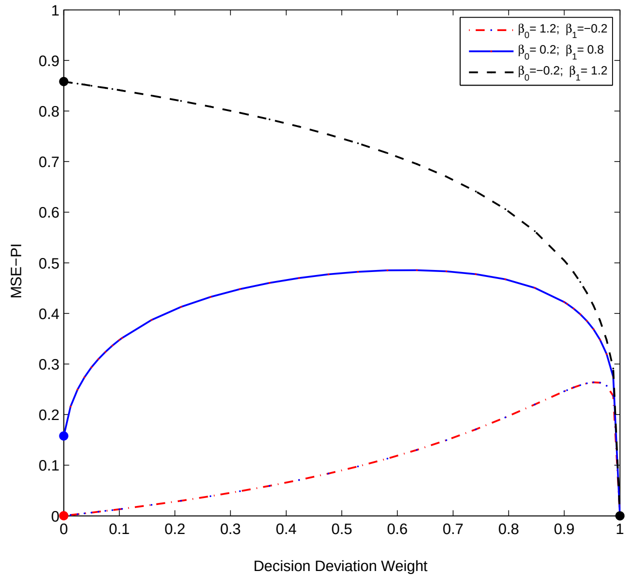
The same intuition holds for any linear replenishment policy. It's worth noting that the evolution patterns of the demand signals and decision deviation signals are different as illustrated above for other ordering policy that is linear in demands and demand signals, i.e. the myopic order up to policy, the ConDI policy with retailer's demand forecast being optimal (we assume that it is linear in past H demands due to practice in our study) and the generalized order-up-to policy introduced by Chen and Lee (2009) (see Online Companion for more discussion), except for the "i.i.d demand replacement" (of which we care less since forecasting beyond zero period is constant). Therefore, the distinct propagation patterns obscure the detailed information structure, which drives positive value of information sharing for any linear inventory policy under any forecast lead time.

Proposition 3 illustrates the value of information sharing when both demand and decision uncertainties are nonzero. If there is no decision deviation, Proposition 1 demonstrates the sufficient and necessary condition of positive value of information sharing. The following proposition, on the other hand, considers another extreme case when demand uncertainty is zero.

Proposition 4 *When the demand shock is zero, the value of information sharing is zero for any forecast lead time.*

In absence of demand shock, the centered order is reduced to an MA process with respect to decision deviations, $O_t - \pi^{-1}(B)\mu = \kappa(B)\delta_t$. For the value of sharing downstream information to be zero, the centered order must be invertible relative to δ_t (equivalent to $\kappa(B)$ has an invertible representation). The unique root of $\kappa(z^*) = 0$ lies on the unit circle, which Plosser and Schwert (1997) defined as strictly non-invertibility. When $|z^*| = 1$, there is no corresponding invertible representation. The author shows that the univariate MA parameter's estimator is asymptotically similar to the invertible processes, which indicates parameters $\kappa(B)$ can be correctly estimated from historical orders. Therefore, we can still apply the result for the invertible process and conclude that the value of information sharing is zero.

Figure 1 The MSE percentage improvement against the decision deviation weight for an ARIMA(0,1,1) demand with $\lambda = 0.5$ and a ConDI policy with order smoothing with $\gamma = 0.8$ and $\Gamma = 2$.

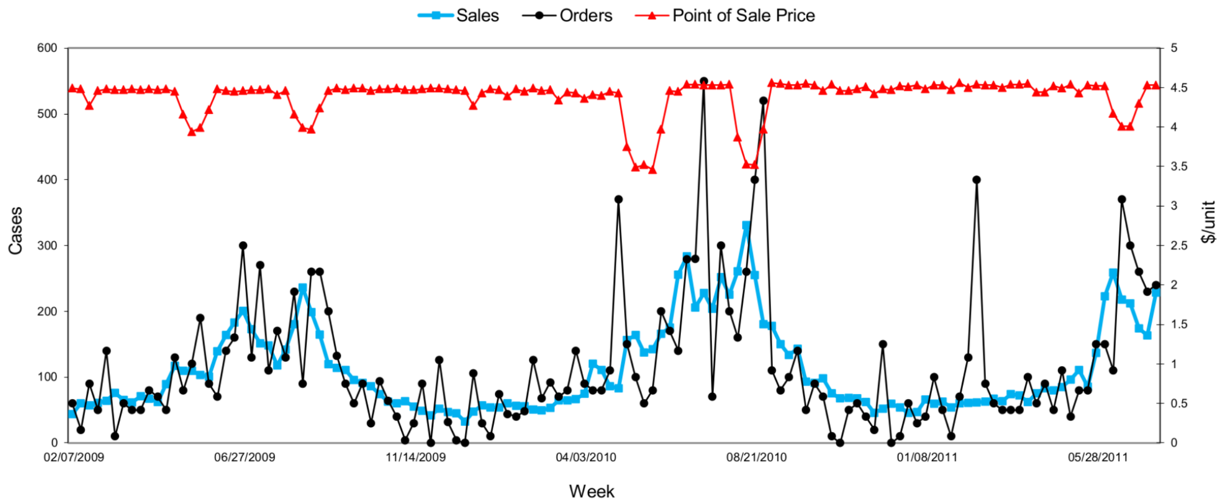


To summarize the above theoretical findings, we characterize the value of information sharing with a numerical analysis. We apply the MSE percentage improvement metric introduced in Section 3. Let $MSE\text{-}PI$ denote the MSE percentage improvement over no information sharing. Recall that $MSE\text{-}PI = (\mathrm{Var}(O_{t+h} - \hat{O}_{t,t+h}|\Omega_t^O) - \mathrm{Var}(O_{t+h} - \hat{O}_{t,t+h}|\Omega_t^O \cup \Omega_t^D))/\mathrm{Var}(O_{t+h} - \hat{O}_{t,t+h}|\Omega_t^O)$, which takes value between 0 and 1.

Figure 1 displays the MSE-PI of the 1-step-ahead forecast with respect to the relative weight of the decision deviation under several sets of policy parameters. We define $\sigma_\delta^2/(\sigma_\epsilon^2 + \sigma_\delta^2)$ as the deviation decision weight. The retailer places weights β_i on historical demands to determine their future forecast as in equation (9). Keeping the DOI level and the order smoothing level fixed, we choose three sets of policy weight β_i $(i = 0, 1)$ that correspond to three lines in Figure 1. We consider the retailer faces an ARIMA$(0,1,1)$ demand, $D_t = D_{t-1} + \epsilon_t - \lambda\epsilon_{t-1}$, with the MA parameter $\lambda = 0.5$. The policy parameter $\psi(B)$ is non-invertible for the top two processes while invertible for the bottom one.

Our theoretical prediction aligns with the numerical observations. When the decision uncertainty is zero, the value of information sharing is positive for the first two and zero for the last policy parameters. This pattern is consistent with Proposition 1. Note that the studies in the literature correspond to the points on the vertical axis where the decision deviation weight is zero. As decision deviation become dominant, there is no gain from sharing the downstream sales information, which coincides with Proposition 4. When the decision deviation and demand uncertainty both exist, Figure 1 presents a strictly positive value of information sharing, which agrees with Proposition 3.

Figure 2 **Summary of sales, orders and point-of-sale price for product PD OR.**



6. Data, Estimation and Model Validation

6.1. Data

As we introduced in Section 3, our data set is provided by a leading supplier in the beverage and snack food industry and consists of three elements corresponding to a specific retail customer: (1) the retailer's sales, (2) the retailer's orders to the supplier and (3) the products' retail price. The data spans over 126 weeks between 2009 and 2011. To be specific, the raw data consists of the retailer's sales from its six distribution centers to local stores, and orders from the retailer's six distribution centers to the supplier's distribution center. For the purpose of our analysis, we work with the aggregated sales and orders. We calculate the retailer's inventory using the fundamental law of material conservation, given sales and orders. Without information sharing, the supplier only observes the retailer's order and the price plan. With information sharing, the supplier observes additional information: the retailer's sales. We eliminate untrustworthy data, the new-entering products that have not reached the stationary state or obsolete products that are existing the market. After cleaning the data, we have 51 product lines in total: 19 orange juice products and 32 sports drink products.

We summarize the sales, orders and price of a specific product over 126 weeks in Figure 2. It shows the bullwhip effect: the upstream order has larger volatility than the downstream sales. Further, when there is a price promotion, the demand experiences a spike during the discount activity and suffers a slump as price returns to normal.

For the purpose of our study, we shall classify the products into low-promotional and high-promotional products and focus on the former[6]. This classification is based on the price discount

[6] The Online Companion provides a discussion on the promotional products.

and frequency of discount being offered on the product. From the data, we observe that the beverage products are frequently on sale. A promotional activity can last for several weeks. During a promotional activity, all retailer's local stores execute the same price discount plan. Negotiating at the beginning of each year, the supplier has a fixed price plan throughout the year. Thus the future price can help predict demand changes for promotional products. The spikes in orders caused by promotions result in a non-constant demand mean and perhaps a time-variant covariance matrix, breaking the stationary assumption that we use to develop our theory.

To make the above precise, we define promotional depth metric to capture price discount and frequency. Promotional depth sums every promotion activity's price discount measured as a percentage within the last 26 weeks in our data, $\sum_i \text{discount rate}_i$ where $i \leq$ the number of activities in 26 weeks. We define the low-promotional product as those with positive depth ≤ 0.15 (or equivalently no promotional activity or one promotional activity), and the high-promotional product as those with higher promotional depth. The low-promotional items contain 14 product lines and occupy 20% of the total ordering volume of all products. In the rest of this section, we will only discuss the methods and results for these 14 low-promotional items.

6.2. Estimation Procedure and Parameter Results

In Section 3, we presented the forecast accuracy of the policy structure method and concluded that the forecast improvement of adopting such method to incorporate the downstream demand is statistically significant. In this section, we first describe the estimation of the *policy structure method*. Next, we show the estimated demand and policy parameters. Note that our analysis only uses sales and orders data.

We assume that the retailer adopts the ConDI policy with order smoothing. In each period, the order is $D_t + \gamma(\Gamma m_t - I_{t-1}) + \delta_t$. Recall that $m_t = \sum_{i=0}^{H} a_i D_{t-i}$. In practice, the retailer's forecast usually accounts for last months' demands. Thus, we let $H = 3$. We rewrite equation (16) as $O_t = (1+\gamma\Gamma\beta_0)D_t + \gamma\Gamma\beta_1 D_{t-1} + \gamma\Gamma\beta_2 D_{t-2} + \gamma\Gamma\beta_3 D_{t-3} - \gamma\Gamma I_{t-1} + \delta_t$. The estimating equation then becomes

$$O_t = c_0 D_t + c_1 D_{t-1} + c_2 D_{t-2} + c_3 D_{t-3} + c_{inv} I_{t-1} + \delta_t \tag{22}$$

We run a linear regression of equation (22) to estimate the policy parameters for each week in the test period. We apply the step-wise variable selection method to only include variables with $p < 0.05$ in the regression. The idiosyncratic shock in the order equation is the decision deviation. If δ_t is positive, the retailer orders more than what our policy predicts and vice versa.

To forecast supplier's order in $t+1$, we first forecast future demands. We fit the ARIMA model to forecast $\hat{D}_{t,t+1}$. With the parameters estimated from equation (22), the order prediction for period $t+1$ uses $\hat{D}_{t,t+1}$ and D_s where $s \leq t$ and I_t:

$$\hat{O}_{t,t+1} = c_0 \hat{D}_{t,t+1} + c_1 D_t + c_2 D_{t-1} + c_3 D_{t-2} + c_{inv} I_t. \tag{23}$$

Table 2 Estimated demand and policy parameters. The number in parenthesis denotes the standard error of the estimate.

		Demand parameters				Policy parameters					
Brand	Product	(p,d,q)	λ_1	λ_2	λ_3	c_0	c_1	c_2	c_3	c_{inv}	DOI
Orange	128 OR	$(0,1,1)$	0.93			1.30	0.27			-0.63	6.36
Juice			(0.04)			(0.18)	(0.16)			(0.12)	
	128 ORCA	$(0,1,1)$	0.93			1.33	0.30			-0.53	8.48
			(0.04)			(0.18)	(0.17)			(0.12)	
	12 OR	$(0,1,2)$	0.48	0.33		1.46		0.60		-0.87	8.56
			(0.10)	(0.10)		(0.17)		(0.18)		(0.12)	
	12 ORCA	$(0,1,2)$	0.28	0.25		0.97	1.09		0.36	-0.86	11.54
			(0.10)	(0.10)		(0.21)	(0.23)		(0.19)	(0.12)	
	59 ORST	$(0,1,1)$	0.72			1.55				-0.39	9.76
			(0.07)			(0.10)				(0.07)	
	59 ORPC	$(0,1,1)$	0.8			1.84			-0.46	-0.28	9.58
			(0.07)			(0.18)			(0.22)	(0.08)	
Sports	500 BR	$(0,1,3)$	0.12	0.16	0.49		1.17	0.55		-0.35	14.50
Drink			(0.09)	(0.09)	(0.09)		(0.29)	(0.33)		(0.07)	
	500 GP	$(0,1,0)$				0.39	0.67	0.63		-0.36	13.43
						(0.25)	(0.38)	(0.29)		(0.07)	
	PD LL	$(0,1,0)$					0.61	1.11		-0.24	21.06
							(0.40)	(0.44)		(0.06)	
	PD OR	$(0,1,1)$	0.3				1.37		0.39	-0.29	18.35
			(0.10)				(0.16)		(0.23)	(0.07)	
	PD FRZ	$(0,1,1)$	0.32				1.27		0.47	-0.32	16.22
			(0.09)				(0.16)		(0.22)	(0.07)	
	1GAL GLC	$(0,1,2)$	0.2	0.43		0.76	1.17		-0.54	-0.22	12.38
			(0.09)	(0.09)		(0.25)	(0.28)		(0.21)	(0.08)	
	1GAL FRT	$(0,1,2)$	0.37	0.36			1.49			-0.25	13.38
			(0.10)	(0.10)			(0.13)			(0.07)	
	1GAL OR	$(0,1,2)$	0.29	0.34			1.52			-0.30	12.33
			(0.10)	(0.10)			(0.12)			(0.07)	

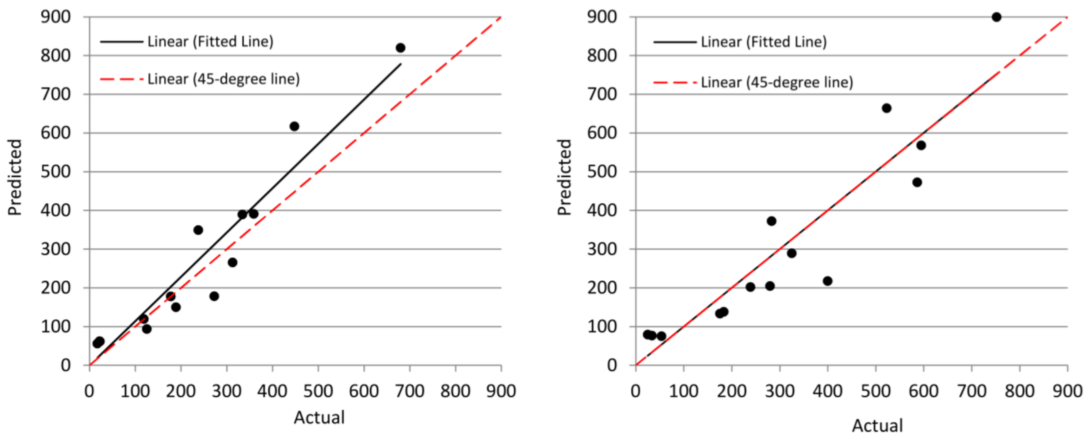
We present the demand and policy parameters in Table 2. For all products, demand has $d = 1$, which implies that the first-order differenced demand is an ARMA process. The transportation lead time from the supplier to the retailer is one week, thus we consider the case that $L_R = 1$. Therefore, $\sum_{i=0}^{H} a_i = 1$ and $\Gamma = -c_{inv}^{-1}(\sum_{i=0}^{H} c_i - 1)$. The last column displays the DOI level. In practice, the retailer targets at a lower DOI level for the orange juice brand and a higher DOI for the sports drink brand. Our estimated DOI level is consistent with the actual target level claimed by the decision maker.

6.3. Model Validation

In this section, we validate whether the signal propagation agrees with our theory predictions. To this end, we compare the predicted and actual root mean squared forecast error, separately under the *policy structure method* and NoInfoSharing method. To be specific, we first derive the actual forecast. We follow the above estimation procedure to obtain the actual forecast under information sharing. We fit the ARIMA model and forecast order using only order information without information sharing. Second, based on the theory we explained in section 5, we calculate the root mean square error, which we refer to as the predicted value.

We next elaborate on the procedure used to obtain the predicted value. We calculate the in-sample mean squared error from demand when fitting the ARIMA model in Table 2, and we use

Figure 3 **Actual vs predicted root mean squared forecast error with information sharing (left) and without information sharing (right).**



it as σ_ϵ^2. We calculate the in-sample mean squared error from estimating policy in equation (22), and we use it as σ_δ^2. Together with the estimated demand and policy parameters, we know all the parameters in the model[7]. We empirically test the correlation between contemporaneous shocks δ_t and ϵ_t. The result suggests weak correlation or no correlation. Therefore, when there is information sharing, the mean squared forecast error is $c_0^2\sigma_\epsilon^2 + \sigma_\delta^2$. In absence of information sharing, we calculate the mean squared error by applying the Innovation Algorithm used in the time series literature (see Online Technical Companion).

We present the results in Figure 3. We plot the fitted against actual root mean squared prediction error under both information sharing (left) and no information sharing (right) case. A perfect model fit would lead to the points lying on the 45-degree dashed line in the figure. The fitted points from our model are overall close to the 45-degree line for both with information and no information, indicating a good fit. We fit a regression of the theoretical prediction on the actual observation. The 95% confidence interval is $[1.01, 1.27]$ under the information sharing setting and $[0.86, 1.14]$ under the no information sharing setting.

The good fit indicates that our theoretical model with decision deviations can well explain how demands prorogate upstream, and thus well predict the value of information sharing. In Section 5, we proved that the presence of decision deviations guarantees strictly positive value under any forecast lead time. In the following section, we will investigate how the value of sharing changes with respect to the demand, policy and lead time on the value of information sharing.

[7] In the policy parameter sector of Table 2, for some products, the coefficient of current week's demand is zero, which means the retailer's replenishment fulfillment places zero weight on current week's demand. This is unlikely to occur in practice. Our estimation shows zero coefficient because the retailer may replenish inventory during the week, but our data set consists of system's snapshots at the end of the each week. If the retailer replenishes certain products always on Monday, the current week's order should be a linear combination of past weeks' demand, not including the current week (since current week's demand has not been realized yet). Therefore, for products with zero c_0 in Table 2, we interpret a_1 as the actual coefficient of current week (shift a_2 and a_3 in the same way).

7. Properties of the Value of Information Sharing

While we have shown that the value is positive, we have not specified its magnitude and how it changes relative to other key variables. In this section, we investigate the impact of demand on the value of information sharing. We study how the value changes with respect to the parameter of an ARIMA$(0,1,1)$ demand process. (We focus on this particular form of ARIMA demand process as 8 out of 14 products in the data have this structure). We then show that the theoretically obtained relation is consistent with our empirical observations. We focus on the 1-step-ahead forecast. We theoretically analyze two special cases to understand the intuition and resort to numerical studies for more involved settings.

We analyze a simple yet reasonable model to derive the theoretical prediction. The empirical estimation suggests that 8 out of 14 products follow an ARIMA$(0,1,1)$ demand. Therefore, in this section, we assume that the demand follows an ARIMA$(0,1,1)$ process with parameter $\lambda \in [0,1)$,

$$D_t = D_{t-1} + \epsilon_t - \lambda \epsilon_{t-1},$$

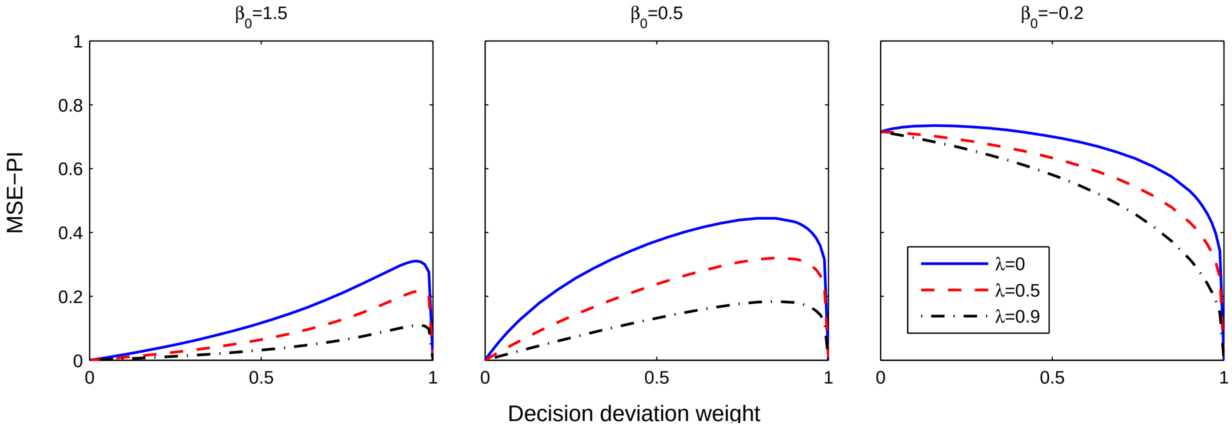
which can be equivalently written as $D_t = (1-\lambda)\sum_{i=1}^{\infty} \lambda^{i-1} D_{t-i} + \epsilon_t$. The current observation is a weighted average of historical observations with exponentially decaying coefficients. Values of λ closer to one put greater weight on recent data and thus react more intensely to recent variations, while processes with λ closer to zero smooth the weight on past observations and thus are less responsive to recent changes. Therefore, the process trends more slowly with a smaller λ. For example, the products that we study can be classified according to λ. Orange juice is an everyday drink for consumers since it is usually served with breakfast. A sports drink is designed for rehydration and energy-providing, which is mainly consumed for exercising. Thus its consumption is influenced by weather, temperature and sport events. The data exhibits a clearer slowly trending pattern in demand for sports drinks. Consistent with the above analysis, orange juice products has larger λ while sports drink have smaller λ according to our demand parameter estimations. We refer to demands with small λ as slowly trending demands.

Recall that the retailer's future demand forecast is a linear combination of historical $H+1$ periods demands. In the rest of this section, we assume the retailer's order relies on current and last week's demand, $H=1$. The order-up-to level of the ConDI policy becomes $\Gamma m_t = \Gamma\beta_0 D_t + \Gamma\beta_1 D_{t-1}$ ($\beta_0+\beta_1 = 1$ due to $\beta_0+\beta_1 = L_R = 1$). We will refer to the two parameters β_0 and β_1 as policy weights. The order can be written as a summation of two processes in equation (21)

$$X_t^1 = (1+\gamma\Gamma\beta_0 + \gamma\Gamma\beta_1 B)(1+\lambda B)\epsilon_t - \gamma^2 \sum_{i=1}^{\infty}(1-\gamma)^{i-1}(\Gamma\beta_0 + \Gamma\beta_1 B)(1+\lambda B)\epsilon_{t-i},$$

$$X_t^2 = \delta_t - \delta_{t-1} - \sum_{i=1}^{\infty}\gamma(1-\gamma)^{i-1}(\delta_{t-i} - \delta_{t-i-1}),$$

Figure 4 **The MSE percentage improvement with respect to an ARIMA(0,1,1) demand with λ and a ConDI policy with order smoothing with β_0, γ and $\Gamma=2$.**



where ϵ_t is the demand shock and δ_t is the decision deviation. We assume ϵ_t is independent with δ_s for any s.

For the purpose of our theoretical analysis, we focus on processes X_t^1 and X_t^2 with degree smaller or equal to 3. When the degree of either process exceeds 3, the complexity of the problem precludes analytically tractable solutions and necessitates numerical analysis. Therefore, we first focus on two simple policies: (1) the retailer follows a demand replacement policy ($\gamma=0$) and (2) the retailer adopt a ConDI policy ($\gamma=1$) with zero weight on previous week's demand ($a_1=0$). Under (1), the order process is $O_t=D_t+\delta_t$ and under (2), the order process becomes $O_t=(1+\Gamma\beta_0)D_t-\Gamma\beta_0 D_{t-1}+\delta_t-\delta_{t-1}$. The following proposition demonstrates that under the demand replacement policy, the value strictly decreases with λ.

Proposition 5 *The value of information sharing under the 1-step-ahead forecast strictly decreases with λ if (1) the retailer follows a demand replacement policy, or (2) the retailer follows a ConDI policy with $\beta_1=0$.*

To further explore the demand's impact under other parameters, we conduct numerical studies. Figure 4 presents the relation of MSE-PI with respect to λ under three policy weight parameters. The DOI level is set to 2 and the order smoothing level is set to 0.5. In each sub-figure, the three lines from top to bottom correspond to $\lambda=0, 0.5$ and 0.9. The three columns from left to right correspond to $\beta_0=1.5, 0.5$ and -0.2.

Consistent with the theoretical prediction, the value of information sharing with demand parameters closer to zero dominates those with larger λ. This indicates that the products with slowly trending demands have strictly larger forecast accuracy improvement, regardless of the decision deviation weight, policy and demand parameter. Let us revisit the empirically obtained MSE-PI results in the last column of Table 1 in Section 3. The two orange drinks 12 OR and 12 ORCA have much smaller bottle volume compared to other orange juice products. In Table 2 in Section 6, their λ is closer to zero which differs substantially from the other orange drinks. Thus, their demand structure is closer to the sports drink products. Consistent with our theoretical prediction, the percentage improvement of the sports drink and the above two products are in general larger than the rest of orange juice. In short, our theory can provide correct mapping from the demand pattern to the potential gain from information sharing.

The result implies that it's more worthwhile for managers to invest in the information sharing system for products with slow trending consumption under the one-step-ahead forecast lead time. It's worth noting that forecasting beyond one period might reverse the relation of the value of information sharing and demand parameter λ. We recommend that the managers run a numerical study to validate the potential gain based on demand and policy characteristics.

8. Conclusion and Discussions

This paper empirically evaluates the supplier's forecast improvement by incorporating downstream retail sales data and supports the observations with an extended theoretical model. Table 1 in Section 3 summarizes our main empirical findings. Overall, the forecast accuracy improvement can be statistically significant, even when including demand data in a naive way. We further show that a more refined inclusion of demand (by modeling the underlying policy structure along with the demand) yields the highest forecast accuracy and its forecast improvement over the NoInfoSharing method is statistically significant across all products. Our observations highlight the positive value to suppliers from incorporating retailers' sales data: 7.1% to 81.1% MSE percentage improvement across 14 products and 40% MAPE percentage improvement on an overall level, which is regarded as a significant improvement by the CPG company we studied.

We also revisit and extend the theoretical model in the existing literature. Until now, the theoretical literature showed no value of information sharing for 10 out of 14 products. We recognize that the key assumption in the theoretical model is that the decision maker strictly and consistently follows the specified replenishment policy, which in practice is rarely the case. A decision maker may implement adjustments according to private information that we do not observe. Following the same spirit as the "error term" defined by Rust (1997), we introduce "decision deviations" that stem from a state variable observable to the agent but not to us. Our extended theory yields qualitatively different results than the previous literature. We demonstrate that if both demand shock and decision shock are nonzero, the value of information sharing is strictly positive for any forecast lead time. We identify that the distinct evolution patterns of demand process and decision deviation process drive such conclusion. As both processes propagate together in different manners, the detailed information is lost and is replaced with an order signal with larger variance.

Our extended theory reconciles our empirical observations. Our paper therefore underscores the importance of extending the theoretical model by recognizing that the decision maker may deviate from the exact policy, a phenomenon that is common in practice and is absent in earlier theoretical models. We not only show that the value is positive, but also investigate the impact of demand characteristics on the magnitude of the value of information sharing. We suggest that managers invest in information sharing systems for products with slow trending consumption. This shows another contribution of our framework: we provide guidelines for evaluating the potential gain of information sharing.

Our study focuses on a specific linear and stationary inventory policy with a stationary demand process. The conclusion regarding the strictly positive value of information sharing can be generalized to both broader linear and stationary inventory policies and nonstationary demand processes. For any linear and stationary inventory policy and stationary demand, the evolution patterns of the demand process and decision deviation process are different because of the distinct way that they accumulate in the order decision. This implies that if a retailer follows the generalized order-up-to policy under the MMFE demand (studied in Chen and Lee 2009), the value of sharing the retailer's demand forecast revision is always positive for any forecast lead time. It's worth noting that the information shared by the retailer is no longer sales history but sales forecast revision history (based on the MMFE structure). In this paper, we restrict our attention to low-promotional products, the demand of which follows the stationary process. The demand of high-promotional products, however, can become nonstationary due to the spikes and slumps caused by the promotions. The nonstationary demand indicates that the order structure (in demand signals and decision deviation signals) changes over time. Therefore, the optimal estimator for the order structure obtained in the current period might be suboptimal for the next period, if the supplier has access to only the order (and price schedule) information. The suboptimal estimator together with the distinct evolution pattern reinforce our conclusion: the value of information is strictly positive.

Our model demonstrates that the decision deviation from a linear policy can allow the supplier to reap higher benefit from incorporating downstream sales data. We believe that our model can well represent many industries in practice, but our analysis has limitations and future work is needed to test the robustness of our results. In particular, future theoretical research should explore non-linear policies such as the (s, S) policy and forecasting multiple products with correlated demands. The former breaks the affine structure and thus requires a re-examination via a non-linear time series model or a proper approximation.

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9. Appendix

Proof of Theorem 2: **With information sharing.** The process i is $X_t^i = \chi_i(B)\epsilon_t^i$ with coefficient $\chi_i(B) = 1 + \lambda_1^i B + \lambda_2^i B^2 + \cdots + \lambda_{q_i}^i B^{q_i}$. Recall that $\Omega_t^{X^i} = \text{sp}\{\epsilon_1^i, ..., \epsilon_t^i\}$ is the plane containing the historical shocks $\epsilon_1^i, ..., \epsilon_t^i$. According to the definition, $\epsilon_{t+1}^i \perp \Omega_t^{X_i}$. Since we assume $\epsilon_t^i \perp \epsilon_{t-k}^j$ for any $k > 0$, the general orthogonal condition is

$$\epsilon_{t+1}^i \perp \Omega_t^{X_j}, \forall i, j. \tag{24}$$

The future forecast of a process depends only on its realized historical shocks. Therefore, with full information, it is optimal to generate predictors for each process and aggregate the optimal individual forecast as the total prediction.

The h-step-ahead forecast of process X_t^i made in period t is$\hat{X}_{t,t+h}^i = \lambda_h^i \epsilon_t + \lambda_{h+1}^i \epsilon_{t-1}^i + \cdots + \lambda_{q_i}^i \epsilon_{t+h-q_i}^i$. The h-step-ahead mean squared forecast error is $\text{Var}(S_{t+h} - \hat{S}_{t,t+h} | \cup_i \Omega_t^{X^i}) = \text{Var}(S_{t+h} - \sum_{i=1}^N \hat{X}_{t,t+h}^i)$.

Without information sharing. In absence of demand information, the order process is an MA model. We consider the invertible MA representation and define it as $S_t = \chi_S(B)\eta_t$, where $\{\eta_t\}$ is the white noise series, the MA coefficient is $\chi_S(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_{q_S} B^{q_S}$. Recall that $\Omega_t^S = \text{sp}\{\eta_1, ..., \eta_t\}$ is the plane containing the order process signals $\eta_1, ..., \eta_t$. Since $\chi_S(B)\eta_t = \sum_{i=1}^N \chi_i(B)\epsilon_t^i$, then $\eta_t = \chi_S^{-1}(B) \sum_{i=1}^N \chi_i(B)\epsilon_t^i$. Since η_t is a linear combination of $\epsilon_s^i, s \leq t$, then $\Omega_t^S \in \cup_i \Omega_t^{X^i}$.

According to the orthogonal condition in equation (24), we have $\epsilon_{t+1}^j \perp \cup_i \Omega_t^{X^i}$ for any j. Since $\Omega_t^S \in \cup_i \Omega_t^{X^i}$, then

$$\epsilon_{t+1}^i \perp \Omega_t^S, \forall i. \tag{25}$$

The h-step-ahead forecast of process S_t made in period t is$\hat{S}_{t,t+h} = \theta_h \eta_t + \theta_{h+1}\eta_{t-1} + \cdots + \theta_{q_S}\eta_{t+h-q_S}$. We abbreviate the h-step-ahead mean squared error under no information sharing $\text{Var}(S_{t+h} - \hat{S}_{t,t+h} | \Omega_t^S)$ as $\text{Var}(S_{t+h} - \hat{S}_{t,t+1})$ in the rest of the proof.

The Value of Information Sharing. Next we prove that if $\chi_i(B) \neq \chi_j(B)$ for any i, j, then $\sum_{i=1}^N \hat{X}_{t,t+h}^i \neq \hat{S}_{t,t+h}$ for any finite forecast lead time $h \leq \max_i\{q_i\}$. We rewrite $\text{Var}(S_{t+1} - \hat{S}_{t,t+1})$ as $\text{Var}(S_{t+1} - \sum_{i=1}^N \hat{X}_{t,t+1}^i + \sum_{i=1}^N \hat{X}_{t,t+1}^i - \hat{S}_{t,t+1})$. According to the orthogonal condition in equation (24) and (25), $\text{Var}(S_{t+1} - \hat{S}_{t,t+1})$ can be simplified to

$$\text{Var}(S_{t+h} - \hat{S}_{t,t+h}) = \text{Var}(S_{t+h} - \sum_{i=1}^N \hat{X}_{t,t+h}^i) + \text{Var}(\sum_{i=1}^N \hat{X}_{t,t+h}^i - \hat{S}_{t,t+h}). \tag{26}$$

$\text{Var}(S_{t,t+h} - \hat{S}_{t,t+h}) > \text{Var}(S_{t+h} - \sum_{i=1}^N \hat{X}_{t,t+h}^i)$ if and only if $\sum_{i=1}^N \hat{X}_{t,t+1}^i \neq \hat{S}_{t,t+h}$.

If $q_S < \max_i\{q_i\}$, then for the forecast lead time h that satisfies $q_S < h \leq \max_i\{q_i\}$, $\hat{X}_{t,t+h}^i \neq 0$ for $q_i \geq h$ and $\hat{S}_{t,t+h} = 0$. Therefore, $\sum_{i=1}^N \hat{X}_{t,t+h}^i \neq \hat{S}_{t,t+h}$. Suppose that there exists a finite forecast lead time $h \le q_S \le \max_i\{q_i\}$ (the latter inequality is from the MA process aggregation) such that $\sum_{i=1}^N \hat{X}^i_{t,t+h} = \hat{S}_{t,t+h}$. This is equivalent to $\sum_{i=1}^N \hat{X}^i_{t-h,t} = \hat{S}_{t-h,t}$, which can be expanded as

$$\theta_h \eta_{t-h} + \theta_{h+1}\eta_{t-h-1} + \cdots + \theta_{q_S}\eta_{t-q_S} = \sum_{i=1}^N \lambda^i_h \epsilon_{t-h} + \lambda^i_{h+1}\epsilon^i_{t-h-1} + \cdots + \lambda^i_{q_i}\epsilon^i_{t-q_i}. \tag{27}$$

For notational convenience, let $\lambda^i_j = 0$ for $j > q_i$. Since $\eta_t + \theta_1\eta_{t-1} + \theta_2\eta_{t-2} + \cdots + \theta_{q_S}\eta_{t-q_S} = \sum_{i=1}^N(\epsilon^i_t + \lambda^i_1\epsilon^i_{t-1} + \cdots + \lambda^i_{q_i}\epsilon^i_{t-q_i})$, we subtract equation (27) from it,

$$\eta_t + \theta_1\eta_{t-1} + \cdots + \theta_{h-1}\eta_{t-h+1} = \sum_{i=1}^N \epsilon^i_t + \lambda^i_1\epsilon^i_{t-1} + \cdots + \lambda^i_{h-1}\epsilon^i_{t-h+1}. \tag{28}$$

We replace η_{t-j} with $\chi_S^{-1}(B)\sum_{i=1}^N \chi_i(B)\epsilon^i_{t-j}$ for all j, equation (28) becomes

$$\eta_t + \theta_1\eta_{t-1} + \cdots + \theta_{h-1}\eta_{t-h+1} = \sum_{j=0}^{h-1} \theta_j \chi_S^{-1}(B) \sum_{i=1}^N \chi_i(B)\epsilon^i_{t-j}. \tag{29}$$

We denote the degree of $\chi_S^{-1}(B)$ as q_S^{inv}. Since $q_S \ge 0$ and $\chi_S(B)$ is the invertible representation (which guarantees nonnegative q_S^{inv}), then $q_S^{inv} \ge 0$. Therefore, according to the above equation, $\chi_S(B)\eta_t$ is the summation of N processes with degree $q_i + q_S^{inv} + \min\{h-1, q_S\}$ relative to shock ϵ^i_t. Then from equation (28), the degree with respect to ϵ^i_t is $\min\{h-1, q_i\}$. When there exist two processes with different coefficients, then we can find a pair of processes with different coefficients, one of which has the largest degree $\max_i\{q_i\}$ and we denote the process as X^k_t. Since $h < \max_i\{q_i\}$, X^k_t has a degree of $h-1$ with respect to ϵ^k_t according to equation (28). Since $h \le q_S$, X^k_t has a degree of $q_i + q_S^{inv} + h - 1$ with respect to ϵ^k_t in equation (28) is. Since $q_k + q_S^{inv} + h - 1 > h - 1$, we have reached a contradiction. As a result, for any finite forecast lead time $h \le \max_i\{q_i\}$, we have $\sum_{i=1}^N \hat{X}^i_{t,t+h} \ne \hat{S}_{t,t+h}$ and according to equation (26), $\mathrm{Var}(S_{t,t+h} - \hat{S}_{t,t+h}) > \mathrm{Var}(S_{t+h} - \sum_{i=1}^N \hat{X}^i_{t,t+h})$ for all $h \le q_S$.

The contrapositive of the above statement is that if there exists a forecast lead time $h \le q_S$ such that $\mathrm{Var}(S_{t,t+h} - \hat{S}_{t,t+h}) = \mathrm{Var}(S_{t+h} - \sum_{i=1}^N \hat{X}^i_{t,t+h})$, then $\chi_i(B) = \chi_j(B)$ for any i,j. If $\chi_i(B) = \chi_j(B)$ for any i,j, then it is obvious that the aggregated process has the same parameters and $\eta_t = \sum_{i=1}^N \epsilon^i_t$. Therefore, $\mathrm{Var}(S_{t+h} - \hat{S}_{t,t+h}| \cup_i \Omega^{X^i}_t) = \mathrm{Var}(S_{t+h} - \hat{S}_{t,t+h}|\Omega^S_t)$ for any forecast lead time h including $h = 1$. ■

Proof of Proposition 5: **The retailer follows the demand replacement policy.** Recall that the centered order is the summation of two MA processes $O_t - O_{t-1} = \epsilon_t - \lambda\epsilon_{t-1} + \delta_t - \delta_{t-1}$. We denote the aggregate MA process (or the order process) as $S_t = \eta_t + \theta\eta_{t-1}$, where $\mathrm{Var}(\eta_t) = v$. It satisfies the covariance equations

$$\begin{aligned} -\lambda\sigma^2_\epsilon - \sigma^2_\delta &= v\theta, \\ (1+\lambda^2)\sigma^2_\epsilon + 2\sigma^2_\delta &= v(1+\theta^2). \end{aligned} \tag{30}$$

We use equation (30) to solve for θ and v given parameters σ_ϵ, σ_ϵ and λ. We substitute θ with v in equation (30) to obtain a function $f(v,\lambda)$ with variable v and parameter λ such that the solutions of $f(v,\lambda)=0$ is the variance of η_t. The function $f(v,\lambda)$ satisfies $f(v,\lambda)=v^2-((1+\lambda^2)\sigma_\epsilon^2+2\sigma_\delta^2)v+(-\lambda\sigma_\epsilon^2-\sigma_\delta^2)^2$.

The aggregate process S_t has an invertible representation and a noninvertible representation. Each representation corresponds to a solution of (30). Therefore, fixing λ, $f(v,\lambda)=0$ has two solutions: the variance of the invertible process and the noninvertible process. We denote the former as v^*. The value of information sharing is $1-(\sigma_\epsilon^2+\sigma_\delta^2)/v^*$. We will prove the invertible white noise variance of the aggregated process is decreasing in λ.

We first take derivatives of $f(v,\lambda)$ with respect to v, $\partial f(v,\lambda)/\partial v = 2v-((1+\lambda^2)\sigma_\epsilon^2+2\sigma_\delta^2)$. Since the invertible solution θ of (30) is smaller than 1, $2v^*=2((1+\lambda^2)\sigma_\epsilon^2+2\sigma_\delta^2)/(1+\theta^2) > (1+\lambda^2)\sigma_\epsilon^2+2\sigma_\delta^2$. As a result, $\partial f(v,\lambda)/\partial v>0$ at v^*. Since $f(v,\lambda)$ is continuous, $\partial f(v,\lambda)/\partial v>0$ in an open interval of $v^*(\lambda)$. v^* is decreasing in λ. Therefore, it suffices to show that $\partial f(v,\lambda)/\partial\lambda>0$. We have

$$\frac{\partial f(v,\lambda)}{\partial\lambda}=2\sigma_\epsilon^2(-\lambda v+\lambda\sigma_\epsilon^2+\sigma_\delta^2)=-2\sigma_\epsilon^2 v(\theta+\lambda). \tag{31}$$

The covariance generating functions of the two MA processes are $g_\epsilon=\sigma_\epsilon^2(1-\lambda z)(1-\lambda z^{-1})$ and $g_\delta=\sigma_\delta^2(1-z)(1-z^{-1})$, where $z=\cos(\omega)-i\sin(\omega)=e^{-i\omega}$. The covariance generating function for the aggregated process is $g_\eta=v(1+\theta z)(1+\theta z^{-1})$, where $g_\eta=g_\epsilon+g_\delta$,

$$\sigma_\epsilon^2(1-\lambda z)(1-\lambda z^{-1})+\sigma_\delta^2(1-z)(1-z^{-1})=v(1+\theta z)(1+\theta z^{-1}). \tag{32}$$

Let $z=1$, equation(32) then becomes $\sigma_\epsilon^2(1-\lambda)^2=v(1+\theta)^2$. Since $v^*>\sigma_\epsilon^2+\sigma_\delta^2$, then $(1+\theta)^2<(1-\lambda)^2$. Since $1+\theta$ and $1-\lambda$ are both positive, $\theta+\lambda<0$. Therefore, the right hand side of equation (31) is positive and v^* is decreasing in λ.

The retailer follows the ConDI policy. The order under the ConDI policy is $(1+\Gamma\beta_0)\epsilon_t-(\lambda+\lambda\Gamma\beta_0+\Gamma\beta_0)\epsilon_{t-1}+\lambda\Gamma\beta_0\epsilon_{t-2}+\delta_t-2\delta_{t-1}+\delta_{t-2}$. Let $\alpha=\Gamma\beta_0/(1+\Gamma\beta_0)$. We denote the aggregate process as $S_t=\eta_t+\theta_1\eta_{t-1}+\theta_2\eta_{t-2}$, with the covariance equations

$$\begin{aligned} \sigma_\delta^2+\alpha\lambda\sigma_\epsilon^2 &= \theta_2 v, \\ -4\sigma_\delta^2-(\lambda+a)(1+\alpha\lambda)\sigma_\epsilon^2 &= \theta_1(1+\theta_2)v, \\ 6\sigma_\delta^2+(1+(\lambda+a)^2+\alpha^2\lambda^2)\sigma_\epsilon^2 &= (1+\theta_1^2+\theta_2^2)v. \end{aligned} \tag{33}$$

Following the same spirit as above, we substitute θ_1 and θ_2 with v in equation (33) to obtain a function $f(v,\lambda)$ with variable v and parameter λ. The function $f(v,\lambda)$ satisfies$f(v,\lambda)=v^2(v+\gamma(2))^2+v^2\gamma(1)^2+(v+\gamma(2))^2(\gamma(2)^2-\gamma(0)v)$, where $\gamma(2)\equiv\sigma_\delta^2+\alpha\lambda\sigma_\epsilon^2$, $\gamma(1)\equiv-4\sigma_\delta^2-(\lambda+a)(1+\alpha\lambda)\sigma_\epsilon^2$, $\gamma(0)\equiv 6\sigma_\delta^2+(1+(\lambda+a)^2+\alpha^2\lambda^2)\sigma_\epsilon^2$.

As before, we denote v^* as the invertible white noise variance of the aggregated process. We need to prove that v^* is decreasing in λ. Following the same argument from before, it is equivalent to prove $\partial f(v,\lambda)/\partial\lambda > 0$.

We take derivatives of $f(v,\lambda)$ with respect to λ

$$\begin{aligned}\frac{\partial f(v,\lambda)}{\partial\lambda} &= 2(v+\gamma(2))\gamma'(2)[v^2+(\gamma(2)-\gamma(0))v+2\gamma(2)^2]+2v^2\gamma'(1)\gamma(1)-(v+\gamma(2))^2v\gamma'(0) \quad (34)\\ &= -2v^3\sigma_\epsilon^2(1+\theta_2)[\theta_1(1+2\alpha\lambda+\alpha^2)+(1+\theta_2)(\lambda+\alpha+\lambda\alpha^2)-\alpha(\theta_2+\theta_2^2-\theta_1^2)]\\ &= -2v^3\sigma_\epsilon^2(1+\theta_2)[(\alpha+\lambda-\alpha\lambda+\theta_1+\theta_2)(\alpha\theta_1+1+\theta_2)-(-\alpha\lambda+\theta_2)(1+\alpha)(\theta_1+\theta_2+1)]\end{aligned}$$

For the process $\eta_t+\theta_1\eta_{t-1}+\theta_2\eta_{t-2}$, the invertible solutions of $1+\theta_1 m+\theta_2 m^2$ lie outside the unit circle. Since $1+\theta_1 m+\theta_2 m^2=1$ at $m=0$, then the function takes positive value at $m=1$, $\theta_1+\theta_2+1>0$. Using the same argument, the function $1-(\lambda+\alpha)m+\alpha\lambda m^2$ takes positive value at $m=1$, $1-\lambda-\alpha+\alpha\lambda>0$.

The covariance generating functions satisfy $\sigma_\epsilon^2(1-\lambda z)(1-\alpha z)(1-\lambda z^{-1})(1-\alpha z^{-1})+\sigma_\delta^2(1-z)^2(1-z^{-1})^2=v(1+\theta_1 z+\theta_2 z^2)(1+\theta_1 z^{-1}+\theta_2 z^{-2})$. Let $z=1$, we have $\sigma_\epsilon^2(1-\lambda-\alpha+\alpha\lambda)^2=v(1+\theta_1+\theta_2)^2$. Since $v>\sigma_\epsilon^2+\sigma_\delta^2$, $(1+\theta_1+\theta_2)^2<(1-\lambda-\alpha+\alpha\lambda)^2$ and $1+\theta_1+\theta_2<1-\lambda-\alpha+\alpha\lambda$. Therefore, $\alpha+\lambda-\alpha\lambda+\theta_1+\theta_2<0$. Since $\gamma(2)>0$ and $\gamma(1)<0$, θ_2 is positive and θ_1 is negative. Since $\theta_1+\theta_2+1>0$ and $\theta_1<0$, $\alpha\theta_1+1+\theta_2>\theta_1+\theta_2+1>0$. Therefore, $(\alpha+\lambda-\alpha\lambda+\theta_1+\theta_2)(\alpha\theta_1+1+\theta_2)<0$.

We next prove $-\alpha\lambda+\theta_2>0$. If $\sigma_\delta/\sigma_\epsilon=0$, then $\theta_2=\alpha\lambda$. If $\sigma_\delta/\sigma_\epsilon\to\infty$, then $\theta_2=1$. As $\sigma_\delta/\sigma_\epsilon$ increases from 0 to ∞, θ_2 changes continuously from $\alpha\lambda$ to 1. If there exists a $\theta_2<\alpha\lambda$, there must be a $\theta_2=\alpha\lambda$ when $\sigma_\delta/\sigma_\epsilon\neq 0$. Then according to equation(33), $v=\sigma_\delta^2/\alpha\lambda+\sigma_\epsilon^2$ and $\theta_1=\alpha\lambda(-4\sigma_\delta^2-(\lambda+\alpha)(1+\alpha\lambda)\sigma_\epsilon^2)/(1+\alpha\lambda)(\alpha\lambda\sigma_\epsilon^2+\sigma_\delta^2)$. Plugging v and θ_1,θ_2 into equation $\sigma_\epsilon^2(1-\lambda-\alpha+\alpha\lambda)^2=v(1+\theta_1+\theta_2)^2$, we have

$$\frac{((1+\alpha\lambda)(1-\lambda)(1-\alpha)-2(1-\alpha\lambda)^2)\alpha\lambda(1-\lambda-\alpha+\alpha\lambda)(1+\alpha\lambda)\sigma_\epsilon^2\sigma_\delta^2-(1-\alpha\lambda)^4\sigma_\delta^4}{\alpha\lambda(1+\alpha\lambda)^2(\alpha\lambda\sigma_\epsilon^2+\sigma_\delta^2)}=0$$

Since $2>1+\alpha\lambda$, $1-\alpha\lambda>1-\lambda$ and $1-\alpha\lambda>1-\alpha$, then $(1+\alpha\lambda)(1-\lambda)(1-\alpha)<0$. Since $1-\alpha\lambda>0$, $1+\lambda-a-a\lambda>0$, $\lambda<0$ and $-(1-\alpha\lambda)^4\sigma_\delta^4<0$, the numerator is negative. Since the denominator is positive, the equation is violated. Therefore $-\alpha\lambda+\theta_2>0$.

The right hand side of equation(34) is positive at v^*. Applying the same argument as before, v^* is decreasing in λ. ■