Mixture Spaces

- The primitives are an arbitrary set \( P \) and an operation \( h : [0, 1] \times P \times P \to P \), we will write \( h_a(p, q) \) instead of \( h(a, p, q) \).

- \( (P, h) \) is a mixture space if the following axioms hold:
  
  - **M1** \( h_1(p, q) = p \) (sure mix)
  - **M2** \( h_a(p, q) = h_{1-a}(q, p) \) (commutativity)
  - **M3** \( h_{ab}(h_b(p, q), q) = h_{ab}(p, q) \) (one-sided distributivity)

- **M1, M2, M3** \( \implies \) **M4**: \( h_c(h_a(p, q), h_b(p, q)) = h_{c(a)(1-c)b}(p, q) \) (two-sided distributivity)

- **M1, M2, M3** \( \implies \) **M0**: \( h_a(p, p) = p \) (trivial mix)

- Careful! The following properties are true in lottery spaces, but **not true** in general mixture spaces:
  
  - \( h_a(h_b(p, q), r) = h_{ab}(p, h_{(1-b)(a-b)}(q, r)) \) (associativity)
  - \( h_a(p, r) = h_a(q, r) \implies p = q \) (determinism)

- A binary relation \( \succeq \) on \( P \) satisfies the von Neumann–Morgenstern (vNM) axioms if:

  - **A1** \( \succeq \) is a preference relation (complete and transitive)
  - **A2** \( p \succ q, a \in (0, 1) \implies h_a(p, r) \succ h_a(q, r) \)
  - **A3** \( p \succ q \implies \exists a, b \in (0, 1) : h_a(p, r) \succ q \succ h_b(p, r) \)

- Given **A1** and **A2**, the following is equivalent to **A3** if the set \( P \) is a topological space:

  - **A3***: For every \( p \in P \), the sets \( \{ q : p \succeq q \}, \{ q : q \succeq p \} \) are closed

**Theorem** (Mixture Space Theorem, Herstein-Milnor 1953). Let \( (P, h) \) be a mixture space and \( \succeq \) binary relation on \( P \). Then:

\[ \succeq \text{ satisfies A1, A2, A3 iff } \succeq \text{ has a linear representation.} \]

The representation is unique up to an affine transformation, i.e., if \( U : P \to \mathbb{R} \) is a linear function that represents \( \succeq \) and \( V \) is some other function that represents \( \succeq \) then \( V = aU + b \), for \( a > 0, b \in \mathbb{R} \).

- **U is linear** if \( U(h_a(p, q)) = aU(p) + (1 - a)U(q) \)

- Before proving the theorem, we showed two lemmas

**Lemma 4.1.** If \( \succeq \) satisfies A1, A2, A3 then:

1. \( 1 \geq a > b \geq 0, p \succ r \implies h_a(p, r) \succ h_b(p, r) \)
2. \( a \in (0, 1), p \sim q \implies h_a(p, q) \sim q; \text{ if } p \succ q \text{ then } p \succ h_a(p, q) \succ q \)
3. \( p \succeq q, r \succeq s \implies h_a(p, r) \succeq h_b(q, s) \) strict if strict.

**Lemma 4.2.** Let \( \alpha(p, q, r) := \sup \{ a \in [0, 1] : q \succeq h_a(p, q) \} \). For any \( p, q, r \) such that \( p \succ q \succ r \), \( h_{\alpha(p,q,r)}(p, r) \sim q \).

We proved the theorem by construction

- Note that if \( p \sim q \) for all \( p, q \in P \) then we can set \( U(p) = 1 \) for all \( p \) and we are done

- Thus assume \( p \succ q \) for some \( p, q \in P \) and for any \( r \in P \) define:

\[
U(r) = \begin{cases} 
\frac{1}{a(r, p)} & \text{if } r \succ p \\
1 & \text{if } r \sim p \\
\frac{\alpha(p, r, p)}{1-a(p, r, p)} & \text{if } r \prec p \\
0 & \text{if } r \prec q
\end{cases}
\]

- Consider each case in turn to show that \( U \) represents \( \succeq \)

To show that \( U \) is linear, we again consider the various cases

- **A1**: \( p \succ q \succ p, \text{ if case 3 for both} \)
- **Note that** \( h_U(p) \succeq p, \text{ and } h_U(q) \succeq q, \text{ by 4.2 and construction of } U \)

- By 4.1(iii) and the above (as well as M4):

\[
h_a(p, q) = h_a(h_U(p)(p, q), h_U(q)(p, q))
= h_{aU(p) + (1-a)U(q)}(p, q),
\]

- Thus \( U(h_a(p, q)) = aU(p) + (1-a)U(q) \)

- That \( \succeq \) iscts and \( U \) represents \( \succeq \) does **not** imply that \( U \) is cts

- If in addition to the above \( U \) is also linear, then \( U \) must also be continuous

- The representation is unique in the followin sense

**Theorem.** If \( U \) is a linear function that represents \( \succeq \) and \( V \neq U \) is linear, then \( V \) represents \( \succeq \) iff \( \exists a > 0, b \in \mathbb{R} \) such that \( V = aU + b \).

**von Neumann-Morgenstern**

- **Let** \( X \) be a finite set of prizes, \( \mathcal{L}(X) \) be lotteries over \( X \)

- An expected utility representation is a utility function \( U \) which represents \( \succeq \) on \( \mathcal{L}(X) \), such that there exists some u: \( X \to \mathbb{R} \) such that \( U(p) = \sum_{x \in X} u(x)p(x) \) for any \( p \in \mathcal{L}(X) \)

**Theorem** (von Neumann-Morgenstern 1947). \( \succeq \) on \( \mathcal{L}(X) \) satisfies A1, A2, A3 iff it has an expected utility representation.

- Proof is a consequence of the mixture space theorem and the following lemma

**Lemma.** If \( U \) is a linear function on \( \mathcal{L}(X) \), then there exists some \( u: X \to \mathbb{R} \) such that \( U(p) = \sum_{x \in X} u(x)p(x) \) for all \( p \in \mathcal{L}(X) \). Conversely, if \( U(p) = \sum_{x \in X} u(x)p(x) \) for all \( p \in \mathcal{L}(X) \), then \( U \) is linear.

- **Proof of lemma** is by induction on the number of prizes

**von Neumann-Morgenstern on Monetary Prizes**

- \( X \) infinite set of prizes, \( X = [w, b] \subseteq \mathbb{R}, w < b \)

- **Let** \( F \) be the set of CDFs on \( X \)

- The axioms are slightly adjusted, as follows:

**A1+M** \( \succeq \) is a preference relation and satisfies monotonicity, i.e., \( x \succ y \implies \delta_x \succ \delta_y \)

**A2** \( p \succ q, a \in (0, 1) \implies h_a(p, r) \succ h_a(q, r) \)
For every $F \in \mathcal{F}$, the sets $\{G : G \succeq F\}$ and $\{G : F \succeq G\}$ are closed under the topology induced by the metric $d(F,G) = \int |F - G| \, dx$

A3* implies we are endowing $F$ with the weak topology

- A3* can be replaced by other notions of weak convergence
- e.g., if $\mu^n \to F$ at every continuity point of $F$ and $\mu^n \succeq G$ for all $n$, then $F \succeq G$

**Theorem.** $\succeq$ on $F$ satisfies A1+M, A2, A3* iff there is a continuous, strictly increasing $u : X \to \mathbb{R}$, such that:

$$U(F) = \int u(x) \, dF(x) \quad \text{represents } \succeq.$$

- $F$ second order stochastically dominates $G$ if $F \neq G$ and $\int_{-\infty}^{x} G(z) \, dz \geq \int_{-\infty}^{x} F(z) \, dz$ for all $x$
- $\succeq$ is risk averse when $F \succ G$, if $F$ second order dominates $G$

**Theorem (Notions of Risk Aversion).** Let $\succeq$ on $F$ satisfy A1+M, A2, A3* and $u$ be the vNM utility index. Then:

$$u \text{ strictly concave } \iff \delta_{\frac{1}{2}y + \frac{1}{2}y} > \frac{1}{2} \delta_{x} + \frac{1}{2} \delta_{y}.$$  

**Anscombe-Aumann**

Let $S = \{1, \ldots, n\}$ be a finite set of states, $X$ be a set of prizes

Let $H$ be a set of acts $f : S \to \mathcal{L}(X)$

We have the following AA axioms:

**AA1-AA3** These are just A1–A3 for $\succeq$ on $H$

**AA4** $x \succ y$ for some $x, y \in X$ (non-degenerate preference)

**AA5** For every $f, g, \tilde{f}, \tilde{g}$, such that there are non-null $i,j \in S$ so that $f_{k} = g_{k}$ for all $k \neq i$, $\tilde{f}_{k} = \tilde{g}_{k}$ for all $k \neq j$ and $f_{i} = \tilde{f}_{i}$, $g_{i} = \tilde{g}_{i}$, we have $f \succeq g$ implies $\tilde{f} \succeq \tilde{g}$ (state separability)

- A state $i \in S$ is null if for all $f, g$ such that $f_{j} \equiv g_{j}$, for $j \neq i$, we have $f \sim g$

- Note that AA5 implies state-separable preferences

**Theorem (Anscombe-Aumann 1963).** $\succeq$ satisfies AA1–AA5 on $H$ iff there exists a non-constant linear $U$ on $\mathcal{L}(X)$, and a probability $\mu$ on $S$ such that:

$$W(f) := \sum_{s \in S} U(f_{s}) \mu(s)$$

represents $\succeq$. $U$ is unique up to a positive affine transformation.

- The key part of the proof is the lemma below

**Lemma.** Function $W : H \to \mathbb{R}$ is linear iff $\exists \{U_{s}\}_{s \in S}$, such that $W(f) = \sum_{s \in S} U_{s}(f_{s})$.

- The rest of the proof proceeds as follows:
  - Using AA5 show that all $U_{s}$ from the lemma represent the same preferences up to positive affine transformation
  - By AA4 there is one non-null state $i \in S$
    - Let the positive affine transformations taking utility from state $j$ to state $i$ be $a_{j} > 0, b_{j}$

- Define $\mu(j) = a_{j}$ and normalize to sum to 1

- We extended this to arbitrary $S$, but restricting to $H^{0}$, the set of simple acts, i.e., acts which yield finite number of prizes

- In this case we needed a slightly stronger axiom AA5:

**AA5** $f, g \in H^{0}$, and $p, q \in \mathcal{L}(X)$, and non-null events $E, \hat{E}$, such that: $f_{s} = g_{s}$ for $s \notin E$, $\hat{f}_{s} = \hat{g}_{s}$ for $s \notin \hat{E}$, and $f_{s} = \hat{f}_{s} = p$, $g_{s} = \hat{g}_{s} = q$ for all $s \in E$, $\hat{s} \notin E$ we have $f \succeq g$ implies $\hat{f} \succeq \hat{g}$

- Event $E \subseteq S$ is null if $f_{s} = g_{s}, \forall s \in S \setminus E$ implies $f \sim g$

**Qualitative Probability**

- We began by looking at some facts from probability theory

**Fact 1.** A finitely-additive probability $\mu$ on $A$, an algebra on $S$, can be extended to $2^{S}$.

**Fact 2.** A $\sigma$-additive probability $\mu$ on $A$, an algebra on $S$, can be extended to $\sigma(A)$.

- A probability $\mu$ on an algebra $A$ is convex-valued if $\forall A \in A$, $r \in [0, 1]$, there is $B \subseteq A$ such that $\mu(B) = r \mu(A)$

- A probability $\mu$ on an algebra $A$ is non-atomic if $\forall \mu(A) > 0$, there is $B \subseteq A$ such that $0 < \mu(B) < \mu(A)$

**Theorem.** Convex-valued $\mu \Rightarrow$ non-atomic $\mu$. The converse is true for $\sigma$-additive $\mu$.

- Preference $\succeq^{*}$ over $A$ is a qualitative probability if:

**Q1** $\succeq^{*}$ is complete and transitive

**Q2** $A \succeq^{*} \emptyset$, for all $A$

**Q3** $S \succeq^{*} \emptyset$

**Q4** $A \succeq^{*} B$ iff $A \cap C \succeq^{*} B \cap C$, when $A \cap C = B \cap C = \emptyset$

- Further, there was an important axioms regarding partitions

**P** $A \succeq^{*} B$ implies $\exists \{A_{1}, \ldots, A_{n}\}$, a partition of $S$, such that $A \succeq^{*} (B \cup A_{i})$ for every $A_{i}$

  - Says that not too many things are different from each other”

- Kreps book shows that a qualitative probability, $\succeq^{*}$, satisfies $P$

  - is both fine and tight:
    - $\succeq^{*}$ is fine if for all $A \succeq^{*} \emptyset$, there is a finite partition of $S$ no member of which is as likely as $A$
    - $\succeq^{*}$ is tight, if for all $A \succeq^{*} B$, there is $C$ such that $A \succeq^{*} (B \cup C) \succeq^{*} B$

- Below are some examples of the above theorem

**Example (Lexicographic).** $A \succeq^{*} B$ iff $|A_{1}| > |B_{1}|$ or if $|A_{1}| = |B_{1}|$ and $|A_{2}| > |B_{2}|$

- Finess fails: For $A = [2, 2.5]$, we have $A \succeq^{*} \emptyset$, by every finite partition includes an element with a positive mass on $S$

**Example (Substitutes).** $A \succeq^{*} B$ iff $|A_{1}| + |A_{2}| > |B_{1}| + |B_{2}|$, or if $|A_{1}| + |A_{2}| = |B_{1}| + |B_{2}|$ and $|A_{1}| > |B_{1}|$
There is some
For all
Theorem. Let \( \preceq^* \) satisfy Q1–Q4 and P. Then \( \exists \mu \) a \( \text{finitely}-\text{additive} \) prob. that represents \( \preceq^* ; \mu \) is unique and convex valued.

Fact*. Every \( \preceq^* \) satisfying Q1–Q4 and P has an \( 2^n \)-equipartition (for every \( n \)), and for \( A \succ^* B \), there is \( C \subset A \) such that \( C \sim^* B \).

The proof of the theorem uses this fact and a few lemmas.

**Lemma 0.** Assume \( A \cap B = \emptyset = C \cap D \). If \( A \succ^* C \) and \( B \succ^* D \), then \( A \cup B \succ^* C \cup D \). Further \( A \succ^* C \) and \( B \succ^* D \) implies \( A \cup B \succ^* C \cup D \).

**Lemma 1.** Let \( a = \{A_1, \ldots, A_n\}, b = \{B_1, \ldots, B_m\} \) be two equips of \( S \). Then (i) \( n = m \) implies \( A_i \sim^* B_j \) for all \( i, j \) (ii) \( n > m \) implies \( A_i \succ^* B_j \) for all \( i, j \) (iii) \( n = 2m \) implies \( A_i \sim^* B_j \cup B_k \).

- Define \( k(n, a, A) = \min \{k : \cup_{i=1}^k A_i \succ^* A\} \), for some \( a = \{A_i\}_{i=1}^{2^n} \)
- Further define \( k(n, A) = \min_{a \in 2^n} k(n, a, A) \)
- By lemma 1, \( k(n, A) = k(n, a, A) \)
- Finally define \( \mu(A) = \lim_{n} \frac{k(n, A)}{2^n} \)

**Lemma 2.** \( \mu(A) \) is a \( \text{finitely}-\text{additive} \) probability.

The final step is to show \( \mu \) represents \( \preceq^* \) and is unique and convex valued.

**Savage**

- Let \( S \) be an arbitrary set of states, \( X \) the set of prizes
- An act \( f : S \to X \) is \textit{simple} if it takes a finite number of prizes
- Let \( \succ^* \) be a relation on \( F \), the set of all simple acts
  - For any \( x \in X \), let \( f \in F \) be the act always returning prize \( x \)
  - Let \( fAg \) denote the act which is the same as \( f \) on states \( A \subset S \) and the same as \( g \) on \( S \setminus A \)

- The Savage axioms are:
  - \( S1 \) \( \succ^* \) is a preference relation
  - \( S2 \) There is some \( x, y \in X, F \) such that \( x \succ^* y \) (non-degeneracy)
  - \( S3 \) \( fAh \succ^* gAh \) implies \( fAh' \succ^* gAh' \) (sure-thing principle)
  - \( S4 \) If \( A \) is non-null, then \( xAh \succ^* yAh \forall h \) iff \( x \succ^* y \) (Kreps sure-thing principle)
  - \( S5 \) \( xAy \succ^* xBy \) implies \( x'Ay' \succ^* x'By' \), for all \( x \succ^* y, x' \succ^* y' \) (states are independent of prizes)
  - \( S6 \) For all \( f \succ^* g, \) \( x \in X \) there is a finite partition of \( S \) such that \( xAf \succ^* g, \) and \( f \succ xAg \) (strong continuity)

- Some useful observations
  - For \( S6 \) to be satisfied \( S \) must be infinite
  - For \( S5 \) to be violated we need at least 3 prizes
  - The existence of an additive representation implies \( S3 \)

- The strict version of \( S4 \) is for all non-null \( A \): \( x \succ^* y \) if \( \exists h. xAh \succ^* yAh \)
- The "for all \( h \)" is important for constructing counterexamples of \( S4 \)
- In the absence of completeness, zero measure sets might not be null, as by definition, \( A \) is null if for all \( f, g, h \): \( fAh \sim gAh \), but the latter may not be computable

**Theorem** (Savage 1954). A binary relation \( \succeq \) on the set of simple savage acts satisfies S1-S6 iff there exists a non-constant function \( u \) on \( X \), and a \( \text{finitely}-\text{additive} \) probability on \( (S, 2^S) \), such that \( W(f) = \sum_{x \in X} u(x) \mu(f^{-1}(x)) \) represents \( \succeq \). Furthermore, \( \mu \) is unique and \( u \) is unique up to an affine transformation.

**Steps of the proof:**
- Pick any \( x \succ^* y \) and define \( \succeq^* \) by \( xAy \succ^* xBy \)
- \( \succeq^* \) is a qualitative probability that satisfies Axiom P, hence there exists a unique convex valued \( \text{finitely}-\text{additive} \) probability \( \mu \) that represents \( \preceq^* \)
- For \( f \in F \), define \( p_f \) by \( p_f(x) = \mu(f^{-1}(x)) \), i.e. fold-down (simple) acts to (simple) lotteries
- Show that \( \phi : F \to \mathcal{L}^0(X) \), defined by \( \phi(f) = p_f \) is onto, i.e. every (simple) lottery \( p \in \mathcal{L}^0(X) \) has an act that "reduces" to it
- \( p_f = p_g \) implies \( f \sim g \)
- Define \( \succeq^0 \) on \( \mathcal{L}^0(X) \) by \( p \succeq^0 q \) iff there are \( f, g \) such that \( f_p = p, \) \( g_q = q, \) and \( f \succeq g \)
- \( \succeq^0 \) satisfies (vNM) A1-A3 and therefore there exists a function \( u : X \to \mathbb{R} \) that represents \( \succeq^0 \) on \( \mathcal{L}^0(X) \)
- Combined together, \( u, \mu \) yield the desired representation: \( W(f) = \sum_{x \in X} u(x) \mu(f^{-1}(x)) \)

**Comments about Finiteness Assumptions**

Throughout this course we worked with finite primitive sets, with occasional excursions into the infinite, in particular:

- vNM: \( X \) is a finite set of prizes
- vNM over monetary prizes: here we considered infinite \( X \), but with the natural order structure, allowing us to approximate elements in \( \mathcal{L}(X) \) by simple lotteries, using an added monotonicity requirement
- A-A: \( S \) is finite and in the homework we extended the result to infinite \( S \) but restricted to \textit{simple} acts, i.e. acts that give a finite number of outcomes (lotteries in \( \mathcal{L}(X) \) in this case)
- Savage: \( S \) is infinite (required for P to hold), but we restrict attention to \textit{simple} savage acts