Secondary Appendix to "Relational Incentive Contracts"

This appendix provides details on the model of optimal contracting with subjective performance measures and discusses the existence of optimal relational contracts.

Appendix C: Subjective Performance Measures.

Consider a full review contract that in its initial period specifies effort e, payments w and $b(y)$, and continuation payoffs $u(y), \pi(y)$ contingent on output, with reversion to static no trade equilibrium following a deviation in payments or in proposing or accepting the contract. Define $W(y) \equiv w + b(y)$ and expected payoffs:

$$\begin{aligned} u &\equiv (1-\delta)\,\mathbb{E}_y\left[W - c \mid e\right] + \delta \mathbb{E}_y\left[u(y) \mid e\right], \\ \pi &\equiv (1-\delta)\,\mathbb{E}_y\left[y - W \mid e\right] + \delta \mathbb{E}_y\left[\pi(y) \mid e\right], \end{aligned}$$

and $s \equiv u + \pi$.

This contract is self-enforcing if and only if *(i)* both parties are willing to participate, $u \geq \overline{u}$ and $\pi \geq \overline{\pi}$; *(ii)* the agent will choose e, i.e.

$$e \in \arg\max_{\tilde{e}} \mathbb{E}_y\left[b(y) + \frac{\delta}{1-\delta}u(y) \mid \tilde{e}\right] - c(\tilde{e});$$

(iii) the principal will truthfully report, i.e. for all y, y'

$$b(y) + \frac{\delta}{1-\delta}\pi(y) = b(y') + \frac{\delta}{1-\delta}\pi(y');$$

(iv) both parties will make payments, i.e. for all y,

$$\begin{aligned} b(y) + \frac{\delta}{1-\delta}u(y) &\geq \frac{\delta}{1-\delta}\overline{u} \\ -b(y) + \frac{\delta}{1-\delta}\pi(y) &\geq \frac{\delta}{1-\delta}\overline{\pi} \end{aligned}$$

and *(v)* for all y, the continuation payoffs $u(y), \pi(y)$ correspond to a self-enforcing contract.

The Equilibrium Payoff Set. Assuming an optimal contract exists that generates surplus s^*, an argument identical to in the proof of Theorem 1 implies that the set of payoffs achievable with a self-enforcing full review contract is equal to $\{(u,\pi) : u \geq \overline{u}, \pi \geq \overline{\pi} \text{ and } u + \pi \leq s^*\}$.

Termination Contracts are Optimal. I now argue constructively, along the lines of Theorem 2, that there is a termination contract that achieves the optimal surplus s^*. Suppose the contract described above is optimal. Then,

$$s^* = (1-\delta)\,\mathbb{E}_y\left[y - c \mid e\right] + \delta \mathbb{E}_y\left[s\left(y\right) \mid e\right]. \tag{1}$$

Note that because s^* is optimal, $s\left(y\right) \leq s^*$ for all y and hence $\mathbb{E}_y\left[y - c \mid e\right] \geq s^*$.

Now, let $u^* \in [\overline{u}, s^* - \overline{\pi}]$ be given and define $\pi^* \equiv s^* - u^*$. I construct a termination contract with these payoffs. Suppose the contract specifies effort e, payments w^* and $b^*\left(y\right)$, and probabilities of continuation $\alpha^*\left(y\right)$. Define $\alpha^*\left(y\right)$ so that the expected continuation surplus following any outcome y is the same as under the original contract:

$$\overline{s} + \alpha^*\left(y\right)\left(s^* - \overline{s}\right) \equiv s\left(y\right). \tag{2}$$

Let $u^*\left(y\right) \equiv \overline{u} + \alpha^*\left(y\right)\left(u^* - \overline{u}\right)$ and $\pi^*\left(y\right) \equiv \overline{\pi} + \alpha^*\left(y\right)\left(\pi^* - \overline{\pi}\right)$ be the expected continuation payoffs following output y. Define $b^*(y)$ to so that agent's expected future payoff following outcome y is the same as under the initial contract:

$$b^*(y) + \frac{\delta}{1-\delta}u^*(y) \equiv b(y) + \frac{\delta}{1-\delta}u(y). \tag{3}$$

Combining (2) and (3) implies that for all y,

$$-b^*(y) + \frac{\delta}{1-\delta}\pi^*(y) \equiv -b(y) + \frac{\delta}{1-\delta}\pi(y).$$

Finally, define the fixed payment w^* so that the agent's expected payoff is u^*:

$$u^* = (1-\delta)\,\mathbb{E}_y\left[w^* + b^*\left(y\right) \mid e\right] + \delta \mathbb{E}_y\left[u^*\left(y\right) \mid e\right].$$

I claim that this termination contract generates surplus $s = s^*$ and is self-enforcing. To see this, observe that the surplus generated, s, satisfies:

$$s \equiv (1-\delta)\,\mathbb{E}_y\left[y - c \mid e\right] + \delta\left\{\overline{s} + \mathbb{E}\left[\alpha^*\left(m\right) \mid e\right]\left(s - \overline{s}\right)\right\}.$$

Substituting (1) and (2) into this expression shows that $s = s^*$. Moreover, this termination contract, with effort e, payments $w, b^*(y)$ and contingent continuation payoffs $u^*\left(y\right)$ and $\pi^*\left(y\right)$ satisfies the constraints *(i)–(v)* by definition. Hence, it is self-enforcing.

Self-Enforcing Termination Contracts. Next I identify simple conditions under which a termination contract is self-enforcing (analogous to Theorem 3). Consider a termination contract with effort e, payments w and $b\left(y\right)$, and continuation probabilities $\alpha\left(y\right)$. The expected payoffs u, π satisfy the recursive equations:

$$\begin{aligned}
u &\equiv (1-\delta)\,\mathbb{E}_y\left[W\left(y\right) - c \mid e\right] + \delta\overline{u} + \delta\mathbb{E}_y\left[\alpha\left(y\right) \mid e\right]\left(u - \overline{u}\right) \\
\pi &\equiv (1-\delta)\,\mathbb{E}_y\left[y - W\left(y\right) \mid e\right] + \delta\overline{\pi} + \delta\mathbb{E}_y\left[\alpha\left(y\right) \mid e\right]\left(\pi - \overline{\pi}\right),
\end{aligned}$$

while the surplus $s \equiv u + \pi$ depends only on the effort and continuation probabilities:

$$s \equiv (1-\delta)\,\mathbb{E}_y\left[y - c \mid e\right] + \delta\overline{s} + \delta\mathbb{E}_y\left[\alpha(y) \mid e\right](s - \overline{s}) \tag{4}$$

This contract is self-enforcing if and only if it satisfies: *(i)* the participation constraints $u \geq \overline{u}, \pi \geq \overline{\pi}$; *(ii)* incentive compatibility for the agent

$$e \in \arg\max_{\tilde{e}} \mathbb{E}_y\left[b(y) + \frac{\delta}{1-\delta}\alpha(y)(u - \overline{u}) \mid \tilde{e}\right] - c(\tilde{e}),$$

and *(iii)* for the principal,

$$-b(y) + \frac{\delta}{1-\delta}\alpha(y)(\pi - \overline{\pi}) \text{ is constant in } y;$$

(iv) willingness to make payments,

$$\begin{aligned} \frac{\delta}{1-\delta}\alpha(y)(\pi - \overline{\pi}) &\geq b(y) \\ \frac{\delta}{1-\delta}\alpha(y)(u - \overline{u}) &\geq -b(y) \end{aligned}$$

and finally *(v)* for all y, $0 \leq \alpha(y) \leq 1$.

The next result reduces these constraints in a manner analogous to Theorem 3.

Lemma 1 *A self-enforcing termination contract can implement effort e with continuation probabilities $\alpha : Y \to [0,1]$ if and only if:*

$$e \in \arg\max_{\tilde{e}} \frac{\delta}{1-\delta}\mathbb{E}_y\left[\alpha(y)(s - \overline{s}) \mid \tilde{e}\right] - c(\tilde{e}) \tag{5}$$

where s is defined by (4) and is greater than or equal to $\overline{s}$.

Proof. ($\Rightarrow$) If a termination contract defined by $e, w, b(y), \alpha(y)$ is self-enforcing, it must satisfy *(i)–(v)* above. Adding the constraints *(ii)* and *(iii)* implies it must satisfy the stated constraint, and from *(i)* it must satisfy $s \geq \overline{s}$.

($\Leftarrow$) Given $e, \alpha : Y \to [0,1]$ satisfying the stated constraint, with $s \geq \overline{s}$, complete the termination contract by adding $b(y) \equiv 0$ for all y and $w \equiv \mathbb{E}\left[y - \overline{\pi} \mid e\right]$. This completed contract gives expected payoffs $u \equiv s - \overline{\pi}$ and $\pi = \overline{\pi}$. To see that it is self-enforcing, observe that it satisfies *(i), (iii)* and *(iv)* by definition, *(v)* because $\alpha(y) \in [0,1]$, and *(ii)* because $b(y) \equiv 0$ and $u - \overline{u} \equiv s - \overline{s}$, so the assumption (5) implies incentive compatibility. *Q.E.D.*

Optimal Incentive Structure. Given the above, an optimal contract solves:

$$\max_{e \in [0,\overline{e}], \alpha : Y \to \mathbb{R}} s \qquad \text{subject to (4),(5),}$$
$$\text{and } 0 \leq \alpha(y) \leq 1 \text{ for all } y \in Y.$$

It is useful to make the following change of variable. Given a termination contract with effort e and continuation probabilities $\alpha : Y \to [0,1]$, define the "per-period" loss following outcome y to be:

$$\tau(y) \equiv (1-\alpha(y))(s-\overline{s}).$$

Making the change of variable, the optimal contract solves:

$$\begin{aligned} &\max_{e\in[0,\overline{e}],\tau:Y\to\mathbb{R}} s = \mathbb{E}_y[y-c \mid e] - \frac{\delta}{1-\delta}\mathbb{E}_y[\tau(y) \mid e] \\ &\text{s.t. } \frac{d}{de}\{-\mathbb{E}_y[\tau(y) \mid e] - c(e)\} = 0, \\ &\quad s-\overline{s} \geq \tau(y) \geq 0 \text{ for all } y. \end{aligned}$$

Note that I have used a first-order condition in place of the agent's effort constraint, a valid substitution under the Mirrlees-Rogerson conditions.

This optimization problem is linear in $\tau(y)$. Given the monotone likelihood ratio property, the solution has $e \leq e^{FB}$ and

$$\tau(y) = \begin{cases} s-\overline{s} & \text{if } y < \hat{y} \\ 0 & \text{if } y \geq \hat{y} \end{cases}$$

for some $\hat{y} \in Y$. Reversing the change of variables shows that under an optimal termination contract $\alpha(y) = 0$ for all $y < \hat{y}$ and $\alpha(y) = 1$ for all $y > \hat{y}$. This proves Theorem 7.

Limit Inefficiency. Consider a solution to the optimal contract defined by a pair $e, \widehat{y}$ and a per-period loss $\tau(y)$ as described above. Given this, and after some algebra, the agent's first-order condition can be written as:

$$-\mathbb{E}_y[\tau(y) \mid e]\frac{F_e(\widehat{y}|e)}{F(\widehat{y}|e)} - c'(e) = 0.$$

Now, because $f > 0$ and f_e/f is continuous on $[\underline{y}, \overline{y}]$, it follows that $f_e/f \in [-l, l]$ for some finite $l > 0$. It follows that $F_e/F \in [-l, l]$. Consequently,

$$\mathbb{E}_y[\tau(y) \mid e] = \frac{c'(e)}{|F_e/F(\widehat{y}|e)|} \geq \frac{c'(e)}{l}.$$

This provides a bound on expected surplus independent of the discount factor:

$$s \leq \mathbb{E}[y-c \mid e] - \frac{c'(e)}{l} < s^{FB}.$$

So even as $\delta \to 1$, an optimal contract cannot approximate the first-best.

Appendix D: Existence of an Optimal Contract.

I now consider the existence of optimal relational contracts in the model of Section I. I show that for the relevant cases studied in the paper, an optimal contract will exist. The argument is closely fashioned on the self-generation construction of Abreu et al. (1990). They prove that in a broad class of games with a finite number of hidden actions and a continuous noisy performance measure, the equilibrium payoff set is compact — this implies the existence of optimal equilibria. I cannot directly adopt their result or proof, but instead use a slightly different argument.

Define $s^{FB} \equiv \mathbb{E}_{\theta,y}\left[y - c \mid e^{FB}(\theta)\right]$ to be the first-best joint surplus. Let $\mathcal{E} \subseteq [\overline{s}, s^{FB}]$ be the set of joint surpluses that are consistent with a self-enforcing contract. To characterize $\mathcal{E}$, I first define an associated contracting problem (Problem C) for each $s \in [\overline{s}, s^{FB}]$.

$$\begin{aligned} \max_{e(\cdot),W(\cdot)} \quad & (1-\delta)\mathbb{E}_{\theta,y}\left[y - c \mid e(\theta)\right] + \delta s \\ \text{subject to} \quad & e(\theta) \in \arg\max_{e} \mathbb{E}_y\left[W(\varphi) \mid e\right] - c(e,\theta) \qquad \text{for all } \theta, \\ & \frac{\delta}{1-\delta}\left(s - \overline{s}\right) \geq \sup_{\varphi} W(\varphi) - \inf_{\varphi} W(\varphi). \end{aligned}$$

Problem C looks for the optimal effort schedule among those that could be enforced given a fixed continuation surplus s. To account for possibility that the parties could forego trade, allow $e(\cdot)$ to be chosen either as a function $e : \Theta \to [0, \overline{e}]$ or as $e(\theta) = \emptyset$, where $c(\emptyset) \equiv 0$ and $\mathbb{E}[y - c | e = \emptyset] \equiv \overline{s}$.

Assumption *Problem C admits a solution for all* $s \in [\overline{s}, s^{FB}]$.

Define $m(s)$ to be the maximized value of Problem C for each $s \in [\overline{s}, s^{FB}]$.

Lemma 2 *The set* $\{s : m(s) \geq s\}$ *has a largest element* s^* *and* $m(s^*) = s^*$.

Proof. First, when $s = \overline{s}$, the only feasible solution to Problem C is to set $e(\theta) = \emptyset$ or $e(\theta) = 0$, so consequently $m(\overline{s}) = \overline{s}$. Second, $m(s)$ is weakly increasing because an increase in s both increases the objective and relaxes the constraints. Third, for all $s \in [\overline{s}, s^{FB}]$, $m(s) \leq s^{FB}$ by the definition of s^{FB}. Thus $m : [\overline{s}, s^{FB}] \to [\overline{s}, s^{FB}]$ is nondecreasing with $m(\overline{s}) = \overline{s}$ and $m\left(s^{FB}\right) \leq s^{FB}$. Tarski's fixed point theorem applied to the function m implies the existence of a largest fixed point. *Q.E.D.*

Lemma 3 $\mathcal{E} \supseteq [\overline{s}, s^*]$.

Proof. I first argue that $s^* \in \mathcal{E}$. Suppose $e(\theta)$ solves Problem C for $s = s^*$. Because $e(\theta)$ is feasible for Problem C, Theorem 3 implies that it can be implemented by a stationary self-enforcing contract. Moreover, as $m(s^*) = s^*$, this stationary contract will generate surplus s^*. Thus $s^* \in \mathcal{E}$ and the result follows from convexity. *Q.E.D.*

Lemma 4 $\mathcal{E} \subseteq [\overline{s}, s^*]$.

Proof. Suppose to the contrary that there is some $s^1 > s^*$ with $s^1 \in \mathcal{E}$. Then the contract that supports s^1 has some initial effort $e(\theta)$ and continuation payoffs $s^1(\varphi)$ such that:

$$s^1 = (1-\delta)\mathbb{E}_{\theta,y}[y - c|e(\theta)] + \delta\mathbb{E}_{\theta,y}[s^1(\varphi)|e(\theta)]$$

Self-enforcement implies that for each φ, $s^1(\varphi) \in \mathcal{E}$.

Suppose that for all φ, $s^1(\varphi) \leq s$. Then $e(\theta)$ must be feasible for Problem C given the parameter s and hence $m(s) \geq s$. But because $s > s^*$ this contradicts the definition of s^*. Thus, it must be that $s^1(\varphi) > s$ for some φ, or in other words that $\sup_\varphi s^1(\varphi) > s$. Consequently since $s > s^*$, $\sup_\varphi s^1(\varphi) > m(\sup_\varphi s^1(\varphi))$. I now argue to a contradiction by constructing a sequence $s^1, s^2, s^3, ...$ converging to some $\hat{s}$ with the property that $\hat{s} > s^*$ and $m(\hat{s}) = \hat{s}$.

Define $s^2 \equiv \sup_\varphi s^1(\varphi) - \varepsilon^2$, where $\varepsilon^2 > 0$ is chosen small enough so that $s^2 > m(\sup_\varphi s^1(\varphi))$. Since $e(\theta)$ must be admissible for Problem C with parameter $\sup_\varphi s^1(\varphi)$, the definition of s^1 above implies that $m(\sup_\varphi s^1(\varphi)) \geq s^1$. Because $m(\sup_\varphi s^1(\varphi)) \equiv m(s^2 + \varepsilon^2)$, it follows that $s^2 > m(s^2 + \varepsilon^2) \geq s^1$.

Now, because $s^1(\varphi) \in \mathcal{E}$ for all φ, and $s^2 < \sup_\varphi s^1(\varphi)$, it must be the case that $s^2 \in \mathcal{E}$. So this same construction can be repeated to find some ε^3 and corresponding s^3 with $s^3 > m(s^3 + \varepsilon^3) \geq s^2$. Moreover, it is possible to take $0 < \varepsilon^3 < \varepsilon^2$. Iterating this process yields an increasing sequence $s^n \to \widehat{s}$ and a decreasing sequence $\varepsilon^n \to 0$ such that for all n,

$$s^{n+1} > m\left(s^{n+1} + \varepsilon^{n+1}\right) \geq s^n.$$

Taking limits implies that $m(\widehat{s}) = \widehat{s}$. Since $\widehat{s} > s^1 > s^*$ this yields a contradiction. *Q.E.D.*

This establishes that if Problem C admits a solution for any s, then $\mathcal{E} = [\overline{s}, s^*]$ so an optimal contract exists. Problem C is a relatively straightforward problem. It is easy to check that if Θ and Y are finite, it will admit a solution. Given that Θ and Y are continuous, the problem needs to be checked separately for different informational conditions. If there is symmetric information, i.e. $\varphi \equiv \{\theta, e, y\}$, then an assumption that S is concave and c is convex ensures a solution. With hidden information, the approach to combining the constraints taken in Section III, combined with concavity, ensures a solution. The moral hazard existence problem is more complicated and is discussed by Holmström (1979). Under the Mirrlees-Rogerson conditions, however, the first-order approach is valid, and a solution certainly exists. Thus, for the cases considered in the paper, existence of an optimal contract is not a problem.