Matching with Contracts
Hatfield and Milgrom (AER, Sep 2005)
summary by N. Anton

Examines connection between two-sided matching and auction/contract theory. Echenique (2012, AER) shows this model can be embedded in the Kelso-Crawford (1982, ECTA) framework. The extension by Hatfield-Kojima (2010, JET) does not embed in the older framework, thus this is a good workhorse model.

**Basic Model**

- Finite sets of doctors and hospitals \(D\) and \(H\), respectively
- \(X\) is a finite set of contracts (generic)
  - A contract \(x \in X\) matches a doctor, \(x_D\), and hospital, \(x_H\)
  - \(x\) may contain more information, e.g., doctor’s wage
  - \(A \subset X\) is an allocation if \(\forall (d, h), (d', h') \in A, d \neq d'\)
- \(\succ_d\) is a total order on \(X_d \cup \{\emptyset\}\), \(X_d := \{x \in X \mid x_D = d\}\)
  - Contract \(x\) is acceptable to doctor \(d\) if \(x \succ_d \emptyset\)
  - Given \(Y \subset X\), let \(C_d(Y) = \max_{\succ_d} (X_d \cap Y \cup \{\emptyset\})\)
    - \(C_d(Y) = \cup_d C_d(Y)\) denotes accepted contracts
    - \(R_d(Y) = Y \setminus C_d(Y)\) denotes rejected contracts
- \(\forall Y \subset X\), hospital preferences are given by a choice correspondence, \(C_h(Y) \subset Y_h := \{x \in Y \mid x_H = h\}\), where for any \(x, x' \in C_h(Y), x \neq x' \Rightarrow x_D \neq x'_D\)
  - Let \(C_h(Y) = \cup_h C_h(Y), R_h(Y) = Y \setminus C_h(Y)\)
- A set of contracts \(A \subset X\) is a stable allocation if
  - \(S1. \ C_d(A) = C_H(A) = A, and\)
  - \(S2. \ \exists h \in H, Y \subset X\) s.t. \(Y = C_h(A \cup Y) \subset C_d(A \cup Y)\)

**Theorem 1.** If \((X_D, X_H) \subset X^2\) solves the system:
\[
X_D = X \setminus R_H(X_H),
\]
\[
X_H = X \setminus R_D(X_D),
\]
then \(A := X_H \cap X_D\) is a stable allocation and \(A = C_H(X_H) = C_d(X_D)\). Conversely, for any stable \(A\), there exist \(X_D, X_H\) satisfying the system such that \(A = X_H \cap X_D\).

**Proof.** \((\Rightarrow)\) \(A := X_H \cap X_D = X_D \setminus R_D(X_D) = C_d(X_D)\); similarly \(A = C_H(X_H)\). By revealed preference \(A = C_d(X_D) = C_d(A) = C_H(A) = C_H(X_H)\), thus \(A\) satisfies \(S1\). Fix \(h\) and some \(Y_h \subset C_d(A \cup Y_h)\). Since \(A = C_d(X_D)\), by doctors’ revealed preferences, \(Y_h \cap X_D \subset C_d(X_D)\) and thus
\[
Y_h \cap R_D(X_D) = Y_h \cap X_D \setminus R_D(X_D) \subset C_d(X_D) \cap R_D(X_D) = \emptyset.
\]
Hence, \(Y_h \subset X \setminus R_D(X_D) = X_H\) and so if \(Y_h \neq C_h(A)\), then \(Y_h \prec h C_h(X_h) = C_h(A)\). Thus \(Y_h \neq C_h(A \cup Y_h)\) and \(S2\) holds.

\((\Leftarrow)\) Let \(A\) be a stable allocation. Note that \(A = C_H(A) = C_d(A)\). Define:
\[
X_D = \bigcup_{d \in D} \{x \in X_d \cup \{\emptyset\} : x \geq d A_d\},
\]
\[
X_H = \bigcup_{d \in D} \{x \in X_d \cup \{\emptyset\} : x \leq d A_d\}.
\]

Clearly, \(X_D \setminus A, X_H \setminus A\) is a partition of \(X\). If \(C_h(X_H) \neq A_h\), then \(Y = C_h(X_H)\) violates \(S2\). Thus \(C_H(X_H) = A\), since \(C_h(X_H) = A_h\) for all \(h\). Hence:
\[
X \setminus R_H(X_H) = X \setminus (X_H \setminus A) = X_D, and
X \setminus R_D(X_D) = X \setminus (X_D \setminus C_D(X_D)) = X_H,
\]
since by definition \(C_D(X_D) = A\).

- Elements of \(X\) are substitutes for \(h\) if \(R_h(Y) \subset R_h(Z)\) for any \(Y \subset Z \subset X\)
  - i.e., if \(R_h\) is isotone (order-preserving) with respect to \(\subset\)
  - New contracts make hospital weakly less interested in old
  - Coincides with demand theory definition in the appropriate setting (Theorem 2)

**Generalized Deferred Acceptance (DA) Algorithm**

- Generalized DA algorithm, \(F : X \times X \to X \times X\)
  \[
  F(X_D, X_H) = (X \setminus R_H(X_H), X \setminus R_D(X \setminus R_H(X_H)))
  \]
- \(R_D\) (by revealed preference) and \(R_H\) (by assumption) are isotone, and thus \(F\) is also isotone wrt \(\geq\), defined as:
  \[
  F(X_D, X_H) \geq F(Y_D, Y_H) \iff Y_D \subset X_D \text{ and } X_H \subset Y_H
  \]

**Theorem (Tarski).** If \((L, \geq)\) is a complete lattice and \(f : L \to L\) is isotone, then \(f\) has a fixed point. Further, if \(P\) is the set of fixed points of \(f\), then \((P, \geq)\) is a complete lattice.

- A special case of this is the existence theorem in the paper

**Theorem 3.** Suppose contracts are substitutes for hospitals. Then:

(a) The set of fixed points of \(F\) is a complete (finite) lattice
(b) Starting at \((X, \emptyset)\), iteration on \(F\) converges monotonically to the highest fixed point \((X_D, X_H)\)
(c) Starting at \((\emptyset, X)\), iteration on \(F\) converges monotonically to the lowest fixed point \((X_D, X_H)\)

- The highest [lowest] fixed point is the stable allocation most preferred by doctors [hospitals] (Theorem 4)

- Starting at \((X, \emptyset)\) is equivalent to doctor-proposing DA
- Starting at \((\emptyset, X)\) is equivalent to hospital-proposing DA

To see equivalence with doctor-proposing DA:

- Define set \(X_H(t)\) as the cumulative contracts offered by doctors to hospitals up to iteration \(t\)
- Define \(X_D(t)\) as the cumulative set of contracts that have not yet been rejected by hospitals up to iteration \(t\)
- \(X_H(t) \cap X_D(t)\) are contracts conditionally accepted
- Using the definition of \(F\), we have that:
  \[
  X_D(t) = X \setminus R_H(X_H(t - 1)),
  X_H(t) = X \setminus R_D(X_D(t)).
  \]

- If \(|H| \geq 2\), and \(R_h\) is not isotone for some \(h\), then there exists a preference profile for doctors and a preference profile for another hospital \(h'\), which has a single job opening, such that no stable match exists (Theorem 5)
Properties of Generalized DA

- Vacancy chain dynamics can be described by adjusting $F$ to reject all offers made to a retiring doctor
- After a doctor retires, a new stable allocation is achieved by iteration on the adjusted $F$; unretired doctors weakly better off and hospitals weakly worse off (Theorem 6)
- The preferences of hospital $h \in H$ satisfy the law of aggregate demand if for all $Y \subset Z$, $|C_h(Y)| \leq |C_h(Z)|$
- If more contracts are offered, each hospital should demand weakly more contracts
- This can be characterized nicely in the matching with wages setting:

**Theorem 7.** If hospital $h$’s preferences are quasi-linear and satisfy the substitutes condition, then they satisfy the law of aggregate demand.

- A rural hospital theorem also holds in matching with contracts

**Theorem 8.** If hospital preferences satisfy substitutes and the law of aggregate demand, then for every stable allocation $(X_D, X_H)$ and every $d \in D$ and $h \in H$, $|C_d(X_D)| = |C_d(X_H)|$ and $|C_h(X_D)| = |C_h(X_H)|$.

- Hospitals and doctors always get the same number of contracts

  - Necessity of law of aggregate demand for this result:

**Theorem 9.** If there exists a hospital $h$, sets $Y \subset Z \subset X$ such that $|C_h(Y)| > |C_h(Z)|$, and $|H| > 1$, then there exist singleton preferences for the other hospitals and doctors such that the number of doctors employed by $h$ is different for two stable matches.

- Truth-telling is optimal in a few senses

Let hospitals’ preferences satisfy substitutes and the law of aggregate demand, and let the matching algorithm produce the doctor-optimal match. Then, fixing the preferences of the other doctors and of all the hospitals, let $x$ be the contract that $d$ obtains by submitting the set of preferences $P_d : z \vdash d$

Let hospital $h$ have preferences such that $|C_h(X)| > |C_h(X \cup \{x\})|$ and let there exist two contracts $y, z$ such that $y_D \neq x_D \neq z_D$ and $y, z \in R_h(X \cup \{x\}) - R_h(X)$. Then if another hospital $h'$ exists, there exist singleton preferences for the hospitals besides $h$ and preferences for the doctors such that it is not a dominant strategy for all doctors to reveal their preferences truthfully.

**Relationship to Proxy Auctions**

- The algorithm described by 1 may not converge if substitutes assumption fails, even if a fixed point exists

The following algorithm is yet another characterization of DA and is suitable for the non-substitutes case, when there is just one hospital (the auctioneer)

$$X_D(t) = X \setminus R_h(X_H(t - 1)),$$

$$X_H(t) = X_H(t - 1) \cup C_D(X_D(t)).$$

**Theorem 15.** Under the substitute assumption, with $X_D(0) = X$ and $X_H(0) = \emptyset$, the sequences of pairs $\{(X_D(t), X_H(t))\}$ generated by the two laws of motion 1 and 2 are identical.

- The difference between the 1 and 2 is the second equation

**Theorem 16.** When the doctor-offering cumulative offer process with a single hospital terminates at time $t$ with outcome $(X_D(t), X_H(t))$, the hospital’s choice $C_H(X_H(t))$ is a stable collection of contracts.

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