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CHEAP TALK WITH ENDOGENOUS CONFLICT OF INTEREST

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This paper analyzes a cheap-talk setting where the conflict of interest between sender and receiver is determined endogenously by the choice of parameters θ_i, one for each agent, prior to the cheap talk phase. Conditions are provided that determine the sign of agent i's inverse demand for θ *without assuming that the most informative equilibrium will necessarily be played in the cheap talk game*. For two popular functional forms of payoffs, we derive analytically tractable approximations for agent i's demand for θ. Under the above conditions, if investors obtain shares in an enterprise by trading in an equity market, then the competitive equilibrium allocation achieves the most information transmission that is possible, given the particular equilibrium selection in the cheap talk game that follows the trading stage. However, an inefficiency lurks in the background: if the receiver's ability to trade is restricted, a competitive market for shares fails to reward the positive externality that the sender provides to other agents by purchasing his shares. In a principal-agent relationship, we show that the optimal contract will not allocate equity in a way that achieves perfect communication. Two extensions are briefly sketched out. First, the scenario where the θ's are acquired covertly rather than overtly. Second, the scenario in which the θ's are traded *after* the sender has received the information.

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1. INTRODUCTION

We study a cheap-talk setting where the conflict of interest between sender (he) and receiver (she) is determined endogenously prior to the cheap talk phase.

For example, consider two large shareholders who purchase their shares on a competitive equity market, and then exchange information regarding a strategic decision for the corporation. Or, consider a CEO/sender who, in addition to exerting effort, also communicates information to the owner/receiver about a payoff-relevant decision for the corporation. In both examples the conflict of interest between sender and receiver is endogenous: in the first example, to the equity holdings acquired by both agents; in the second example, to the compensation scheme (salary plus equity) offered to the CEO.

Formally, we study a model where, before the cheap talk stage, every agent i obtains a certain quantity of θ_i (for example, of shares) at a cost $c_i(\theta_i)$. After paying this cost, each agent is assigned the following payoff function in the cheap talk game:

$$U(a, q_i(\theta_i), \omega),$$

where a is the action taken by the receiver, the function $q_i(\cdot)$ encodes the heterogeneity across agents, and ω represents the unknown (to the receiver) state of the world. In our baseline case, all the agents' θ_i's become publicly observable before the sender learns ω and the cheap talk stage takes place.

Within this setting we obtain several results. First, we characterize the agents' inverse demand functions (their willingness to pay) for θ *without assuming that the most informative equilibrium will necessarily be played in the cheap talk game.* More precisely, we characterize the many inverse-demand functions that arise for a class of equilibrium selections in the subsequent cheap talk game. We start by focusing on equilibrium selections that vary differentiably with θ (Section 3). We provide sufficient conditions on the function U such that *all the inverse demand functions can be signed* (Section 3). These conditions are satisfied in the functional form examples that are commonly used in the applied literature. If these conditions hold, then for any equilibrium selection (i.e., whether the most informative cheap talk equilibrium is played, or the second most informative, etc.) the sender's inverse demand function at any given (θ_S, θ_R) has the opposite sign as the receiver's. Furthermore, both inverse demands equal zero if $q_S(\theta_S) = q_R(\theta_R)$. Intuition for the latter result can be gained by noting that at this point both agents agree as to the optimal choice of a; but this intuition is only partial because mere agreement regarding the optimal a does not imply that both agents' inverse demand for θ should be the same.

In Section 4 we extend the results to equilibrium selections that vary continuously, but not necessarily differentiably, with θ. We call these selections "regular" (an example is the selection that picks out the most informative cheap talk equilibrium as θ varies.) In this case, inverse demand functions can be discontinuous and non-monotonic, but the sign properties are preserved. In Section 5 we *approximate analytically* the inverse demand function associated with the most-informative equilibrium selection. For two functional forms commonly used in applications, we obtain *tractable closed-form approximants to the inverse demand function associated with the most-informative equilibrium selection.* These approximants are continuous, monotonic, and, we show, give rise to an inverse demand that is "informationally close" in a precise sense to the actual (discontinuous and non-monotonic) inverse demand function. We leverage the properties of these approximants to show that, in both functional forms, if the price of θ is constant the receiver will either purchase the minimum possible θ, or else the amount required to achieve perfect alignment with the sender. In contrast, the sender's demand for θ does not have this extremal property.

Equipped with these results, we turn to applications. In the first application (Section 6.1), investors obtain shares in an enterprise by trading in a competitive equity market. Under the assumptions of Proposition 4, we show that the competitive equilibrium allocation generated by *any demand function* arising from a regular selection features perfect alignment, i.e., $q_S(\theta_S^*) = q_R(\theta_R^*)$. This is the allocation that achieves the most information transmission, given the specific cheap talk equilibrium selection that generated the inverse demand function. We interpret this result as saying that competitive markets provide incentives to trade shares that are conducive to maximal information transmission. However, an inefficiency lurks in the background: a competitive market for shares fails to reward the positive externality that the sender, being informed, provides to other agents by purchasing additional shares. If the receiver cannot trade shares, this externality may lead the sender to acquire too few shares and to transmit too little information to the receiver (Section 6.2). This inefficiency points to a conflict between the risk-sharing and the information-transmission motives for trading. This setting may speak to activist investors, who often seek to influence a business'strategy by transmitting information to a controlling shareholder (board, controlling owner). If the controlling shareholder is entrenched, meaning she holds shares partly to exercise control, the activist investor should be buying more shares from a welfare perspective. This is because the share price on the capital market fails to reward the positive informational externality that the activist investor provides by purchasing shares.

The second application is a principal-agent relationship where the shares are granted as compensation by a principal (receiver, board of directors) to an agent (sender, CEO). The sender provides information to the receiver regarding corporate strategy and, in addition, also exerts costly noncontractible effort. Equity can be used as part of the sender's compensation scheme to incentivize two different activities: effort provision and information transmission. We find that, *given any regular cheap talk equilibrium selection*, the optimal contract allocates too much equity to the sender for perfect communication to occur (Section 6.3). In addition, as usual in a principal-agent problem, too much risk (equity) is provided to the agent relative to the first-best risk sharing allocation. This setting adds a third consideration, namely information transmission, to the standard trade-off between moral hazard and risk sharing in the principal-agent literature. In the board-CEO setting, our perspective is different from (though not contradictory of) most of the corporate finance literature's, which is that the CEO might take excessive risk relative to the board's preferences. In our equilibrium analysis, instead, the sender/CEO feels that "too much" of his compensation is tied to the stock price and, consequently, will be more cautious than the board as concerns strategy. Since the board, not the CEO, chooses the strategy, the implemented strategy will not be systematically too cautious; however, in equilibrium the CEO will misreport his signal to the board.

We conclude with two extension of the main model. First, we consider the case in which the θ's are acquired unobservably, and so must be guessed by both sender and receiver in order to play the cheap talk game (Section 6.4). We show that inverse demand functions are lower for both sender and receiver in this scenario, meaning that compared to the overt acquisition case, the incentives that lead toward alignment between sender and receiver are weaker. Finally, we briefly discuss the scenario in which trading takes place after the expert investor has received the information (Section 6.5).

This paper defines the inverse demand function for θ, characterizes its sign, provides a suitable definition of approximate inverse demand function and sufficient conditions for the approximation to be "good." Furthermore, the results do not rely on a particular (e.g., most-informative) equilibrium selection in the cheap talk game. Finally, the incentives for overt and covert acquisition of θ are compared. All this is new, to our knowledge. These results are illustrated through functional-form examples: Example 1 builds on Crawford and Sobel (1982), whereas Example 2 builds on Alonso (2009), and to a lesser degree on Admati and Pfleiderer (2004), Alonso, Dessein and Matouschek (2008), and Kawamura (2015). None of these authors had derived the demand for θ in their settings. Cheap talk with endogenous conflict of interest has appeared in various papers however, often in a setting where sender and/or receiver acquire information at a cost before cheap talk (the closest papers being Deimen and Szalay 2019a, 2019b; but see also Austen-Smith, 1994, and Argenziano et al., 2016). Rantakari (2017) also considers a two-stage game, where first-stage effort influences the distribution of the state of the world, followed by cheap talk. The focus of the paper is the case of two managers, who both generate possible projects and how the CEO decides on which to pursue following the communication.

Turning to applications, three papers are somewhat related to our Section 6.3. Almazan et al. (2008) study the provision of incentives for the dual managerial tasks of effort provision and information transmission, but the setting is quite different. Whereas in Almazan et al. (2008) the information is transmitted to investors, and it is used to decide which price to pay for shares; in our paper, in contrast, the information is transmitted to the principal and is used to choose the value-creating (or destroying) action a. Song and Thakor (2006) is related because though the CEO does not technically engage in cheap talk (in part because both the CEO and the board have interacting career concerns), the CEO nevertheless communicates with the board. Finally, Ross (1974) provides conditions under which Pareto-optimality and full alignment regarding the endogenous choice of a can be achieved between principal and agent. But Ross (1974) assumes a single endogenous action, whereas in Section 6.3 the agent's effort represents a second endogenous action; in addition, Ross (1974) has no cheap talk.

2. MODEL

2.1. *Players*

The state of the world is $\omega \in [0,1]$, where $\omega \sim f$ and $f > 0$ is absolutely continuous. Agents are indexed by $i \in \{S, R, 1, 2, ...\}$. In the cheap talk game, agent S is the sender and agent R is the receiver. "Numbered agents" $1, 2, ...$ play no role in the cheap talk game, but its outcome affects their utility.

2.2. *Timing and Actions*

1. Each agent i obtains $\theta_i \in \mathbb{R}$ (for example, shares purchased in a competitive market, or assigned by a principal as part of an incentive scheme) at a cost $c_i(\theta_i)$. The θ_i's are publicly observed.
2. The sender privately learns ω and engages in cheap talk with the receiver.
3. The receiver chooses action a.
4. Payoffs accrue.

2.3. *Payoffs*

After paying the cost $c_i(\theta)$, an agent i who obtains θ experiences the following utility:

$$U(a, q_i(\theta), \omega). \tag{1}$$

Preference misalignment arises between i and j when $q_i(\theta_i) \neq q_j(\theta_j)$, and it can be mitigated by an opportune choice of θ_i and θ_j.
The cost functions $c_i(\cdot)$ are nondecreasing. The functions $U(\cdot,\cdot,\cdot)$, $q(\cdot)$, and $c_i(\cdot)$ are twice continuously differentiable.

2.4. *Assumptions on utility functions*

The following assumptions are maintained throughout the paper.
Assumption 1 *U is concave in q and strictly concave in a.*
Both concavities arise naturally in applications. Strict concavity in a simplifies the problem of finding the receiver's optimal response.
Assumption 2 ***(U exhibits single-crossing in** a, q**)*** *For any ω and $a' > a$, $U(a, q, \omega) - U(a', q, \omega) = 0$ implies that, for any $q' > q$, $U(a', q', \omega) - U(a, q', \omega) \leq 0$.*
The single-crossing assumption implies that, if ω was known, the optimal action a would be decreasing in q.

Assumption 3 ***(a and ω are complements)*** $U_{a\omega}(a,q,\omega) > 0$.
This assumption helps ensure that , if ω was known, the optimal action a would be decreasing in ω. Together with Assumption 1, Assumption 3 guarantees that all cheap talk equilibria are "essentially" partition equilibria (refer to Lemma 1 in Gordon 2010).

2.5. *Functional-form examples*

A first trivial example is when $q_i(\cdot) \equiv \overline{q}_i$ for all i.[1] In this case there is no meaningful demand for θ, and the standard cheap talk game with fixed biases obtains.
In what follows we provide three non-trivial functional-form examples in which there is a meaningful demand for θ. Lemma B.1 verifies that Assumptions 1-3 hold in all three examples.
Example 1 ***(Crawford-Sobel quadratic preferences)*** *Agent i's preferences are:*

$$-(a-\omega+b_i+\theta)^2,$$

where b_i is given and θ can be adjusted. This is a variant of the functional form in Crawford and Sobel (1982). These preferences have the form (1), with:

$$\begin{aligned} U(a,q,\omega) &= -(a-\omega+q)^2 \\ q_i(\theta) &= b_i+\theta. \end{aligned}$$

Both agents' choice of θ_i affects their ability to communicate truthfully.
Example 2 ***(Markowitz mean-variance preferences, bias arising from share ownership)*** *A number of agents acquire shares in a private investment venture. The rate of return X on investment is a random variable with unknown mean ω and unit variance. After all the shares are allocated, an activist shareholder (sender) learns ω and communicates with a controlling shareholder (receiver) who chooses how much capital a will be invested in the venture. If agent i acquires $\theta \geq 0$ shares, her utility is assumed to have the following mean-variance form:*

$$\mathbb{E}(\theta a X) - \frac{r_i}{2} Var(\theta a X),$$

where $r_i > 0$ denotes i's risk attitudes. Denoting

$$\begin{aligned} U(a,q,\omega) &= qa\omega - \frac{1}{2}(qa)^2 \\ q_i(\theta) &= r_i \cdot \theta, \end{aligned}$$

the agents' utility can be expressed as:

$$\frac{1}{r_i} U(a, q_i(\theta), \omega),$$

which has the form (1), up to the factor $\frac{1}{r_i}$ which is strategically immaterial in the cheap-talk game. This is a variant of the functional form in Alonso (2009).[2] *In this example, preferences in the cheap talk game are perfectly aligned if agents S, R obtain shares in the proportion $\theta_S/\theta_R = r_R/r_S$. In this example, $\rho = (r_R\theta_R)/(r_S\theta_S)$ is a useful measure of "bias."*

[1] To denote that a variable is taken as fixed, we add a bar to it, so $\overline{q}_i$ denotes some fixed value of the variable q_i, and $\overline{q}_{-i}$ denotes some fixed vector of all other agents' q's except agent i.
[2] In Alonso (2009), the functional form emerges endogenously as part of an analysis where decision rights are allocated optimally.

Example 3 ***(bias arising from income level in proportional taxation)*** *The median voter in a large community (receiver) chooses an income tax rate $a \in [0,1]$ to finance a public good. Voter i is endowed with income y_i and can obtain extra income θ at cost $c_i(\theta)$. Aggregate taxable income equals $Q = \int (y_i + \theta_i)\, di$ which can be regarded as a constant from an individual player's perspective. Public good provision is given by $\omega\sqrt{aQ}$, where ω captures factor productivity. After all the θ's have been chosen, one of the voters (sender) learns ω and communicates with the receiver, then the receiver chooses the tax rate a. Given Q, agent i's utility is:*

$$\omega\sqrt{aQ} + (1-a)(y_i+\theta) - c_i(\theta)$$

We may express agent i's preferences as follows:

$$\begin{aligned} U(a,q,\omega) &= \omega\sqrt{aQ} + (1-a)\,q \\ q_i(\theta) &= y_i + \theta. \end{aligned}$$

Thus the agents' preferences have the form (1).

2.6. *Cheap-talk game*

The cheap talk game takes q_R and q_S as parameters. A vector $\Omega = [\omega_0, \omega_1, ..., \omega_N]$ with $\omega_0 = 0, \omega_N = 1$, and $\omega_k < \omega_{k+1}$ is called a partition of the state space.[3] The elements ω_k are called partition cutoffs. Given a partition Ω (not necessarily an equilibrium one), define the function:

$$V_i(q_i, q_{-i}; \Omega) = \sum_{k=1}^{N} \int_{\omega_{k-1}}^{\omega_k} U(a_k^*, q_i, \omega)\ dF(\omega), \tag{2}$$

where

$$a_k^* \in \arg\max_a \int_{\omega_{k-1}}^{\omega_k} U(a, q_R, \omega)\ dF(\omega)\ , \tag{3}$$

for $k = 1, ..., N$.
A partition $\Omega_N = \left[0, \omega_1^*, ...\omega_{N-1}^*, 1\right]$ that solves:

$$U(a_k^*, q_S, \omega_k^*) = U\left(a_{k+1}^*, q_S, \omega_k^*\right) \tag{4}$$

for $k = 1, ..., N-1$ is said to be an N-partition equilibrium in the cheap talk game. When evaluated at an equilibrium, the function (2) is written as:

$$V_i^*(q_i, q_{-i}; \Omega_N), \tag{5}$$

and it represents agent i's payoff in this equilibrium.

[3]Formally, Ω is a set of disjoint intervals of the form: $\{[\omega_0, \omega_1), [\omega_1, \omega_2)..., [\omega_{N-1}, \omega_N]\}$; for short, we notate Ω as the vector of interval cutoffs.

2.7. *The Cheap-talk Equilibrium Correspondence* $\Omega_N(q_R, q_S)$

This section introduces the correspondence $\Omega_N(q_R, q_S)$ whose image is the set of cutoffs of an N-partition equilibrium given (q_R, q_S).[4] This is interpreted as the equilibrium that players expect to play upon reaching the cheap talk stage, among several equilibria that might exist.

Fix (q_R, q_S). Given any partition $[\omega_0, \omega_1, ...\omega_N]$ (not necessarily an equilibrium one), use (3) to express a_k^* as a function $a(\omega_k, \omega_{k-1}, q_R)$, then substitute into (4) to obtain the following function:

$$g(\omega_{k-1}, \omega_k, \omega_{k+1}; q_R, q_S) = U(a(\omega_{k-1}, \omega_k, q_R), q_S, \omega_k) - U(a(\omega_k, \omega_{k+1}, q_R), q_S, \omega_k).$$

Having thus eliminated a_k^*, a_{k+1}^*, the equilibrium conditions (3, 4) may be expressed in vector form as:

$$\mathbf{g}(\omega_1, ..., \omega_{N-1}; q_R, q_S) = \begin{bmatrix} g(0, \omega_1, \omega_2; q_R, q_S) \\ \vdots \\ g(\omega_{k-1}, \omega_k, \omega_{k+1}; q_R, q_S) \\ \vdots \\ g(\omega_{N-2}, \omega_{N-1}, 1; q_R, q_S) \end{bmatrix} = \mathbf{0}, \tag{6}$$

where we have already set $\omega_0 = 0$ and $\omega_N = 1$.

Definition 1 ***(N-partition equilibrium correspondence)*** *Fix $N > 1$. Suppose the functions $\{\mathfrak{w}_k(\cdot,\cdot)\}$ solve the system (6) for every (q_R, q_S) in some open set $\mathfrak{Q} \subset \mathbb{R}^2$ and are differentiable on $\mathfrak{Q}$. The multi-valued mapping $\Omega_N(q_R, q_S) : \mathfrak{Q} \twoheadrightarrow \mathbb{R}$ is an N-partition equilibrium correspondence if its image equals* $\{0, \mathfrak{w}_1(q_R, q_S), ..., \mathfrak{w}_{N-1}(q_R, q_S), 1\}$.

The correspondence Ω_N maps a pair (q_R, q_S) into the set of points that are partition cutoffs of an N-partition equilibrium, for fixed N. For example Ω_2 traces out a two-partition equilibrium as a function of (q_R, q_S) even over a region where a three- or a four-partition equilibrium exists. In Example 1, a unique cheap talk equilibrium $\Omega_2(q_R, q_S) = \{0, \mathfrak{w}_1(q_R, q_S), 1\}$ exists if $|b| = |q_R - q_S| < \frac{1}{4}$. The functional form expression for the partition cutoff $\mathfrak{w}_1(\cdot,\cdot)$ happens to be differentiable over this entire domain (Crawford and Sobel 1982, eq. 23). Therefore $\Omega_2(q_R, q_S)$ is well-defined over the domain $|q_R - q_S| < \frac{1}{4}$. Note that, over a proper subset of $|q_R - q_S| < \frac{1}{4}$, a 3-partition equilibrium correspondence Ω_3 also exists, so over this smaller domain Ω_2 and Ω_3 co-exist.[5]

While Definition 1 imposes differentiability, under mild assumptions on the function g, the implicit function theorem guarantees that any $\mathfrak{w}_k$'s that solve the system (6) are differentiable (refer to Section A.1). Differentiability of the $\mathfrak{w}_k$'s ensures that the function V_i^* is differentiable as a function of q_i (see Lemma B.3).

The following restriction will help sign the inverse demand functions. It says that equilibrium partition cutoffs all move in the same direction as q_R and q_S vary.

Definition 2 ***(comonotonic equilibrium correspondence)*** *An N-partition equilibrium correspondence $\Omega_N(q_R, q_S)$ is **comonotonic** if $\frac{\partial}{\partial q_R}\mathfrak{w}_k(q_R, q_S) \leq 0 \leq \frac{\partial}{\partial q_S}\mathfrak{w}_k(q_R, q_S)$ for all k.*

If a closed-form expression for $\mathfrak{w}_k(q_R, q_S)$ is available, comonotonicity can be verified directly.[6]

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[4] One can also think of $\Omega_N(\cdot,\cdot)$ as an $N+1$-vector valued function.

[5] Technically, every element $w_k(\cdot,\cdot)$ should also be indexed by N, as in $w_k^N(\cdot,\cdot)$, because for a given pair (q_R, q_S) different N's may produce different solutions to the system (6). We will omit this index whenever no confusion can arise.

[6] In Example 1, for instance, the k-th partition cutoff in an N-partition equilibrium takes the form:

$$\mathfrak{w}_k(q_R, q_S) = \frac{k}{N} + 2(q_R - q_S)k(k - N),$$

Else, sufficient conditions for comonotonicity can be derived via the implicit function theorem; this approach is used in Section A.1, where the following Lemma is proved.

Lemma 1 *Suppose $\omega \sim U[0,1]$. Then in Examples 1, 2, and 3, $\Omega_N(\cdot,\cdot)$ is comonotonic for any N.*

Proof. See Section A.1 in the Appendix. ■

3. INVERSE DEMAND FUNCTION OF Q OVER Ω_N

While we are ultimately interested in the willingness to pay for θ, we begin by characterizing the agent's willingness to pay for q. Agent i's inverse demand function evaluated at q captures her willingness to pay to marginally increase q to $q+\varepsilon$. Pathological results can easily be generated by abruptly switching from one cheap talk equilibrium to another as q varies. We will work around this obstacle by restricting attention to continuous equilibrium correspondences. In this section we further restrict attention to correspondences Ω_N, whose functions $\mathfrak{w}_k(q_R, q_S)$ are differentiable.

The inverse demand function will be indexed by the correspondence Ω_N that agents expect to govern play in the cheap talk stage. And, since agent i's payoff in the cheap talk game depends on the sender's and receiver's value of q, inverse demand will also be indexed by the vector $\overline{q}_{-i}$ (even though only the elements $\overline{q}_R$ and $\overline{q}_S$ of the vector $\overline{q}_{-i}$ actually affect i's demand). Ω_N and $\overline{q}_{-i}$ are taken as given throughout this section and the next two.

Definition 3 ***(inverse demand function of** q **over an** N**-partition correspondence)*** *Agent i's inverse demand at q is:*

$$D_i\left(q;\overline{q}_{-i},\Omega_N\right)=\left.\frac{d}{dy}V_i^*\left(y,\overline{q}_{-i};\Omega_N\right)\right|_{y=q}.$$

The derivative is well-defined by Lemma B.3. Inverse demand captures how agent i's payoff V_i^* changes as q is incremented infinitesimally, fixing everyone else's $\overline{q}_{-i}$. Note that, by definition of V_i^*, incrementing q_R or q_S alters the equilibrium strategies as prescribed by equations (3, 4), and these effects are captured in Definition 3. For example, if the sender's q increase by ε, then $\Omega_N(\overline{q}_R, q_S)$ changes to $\Omega_N(\overline{q}_R, q_S+\varepsilon)$, and the receiver's equilibrium strategy changes accordingly. If the receiver marginally changes her q, there is also a direct effect on a^*. The total differential notation reminds the reader of these indirect effects.

We now parse demand into two separate components: a communication and a non-communication component. We abuse notation slightly by denoting the domain of correspondences by (q_i, q_{-i}) rather than by the ordered pair (q_R, q_S).

Definition 4 ***(separate components of the inverse demand function)*** *The* ***non-communication component*** *of agent i's inverse demand function of q is:*

$$D_i^{NoComm}\left(q;\overline{q}_{-i},\Omega_N\right)=\left.\frac{\partial}{\partial y}V_i\left(y,\overline{q}_{-i};\Omega_N\left(q,\overline{q}_{-i}\right)\right)\right|_{y=q}.$$

The inverse demand function's ***communication component*** *is:*

$$D_i^{Comm}\left(q;\overline{q}_{-i},\Omega_N\right)=D_i\left(q;\overline{q}_{-i},\Omega_N\right)-D_i^{NoComm}\left(q;\overline{q}_{-i},\Omega_N\right).$$

(this is eq. 23 in Crawford and Sobel 1982, p. 1441, after replacing $b=q_R-q_S$). Comonotonicity holds because $k\leq N$.

In the definition of D_i^{NoComm}, the equilibrium partition Ω_N is fixed: it does not vary with y.[7] Hence, by equation (3), a^* is unchanged in D_S^{NoComm} as y varies. In D_R^{NoComm} varying y does change a^*, but this variation has vanishing effect due to an envelope condition. In sum, the strategic variables (a^*, Ω_N) are effectively kept fixed when computing D_i^{NoComm}. In this sense, D_i^{NoComm} isolates the "intrinsic" (as opposed to strategic) value of a marginal unit of q.

The communication component of demand isolates the *strategic* value of an additional unit of q: it is the benefit that an additional unit of q brings through the change in (a^*, Ω_N). Note that the communication component is identically zero for all numbered agents, because their choice of q does not affect the cheap talk game. Therefore, the numbered agents' inverse demand coincides identically with its non-communication component.

We now state three alternative assumptions that will help sign both demand components.

Assumption 4 $U_q(a, q, \omega) \geq 0$.

This assumption says that q is valuable regardless of the level of a and ω.

Assumption 5 *(a **and** q **are substitutes**)* $U_{aq}(a, q, \omega) < 0$.

This assumption represents a strenghtening of the single-crossing assumption: it implies that acquiring a larger amount of q makes the agent prefer smaller a's.

Assumption 6 $U_a(a, q, \omega) / U_q(a, q, \omega)$ *is independent of* ω *and greater than zero.*

This assumption is satisfied whenever the function U admits the following representation:

$$U(a, q, \omega) = u(h(a, q), \omega),$$

with the index function $h(\cdot, \cdot)$ being increasing (or decreasing) in both its arguments. Intuitively, Assumption 6 says that, irrespective of the uncertainty generated by ω, a and q are redundant control variables for the receiver. This assumption holds in Examples 1 and 2, but not in Example 3.

Next, we sign the communication demand component. The flavor of the result is that q is valuable up until perfect alignment, but no further.

Lemma 2 ***(sign of communication component of inverse demand function)*** *If* Ω_N *is comonotonic then:*

1.

$$(\overline{q}_S - q) \cdot D_R^{Comm}\left(q; \overline{q}_{-i}, \Omega_N\right) \geq 0.$$

2. *If, moreover, Assumption 5 or Assumption 6 holds, then:*

$$(\overline{q}_R - q) \cdot D_S^{Comm}\left(q; \overline{q}_{-i}, \Omega_N\right) \geq 0.$$

3. *If sender and receiver are perfectly aligned such that* $\overline{q}_R = \overline{q}_S = q$*, then for all* i*:*

$$D_i^{Comm}\left(q; \overline{q}_{-i}, \Omega_N\right) = 0.$$

Proof. Denote $\overline{q}_i = q$ (the q in the lemma's statement) and fix $\left(\overline{q}_i, \overline{q}_{-i}\right)$. For all (q_R, q_S) in a neighborhood of $(\overline{q}_R, \overline{q}_S)$, the equilibrium correspondence $\{\mathfrak{w}_1(q_R, q_S), ...\mathfrak{w}_{N-1}(q_R, q_S)\}$ gives rise to optimal actions $a(\mathfrak{w}_{k-1}(q_R, q_S), \mathfrak{w}_k(q_R, q_S), q_R) = \mathfrak{a}_k(q_R, q_S)$ through the function $a(\cdot, \cdot, \cdot)$ that was defined on page 6. We denote $\mathfrak{w}_k(\overline{q}_R, \overline{q}_S) = \overline{\omega}_k$ and $\mathfrak{a}_k(\overline{q}_R, \overline{q}_S) = \overline{a}_k$, and the differentials of the functions $a, \mathfrak{a}_k$, and $\mathfrak{w}_k$ will be evaluated at $(\overline{q}_R, \overline{q}_S)$ and notated without their arguments.

[7]This is why in Definition 4 it is necessary to replace V_i^* with the function V_i defined in (2). The function V_i^* requires Ω_N to vary with y. Of course the two functions coincide (in level) at the point $y = q$

Receiver

With a slight abuse of notation, replace the vector $\overline{q}_{-i}$ with the scalar $\overline{q}_S$. Then:

$$\begin{aligned}
D_R(\overline{q}_R;\overline{q}_S,\Omega_N) &= \frac{\partial}{\partial y}V_R^*(y,\overline{q}_S;\Omega_N(y,\overline{q}_S))\Big|_{y=\overline{q}_R} \\
&= \frac{\partial}{\partial y}\sum_{k=1}^{N}\int_{\mathfrak{w}_{k-1}(y,\overline{q}_S)}^{\mathfrak{w}_k(y,\overline{q}_S)} U(\mathfrak{a}_k(y,\overline{q}_S),y,\omega)\ dF(\omega)\Big|_{y=\overline{q}_R} \\
&= \sum_{k=1}^{N}\left(\int_{\overline{\omega}_{k-1}}^{\overline{\omega}_k} U_a(\overline{a}_k,\overline{q}_R,\omega)\ dF(\omega)\right)\cdot\left(a_1\frac{\partial\mathfrak{w}_{k-1}}{\partial q_R}+a_2\frac{\partial\mathfrak{w}_k}{\partial q_R}\right) \qquad (7)\\
&\quad +D_R^{NoComm}(\overline{q}_R;\overline{q}_S,\Omega_N)+\sum_{k=1}^{N-1}\left[U(\overline{a}_k,\overline{q}_R,\overline{\omega}_k)-U(\overline{a}_{k+1},\overline{q}_R,\overline{\omega}_k)\right]f(\overline{\omega}_k)\cdot\frac{\partial\mathfrak{w}_k}{\partial q_R}.
\end{aligned}$$

The term in line (7) involving a_3 is part of D_R^{NoComm}; anyway, line (7) vanishes entirely because of the receiver's first order conditions (3). Rewrite using Definition 4:

$$D_R^{Comm}(\overline{q}_R;\overline{q}_S,\Omega_N)=\sum_{k=1}^{N-1}\left[U(\overline{a}_k,\overline{q}_R,\overline{\omega}_k)-U(\overline{a}_{k+1},\overline{q}_R,\overline{\omega}_k)\right]f(\overline{\omega}_k)\cdot\frac{\partial\mathfrak{w}_k}{\partial q_R}.$$

To sign the term in brackets we use the single-crossing assumption. The sender's equilibrium conditions (4) guarantee that:

$$U(\overline{a}_k,\overline{q}_S,\overline{\omega}_k)-U(\overline{a}_{k+1},\overline{q}_S,\overline{\omega}_k)=0.$$

Since $\overline{a}_k<\overline{a}_{k+1}$, Assumption 2 (single-crossing) implies:

$$U(\overline{a}_k,\overline{q}_R,\overline{\omega}_k)-U(\overline{a}_{k+1},\overline{q}_R,\overline{\omega}_k)\geq 0 \quad \text{if } \overline{q}_R>\overline{q}_S.$$

Thus the term in brackets is nonnegative if $\overline{q}_S<\overline{q}_R$. Comonotonicity ensures that $\partial\mathfrak{w}_k/\partial q_R$ is non-positive. Therefore D_R^{Comm} is non-positive if $\overline{q}_S<\overline{q}_R=q$. Conversely, if $\overline{q}_S>\overline{q}_R=q$ the term in brackets is nonpositive and then D_R^{Comm} is nonnegative. This proves part 1. When $\overline{q}_R=\overline{q}_S$, the sender's equilibrium conditions (4) guarantee that $D_R^{Comm}(\overline{q}_S;\overline{q}_S,\Omega_N)=0$, which proves part 3 for the receiver.

Sender

Proceeding as in the previous case, and replacing the vector $\overline{q}_{-i}$ with the scalar $\overline{q}_R$, we get:

$$\begin{aligned}
&D_S(\overline{q}_S;\overline{q}_R,\Omega_N)-D_S^{NoComm}(\overline{q}_S;\overline{q}_R,\Omega_N)\\
&= \sum_{k=1}^{N}\left(\int_{\overline{\omega}_{k-1}}^{\overline{\omega}_k} U_a(\overline{a}_k,\overline{q}_S,\omega)\ dF(\omega)\right)\cdot\frac{\partial\mathfrak{a}_k}{\partial q_S}\\
&\quad +\sum_{k=1}^{N-1}\left[U(\overline{a}_k,\overline{q}_S,\overline{\omega}_k)-U(\overline{a}_{k+1},\overline{q}_S,\overline{\omega}_k)\right]f(\overline{\omega}_k)\cdot\frac{\partial\mathfrak{w}_k}{\partial q_S}.
\end{aligned}$$

The term in bracket vanishes because of the sender's equilibrium conditions (4). Then use Definition 4 to get:

$$D_S^{Comm}(\overline{q}_S;\overline{q}_R,\Omega_N)=\sum_{k=1}^{N}\left(\int_{\overline{\omega}_{k-1}}^{\overline{\omega}_k} U_a(\overline{a}_k,\overline{q}_S,\omega)\ dF(\omega)\right)\cdot\frac{\partial\mathfrak{a}_k}{\partial q_S}. \qquad (8)$$

If Assumption 5 holds and $\overline{q}_S < \overline{q}_R$, then the integral in (8) is larger than:

$$\int_{\overline{\omega}_{k-1}}^{\overline{\omega}_k} U_a\left(\overline{a}_k, \overline{q}_R, \omega\right)\ dF\left(\omega\right) = 0,$$

where equality reflects the receiver's first order conditions (3). Conversely, if $\overline{q}_S > \overline{q}_R$ then the integral in (8) is negative.

Alternatively, suppose Assumption 6 holds. Then the integral in (8) can be written as:

$$\begin{aligned}
(9) \qquad & \frac{U_a\left(\overline{a}_k, \overline{q}_S, \omega\right)}{U_q\left(\overline{a}_k, \overline{q}_S, \omega\right)} \int_{\overline{\omega}_{k-1}}^{\overline{\omega}_k} U_q\left(\overline{a}_k, \overline{q}_S, \omega\right)\ dF\left(\omega\right) \\
\geq \ & \frac{U_a\left(\overline{a}_k, \overline{q}_S, \omega\right)}{U_q\left(\overline{a}_k, \overline{q}_S, \omega\right)} \int_{\overline{\omega}_{k-1}}^{\overline{\omega}_k} U_q\left(\overline{a}_k, \overline{q}_R, \omega\right)\ dF\left(\omega\right) \\
= \ & \frac{U_a\left(\overline{a}_k, \overline{q}_S, \omega\right)}{U_q\left(\overline{a}_k, \overline{q}_S, \omega\right)} \frac{U_q\left(\overline{a}_k, \overline{q}_R, \omega\right)}{U_a\left(\overline{a}_k, \overline{q}_R, \omega\right)} \int_{\overline{\omega}_{k-1}}^{\overline{\omega}_k} U_a\left(\overline{a}_k, \overline{q}_R, \omega\right)\ dF\left(\omega\right) \\
= \ & 0,
\end{aligned}$$

where the inequality holds if and only if $\overline{q}_R > \overline{q}_S$, in light of concavity of U in q (Assumption 1) and positivity of the factors U_a/U_q (Assumption 6). These factors travel in and out of the integral because they are independent of ω (Assumption 6). The final equality reflects the receiver's first order conditions (3).

Summing up, under either Assumption 5 or Assumption 6, the integral in (8) is nonnegative if $\overline{q}_R > \overline{q}_S = q$ and nonpositive if $\overline{q}_R < \overline{q}_S = q$. To complete the proof of part 2 we need to show that the factor

$$\frac{\partial \mathfrak{a}_k}{\partial q_S} = a_1 \frac{\partial \mathfrak{w}_{k-1}}{\partial q_S} + a_2 \frac{\partial \mathfrak{w}_k}{\partial q_S},$$

in (8) is nonnegative. The terms $\frac{\partial \mathfrak{w}_{k-1}}{\partial q_S}$ and $\frac{\partial \mathfrak{w}_k}{\partial q_S}$ are nonnegative by assumption, and the partial derivatives a_1 and a_2 are both positive by Lemma B.2. Thus $\frac{\partial \mathfrak{a}_k}{\partial q_S}$ is nonnegative, as desired.

Finally, when $\overline{q}_S = \overline{q}_R$, the receiver's first order conditions (3) guarantee that the integral in (8) equals zero, and so $D_S^{Comm}\left(\overline{q}_R; \overline{q}_R, \Omega_N\right) = 0$, which proves part 3 for the sender. ■

Part 3 of the above lemma is intuitive: at perfect alignment, communication between receiver and sender is as good as it can be *given* Ω_N, and so neither R nor S has any communication-based incentive to alter q. While intuitive, this result is not obvious because it holds *even when* Ω_N *does not feature perfect communication.* Parts 1 and 2 deal with the sign of inverse demand away from perfect alignment, and give conditions under which closer alignment improves the "strategic component" of payoffs. Assumptions 5 and 6 serve to connect q and a, in such a way that by better aligning q_S with q_R, the sender can commit to "feeling more similarly" with the receiver about a, which permits better communication. Under these conditions, we may say that the "strategic demand for alignment" is positive.

We now sign the non-communication demand component.

Lemma 3 ***(sign of non-communication inverse demand component)***

1. *If Assumption 6 holds, then:*

    (a)

$$D_R^{NoComm}\left(q; \overline{q}_S, \Omega_N\right) \equiv 0.$$

(b) For all agents $i \neq R$:

$$\left(\overline{q}_R - q\right) \cdot D_i^{NoComm}\left(q; \overline{q}_{-i}, \Omega_N\right) \geq 0.$$

2. *If sender and receiver are perfectly aligned such that $\overline{q}_R = \overline{q}_S = q$, then for all i:*

$$D_i^{NoComm}\left(q; \overline{q}_{-i}, \Omega_N\right) \quad \text{is independent of } i.$$

Proof. Using the same notation as in the proof of Lemma 2 (where, recall, $q = \overline{q}_i$) we write, for all $i \neq R$:

$$\begin{aligned} D_i^{NoComm}\left(\overline{q}_i; \overline{q}_{-i}, \Omega_N\right) &= \left.\frac{\partial}{\partial y} V_i\left(y, \overline{q}_{-i}; \Omega_N\left(\overline{q}_R, \overline{q}_S\right)\right)\right|_{y=\overline{q}_i} \\ &= \left.\frac{\partial}{\partial y} \sum_{k=1}^{N} \int_{\overline{\omega}_{k-1}}^{\overline{\omega}_k} U\left(a\left(\overline{\omega}_{k-1}, \overline{\omega}_k, \overline{q}_R\right), y, \omega\right) \, dF(\omega)\right|_{y=\overline{q}_i} \\ &= \sum_k \int_{\overline{\omega}_{k-1}}^{\overline{\omega}_k} U_q\left(\overline{a}_k, \overline{q}_i, \omega\right) \, dF(\omega). \end{aligned} \tag{10}$$

In the case $i = R$ only, the following extra term must be added to expression (10):

$$\sum_{k=1}^{N} \left(\int_{\overline{\omega}_{k-1}}^{\overline{\omega}_k} U_a\left(\overline{a}_k, \overline{q}_R, \omega\right) \, dF(\omega) \right) \cdot a_3\left(\overline{\omega}_{k-1}, \overline{\omega}_k, \overline{q}_R\right),$$

but this extra term vanishes as in expression (7).

Let us prove part 2 first. Given $\overline{q}_R = \overline{q}_S = q$, $\overline{\omega}_k$ and $\overline{a}_k$ are fully determined. Then expression (10) only depends on the value of $\overline{q}_i$, and so the result follows. Let us now prove part 1. The integral in expression (10) is the same as that in expression (9) after replacing $\overline{q}_S$ with $\overline{q}_i$. Repeat the steps following expression (9) to show that the integral vanishes for $\overline{q}_i = \overline{q}_R$, and otherwise has the same sign as $\left(\overline{q}_R - \overline{q}_i\right)$. ■

Unlike for the "strategic component" of payoffs, there is no reason to expect that the "intrinsic" payoff from a higher value of q should improve with closer alignment between sender and receiver. Hence, "intrinsic demand for alignment" need not be positive.[8] Nevertheless, if Assumption 6 holds, Lemma 3 shows that willingness to pay for alignment is positive for all agents except the receiver, for whom it is zero. The intuition is as follows. Since the receiver sets a optimally, she cannot gain by perturbing a. When Assumption 6 holds, q and a are redundant control variables, and so she cannot gain by perturbing q, either. Hence intrinsic demand for alignment is zero for the receiver. As for the sender, his "intrinsic" payoff from q improves by moving q closer to $\overline{q}_R$ because a is chosen to "fit" $\overline{q}_R$.

The next result combines Lemmas 2 and 3 to characterize the sign of the inverse demand functions along a given N-partition correspondence.

Proposition 1 ***(sign of inverse demand function of*** q ***over an*** N***-partition correspondence)*** *Suppose Ω_N is comonotonic.*

1. *If sender and receiver are perfectly aligned such that $\overline{q}_R = \overline{q}_S = q$, then for all i:*

$$D_i\left(q; \overline{q}_{-i}, \Omega_N\right) = p \text{ independent of } i.$$

[8] For example, suppose that all agents have a lot of value for q independent of the cheap talk game, then demand for either θ or q will be positive for all agents irrespective of alignment.

2. If Assumption 6 holds, then:

(a)

$$\left(\overline{q}_S - q\right) \cdot D_R\left(q; \overline{q}_S, \Omega_N\right) \geq 0.$$

(b) For all agents $i \neq R$:

$$\left(\overline{q}_R - q\right) \cdot D_i\left(q; \overline{q}_{-i}, \Omega_N\right) \geq 0.$$

3. If Assumptions 4 and 5 hold, then $D_R\left(q; \overline{q}_{-i}, \Omega_N\right) \geq 0$ for $q \leq \overline{q}_S$, and, for all other agents $i \neq R$, $D_i\left(q; \overline{q}_{-i}, \Omega_N\right) \geq 0$ if $q \leq \overline{q}_R$.

4. Suppose $\Omega_N\left(\cdot, \overline{q}_{-i}\right)$ is defined over the interval $[c, d]$. Then:

$$V_i^*\left(d, \overline{q}_{-i}; \Omega_N\right) = V_i^*\left(c, \overline{q}_{-i}; \Omega_N\right) + \int_c^d D_i\left(q; \overline{q}_{-i}, \Omega_N\right)\, dq.$$

Proof. Parts 1 and 2 hold because inverse demand is given by: $D_i^{Comm} + D_i^{NoComm}$, and the two separate components have been signed in Lemmas 2 and 3. (Recall also that $D_i^{Comm} \equiv 0$ for $i \neq R, S$). Part 3 holds because $D_i^{NoComm} \geq 0$ for all q if Assumption 4 holds (refer to eq. 10). Part 4 is proved in Lemma B.3. ■

Proposition 1 part 1 says that all agents have the same marginal willingness to pay for q at the perfect alignment point. What about their willingness to pay for θ? Agent i's willingness to pay for θ is:

$$D_i\left(q_i(\theta); \overline{q}_{-i}, \Omega_N\right) \cdot q_i'(\theta).$$

At the perfect alignment point the first term is independent of i, and so any two agents i, j have the same willingness to pay for θ if and only if $q_i'(\theta) = q_j'(\theta)$. An important exception is the case in part 2, where the first term vanishes at perfect alignment; then, no restrictions on the functions $q_i(\cdot)$ are required for all agents' willingness to pay for θ to equate (to zero) at the perfect alignment point.

Part 3 shows a case where willingness to pay is nonnegative at least up to the point where q_i which attains the perfect alignment level, and possibly further. If willingness to pay is strictly positive at the perfect alignment point, then at that point the incentives to acquire θ cannot be aligned unless $q_i'(\theta)$ is independent of i.

4. EXTENDING THE RESULTS TO THE REGULAR EQUILIBRIUM CORRESPONDENCE Ω

All the results so far have focused on the correspondence Ω_N. This has restricted the analysis to equilibria with a fixed number N of partition elements as (q_R, q_S) vary. We now introduce a more general equilibrium correspondence Ω.

Definition 5 ***(regular equilibrium correspondence)*** *A multi-valued mapping $\Omega : \mathbb{R}^2 \twoheadrightarrow \mathbb{R}$ is a* ***regular equilibrium correspondence*** *if it is continuous and if, for every $\overline{q}_{-i}$, the domain of $\Omega\left(\cdot, \overline{q}_{-i}\right)$ can be partitioned into a (countable) number of adjacent intervals $[c_\tau, d_\tau)$ where each (c_τ, d_τ) is the largest open interval such that on it, Ω coincides with a comonotonic N-partition correspondence Ω_N.*

A regular equilibrium correspondence is constructed by pasting together several "components" Ω_N's. These components are pasted together continuously, but not necessarily differentiably, at the boundaries of the disjoint intervals (c_τ, d_τ). By construction, Ω has the same number of equilibrium partition elements within each interval (c_τ, d_τ). Across adjacent intervals either the number N of partition elements changes, or the functions $\mathfrak{w}_k\left(\cdot, \overline{q}_{-i}\right)$ from one interval merge non-differentiably into those of the other interval, or both. Typically, the number of partition elements changes from N to $N+1$ between two adjacent intervals.[9,10]

Allowing the number of partition elements N to vary along the equilibrium correspondence is necessary to permit, among others, an important equilibrium selection: that which picks out the most informative equilibrium for every pair (q_R, q_S). For instance, in Crawford and Sobel's (1982) quadratic example, the equilibrium correspondence that picks out the cheap talk equilibrium with the largest number of partition elements N for any given (q_R, q_S), is regular. Another regular equilibrium correspondence (perhaps, a less interesting one) is the one obtained by adding the qualifier "provided that $N \leq 5$" to the previous sentence. At the point of perfect alignment $q_R = q_S$, continuity of Ω pins down a single "limit" cheap talk equilibrium among many that exist.

We now informally introduce the notion of "inverse demand over the regular equilibrium correspondence Ω," with the obvious meaning that Ω_N is replaced by Ω in Definition 3. Clearly, all the characterizations obtained in Proposition 1 also characterize inverse demand over Ω, *except at the boundaries of the intervals* $[c_\tau, d_\tau)$, where differentiability may fail and so inverse demand may not be well-defined. Hence the following extension obtains.

Corollary 1 ***(Proposition 1 extends to Ω almost everywhere)*** *Let Ω be a regular equilibrium correspondence, and replace Ω_N with Ω in the statement of Proposition 1. The result in part 1 holds if Ω is differentiable at the point of perfect alignment. The results in parts 2 and 3 hold almost everywhere.*

The next result applies Corollary 1 to Examples 1-3.

Corollary 2 ***(applying Corollary 1 to Examples 1-3)*** *Let Ω be a regular equilibrium correspondence. In Examples 1, 2 the inverse demand function over Ω has the properties in Proposition 1 part 2. In Example 3, the inverse demand function over Ω has the properties in Proposition 1 part 3.*

Proof. By Lemma B.1, the functional forms of U in Examples 1 and 2 satisfy Assumption 6, and the functional form of U in Example 3 satisfies Assumptions 4 and 5. ■

This corollary highlights the fact that Example 3 is different from the other examples: since Assumption 6 does not hold, inverse demand can only be signed in a portion of the parameter region

[9] An example can help see why non-differentiability arises when N changes. Refer back to Crawford and Sobel (1982). By their equation (23) the k-th partition cutoff in an N-partition equibrium takes the form:

$$\mathfrak{w}_k^N(b) = \frac{k}{N} + 2bk(k-N).$$

As the bias parameter $b \uparrow 1/12$, the 3-partition equilibrium "merges into" into the 2-partition equilibrium and "dies," meaning that $\mathfrak{w}_2^3(b) \to \mathfrak{w}_1^2(b) = 1/3$ and $\mathfrak{w}_1^3(b) \to 0$. However, the merge does not happen differentiably. Indeed, at $b = 1/12$ we have:

$$\frac{\partial}{\partial b}\mathfrak{w}_2^3(b) = -4 \neq \frac{\partial}{\partial b}\mathfrak{w}_1^2(b).$$

[10] Note that the boundaries of these intervals cannot be chosen arbitrarily: because Ω must be continuous, the pasting together of the Ω_N's can only take place at those specific values q_i where the partition cutoffs $\mathfrak{w}_k\left(\cdot, \overline{q}_{-i}\right)$ of one Ω_N "continuously morph" into those of another one; for example, where a 3-partition equilibrium correspondence morphs into a 4-partition equilibrium correspondence.

q. Observe that in the above corollary, regular equilibrium correspondences are easy to construct if $\omega \sim U[0,1]$, because then Lemma 1 guarantees that any Ω_N is comonotonic.

The result in part 4 of Proposition 1 extends as follows.

Corollary 3 ***(slope of V^* given Ω)*** *Let Ω be a regular equilibrium correspondence. $V_i^*\left(\cdot,\overline{q}_{-i};\Omega\right)$ is non-decreasing in any interval where $D_i\left(\cdot;\overline{q}_{-i},\Omega_N\right)\geq 0$ almost everywhere; it is non-increasing in any interval where $D_i\left(\cdot;\overline{q}_{-i},\Omega_N\right)\leq 0$ almost everywhere.*

Proof. See Online Appendix B. ■

This result provides a sort of "fundamental theorem of calculus" for Ω.[11] This is a key result in this paper, and is why we require Ω to be continuous in Definition 5. This result implies, for example, that under the assumptions of Proposition 1 part 2 the function $V_i^*\left(\cdot,\overline{q}_{-i};\Omega\right)$ is quasi-concave with a global maximum at $\overline{q}_{-i}$ for $i=R,S$.

5. APPROXIMATING INVERSE DEMAND FUNCTIONS OVER Ω

Even when Ω is regular, inverse demand functions need not be well-defined at the boundaries of the intervals $[c_\tau,d_\tau)$. At those points Ω switches from one Ω_N to another, and the function $V_i^*\left(\cdot,\overline{q}_{-i};\Omega\right)$, while continuous, exhibits kinks (refer to the green "bumpy" lines in Figure 1).

A discontinuous inverse demand function does not pose conceptual problems: one can still evaluate whether the marginal benefit of q exceeds a given price or not. Practically, however, it is easier to work with continuous inverse demand functions. To this end, we seek to approximate V_i^* with a *continuously differentiable* function $\widehat{V}_i$ that yields an "approximately correct" demand for θ. The next definition spells out the required features of such an approximant.

Definition 6 ***(informational approximants to expected payoff)*** *A continuously differentiable function $\widehat{V}_i$* ***is an informational approximant of*** *V_i^* on an interval Q if $\widehat{V}_i\geq V_i^*$ on Q and if for every $[c_\tau,d_\tau)\subseteq Q$ defined in Definition 5:*

$$\min_{y\in[c_\tau,d_\tau)}\widehat{V}_i\left(y;\overline{q}_{-i},\Omega\right)-V_i^*\left(y,\overline{q}_{-i};\Omega\right)=0.$$

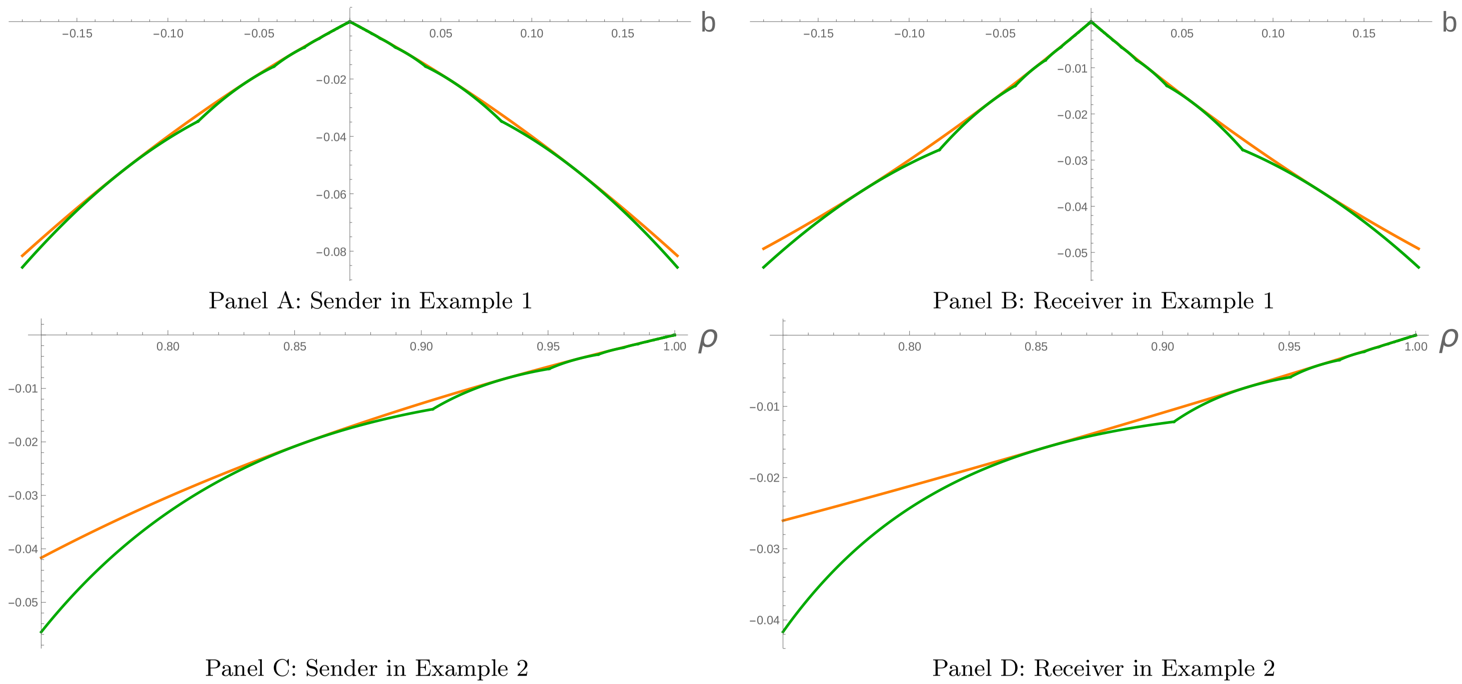
The approximant $\widehat{V}_i$ is a function of a single variable y and is parameterized by $\overline{q}_{-i},\Omega$. The function $\widehat{V}_i$ is differentiable over Q by definition. The function $\widehat{V}_i$ always lies (weakly) above V_i^* and touches V_i^* at least once in every half-open interval where Ω coincides with a different Ω_N.

Typically, many informational approximants to a given payoff function will exist, and there is no recipe for finding an analytically tractable one. Nevertheless, for examples 1 and 2 we able to provide analytically tractable approximants when Ω is the most informative equilibrium correspondence. These approximants are are depicted as the orange lines in Figure 1. Note that the orange function is not differentiable at the perfect alignment point so, technically, it must be regarded as two distinct approximants defined on the two intervals abutting the perfect alignment point $\overline{q}_{-i}$ from the left and the right.[12]

We now define the approximate inverse demand function of θ.

[11]Corollary 3 is not an immediate consequence of the fundamental theorem of calculus because the theorem's assumptions (absolute continuity of D_i) are not invoked here.

[12]When Ω is the is the most informative equilibrium correspondence, no informational approximant can be expected to be differentiable at the perfect alignment point. To understand why, observe that when Ω is the is the most



Panel A: Sender in Example 1

Panel B: Receiver in Example 1

Panel C: Sender in Example 2

Panel D: Receiver in Example 2

FIGURE 1.— Value functions V_i^* (green, bumpy lines) at the most informative equilibrium, and their informational approximants $\hat{V}_i$ (orange, smooth lines) in Examples 1 and 2. Approximants lie above the value functions, and touch them once at every bump. Functional form expressions for displayed approximants provided in the proof of Proposition 3 and in Antic and Persico (2019). Example 2 graphed in case $\rho < 1$ only because value functions in case $\rho > 1$ are smooth and require no approximants.

Definition 7 ***(approximate inverse demand function of*** θ ***given*** $\widehat{V}_i$***)*** *Agent i's approximate inverse demand function of* θ *given* $\widehat{V}_i$ *is:*

$$\frac{d}{d\theta}\widehat{V}_i\left(q_i\left(\theta\right);\overline{q}_{-i},\Omega\right).$$

Note that the derivative exists and is continuous because $\widehat{V}_i$ is continuously differentiable by assumption.

We now define the *demand for* θ, which is how much θ agent i wants to purchase at a given price. This is different from the *inverse demand function of* θ, which indicates the price at which agent i is willing to acquire a marginal unit of θ. The next definition abuses notation by omitting the dependence of demand on $\overline{q}_{-i}, \Omega$, and $\widehat{V}_i$.

Definition 8 ***(approximate and actual demands for*** θ***)*** *Let* $\widehat{V}_i$ *be an informational approximant of* V_i^* *on the interval* $[q_i(\theta_L), q_i(\theta_H)]$. *Agent i's demands for* θ *given price p are:*

$$\widehat{\theta}_i\left(p\right) \in \arg\max_{\theta\in[\theta_L,\theta_H]}\widehat{V}_i\left(q_i\left(\theta\right);\overline{q}_{-i},\Omega\right)-p\theta \textit{approximate} \tag{11}$$

$$\theta_i^*\left(p\right) \in \arg\max_{\theta\in[\theta_L,\theta_H]}V_i^*\left(q_i\left(\theta\right),\overline{q}_{-i};\Omega\right)-p\theta.\textit{actual} \tag{12}$$

The above definition allows demand to be multi-valued, to capture the case of multiple optimal choices. We want our approximants to generate an "approximately correct" demand for θ. Let us define "approximately."

Definition 9 ***(informational closeness)*** *Agent i's approximate demand for* θ *is informationally close to actual demand if, for all p, any two* $\widehat{\theta}_i(p), \theta_i^*(p)$ *are mapped by* $q_i(\cdot)$ *into a single interval* $[c_\tau, d_\tau)$ *defined in Definition 5, or into two adjacent intervals.*

When $\widehat{\theta}_i(p)$ and $\theta_i^*(p)$ are mapped into the same interval, they are informationally similar in the sense that give rise to equilibria with the same number of partition elements. When the two intervals are adjacent, continuity of Ω implies that there is no change in the informational content of the equilibria at the point where q crosses over from one interval to the next. In this sense, the informational content of the equilibria in two adjacent intervals is similar. In most applications, two adjacent intervals will have a similar (i.e., within one) number of partition elements.

We seek to provide conditions under which informational approximants give rise to a demand for θ that is "informationally close" to actual demand.

Proposition 2 ***(conditions when approximate demand is informationally close to actual demand for*** θ***).*** *Let* $\widehat{V}_i$ *be an informational approximant of* V_i^* *on the interval* $[q_i(\theta_L), q_i(\theta_H)]$.

1. *If i's approximate inverse demand function is decreasing in* θ *then approximate demand is informationally close to actual demand.*
2. *If i's approximate inverse demand is increasing in* θ *then approximate demand equals* θ_L *or* θ_H, *and actual demand is informationally extremal for all p, meaning that* $\theta_i^*(p)$ *is informationally close either to* θ_L *or to* θ_H.

Proof. See Appendix A.2. ■

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informative equilibrium correspondence, the points c_τ where V_i^* is non-differentiable accumulate around $\overline{q}_{-i}$. Thus $\widehat{V}_i$ is required to touch V_i^* from above more and more frequently as $q_i \to \overline{q}_{-i}$. This stringent requirement reduces the degrees of freedom in choosing the shape of $\widehat{V}_i$ to the point that its differentiability at $\overline{q}_{-i}$ can no longer be ensured. This can be verified graphically by looking at Figure 1: the orange approximants needs to smooth out an infinite accumulation of green bumps in the neighborhood of $q_R = q_S$, and this forces the left- and right-approximants to approach the limiting point $q_R = q_S$ with different derivatives.

Proposition 2 says that, if approximate inverse demand is either increasing or decreasing, then for the purpose of computing the amount of information exchanged in the cheap talk equilibrium, there is relatively little loss in working with informational approximants.

The next proposition obtains tractable informational approximants for Examples 1 and 2, restricting attention to the set of parameters where some communication is possible.

Proposition 3 ***(Tractable informational approximants for Examples 1 and 2 using the most informative equilibrium selection)*** *Let Ω be the most informative equilibrium correspondence.*

1. *Consider Example 1, and assume $|q_R - q_S| < \frac{1}{4}$ so that non-babbling equilibria exist. Given $\overline{q}_R$, a sender's approximate inverse demand function is:*

$$\frac{4}{3}\left(\overline{q}_R - q_S(\theta)\right) + \frac{1}{3} sgn\left(\overline{q}_R - q_S(\theta)\right),$$

   which is decreasing in θ.

   Given $\overline{q}_S$, a receiver's approximate inverse demand is increasing in θ everywhere except at $q_R(\theta) = \overline{q}_S$.

2. *Consider Example 2, and denote $\rho = q_R/q_S = (r_R\theta_R)/(r_S\theta_S)$. Assume $3/4 < \rho$ so that non-babbling equilibria exist. Given $\overline{q}_R$, a sender's approximate inverse demand function is:*

$$\begin{array}{lll} -\dfrac{\overline{q}_R}{\left(q_S(\theta) - 4\overline{q}_R\right)^2} & \textit{for } \rho \in & (3/4, 1] \\ \dfrac{8\overline{q}_R^2 - 8\overline{q}_R q_S(\theta) + q_S(\theta)^2}{2\overline{q}_R\left(q_S(\theta) - 4\overline{q}_R\right)^2} & \textit{for } \rho > & 1 \end{array}$$

   which is decreasing in θ.

   Given $\overline{q}_S$, a receiver's approximate inverse demand is increasing in θ everywhere except at $q_R(\theta) = \overline{q}_S$.

3. *In both cases listed above: the sender's approximate demand is informationally close to actual demand, and the receiver's actual demand for θ is either informationally extremal or such that $q_R(\theta) = \overline{q}_S$.*

Proof. See Appendix A.2 ■

Parts 1 and 2 provide functional form expression for the approximate demand functions. These functional forms can be useful in applications because they do not depend on the number of partition elements, and thus obviate the need to compute the equilibrium of the cheap talk game. Proposition 3 part 3 says that these approximations are "good."

Proposition 3 reveals that the receiver's inverse demand is increasing, whereas the sender's is decreasing. Figure 2 depicts approximate inverse demand functions for Examples 1 and 2. Consistent with Corollary 2, sender's and receiver's inverse demands have opposite signs.

Why are the approximate inverse demand functions discontinuous at the point of perfect alignment? As discussed in footnote 12, the problem doesn't lie with the choice of approximant, but with the fact that we are approximating the most informative equilibrium selection. In this case, Figure 1 reveals that the function V_i^* itself has a kink at $q_R = q_S$. This kink would be absent if we picked a less-informative equilibrium correspondence – for example, the correspondence that picks out the cheap talk equilibrium with the largest number of partition elements N provided that $N \leq 5$. For this correspondence the function V_i^* is smooth around the perfect-alignment point, and the actual

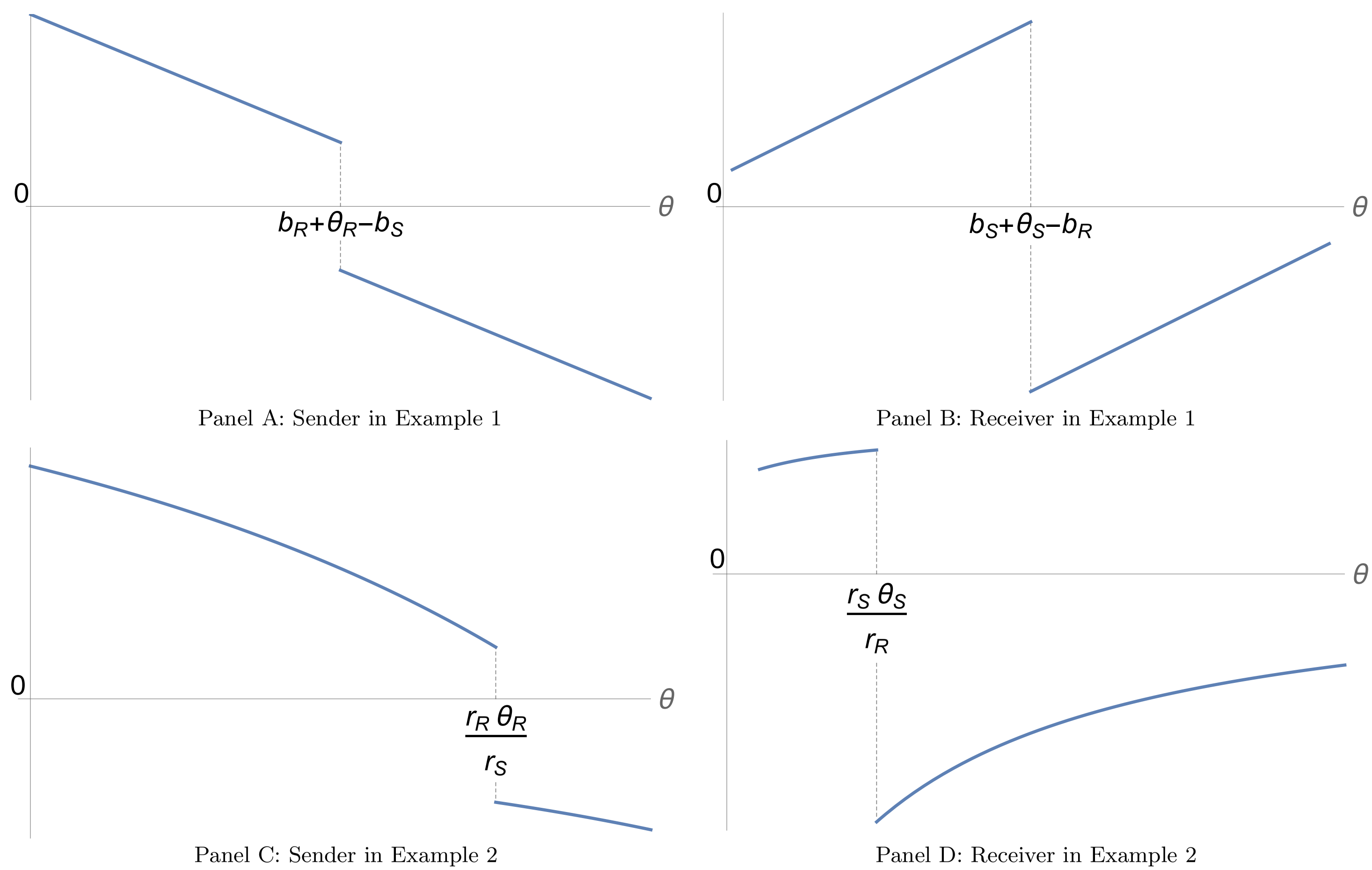

FIGURE 2.— Sender's and receiver's approximate inverse demands in Examples 1 and 2. These are the derivative of the informational approximants $\hat{V}_i$ (orange, smooth lines) in Figure 1. As predicted in Proposition 1, the inverse demands have opposite signs. Because the approximate inverse demands graphed in this picture are the ones associated with the most-informative equilibrium selection, the functions are discontinuous at the point where sender and receiver are perfectly aligned.

inverse demand function is continuous. In this case one can easily construct approximate inverse demand functions that are continuous at the perfect-alignment point.
Proposition 3 focuses on the set of parameters where some communication is possible because no approximation is required elsewhere (where no communication is possible, inverse demand is continuously differentiable.) It is good to remember, therefore, that above functional forms are only good approximations in the parameter region where some communication is possible.

6. APPLICATIONS AND EXTENSIONS

In this section the results of the previous sections are put to use in a number of applications.

6.1. *Competitive Trading of θ*

This section studies the scenario where an aggregate supply of $\theta = 1$ is traded competitively before the cheap-talk game. We assume that agent i funds her purchases at an exogenously specified interest rate λ_i. From now on we normalize $\lambda_i = 0$, meaning that the interest rate is the same for all agents and equilibrium prices are measured net of the interest rate.

Definition 10 *A competitive equilibrium given an equilibrium correspondence Ω is a price p^* at which $\sum_i \theta_i^* (p^*) = 1$, where $\theta_i^* (p^*)$ is as in Definition 8 with $\overline{q}_{-i} = q_{-i} \left(\theta_{-i}^* (p^*)\right)$.*

In a competitive equilibrium, agents trade at a market-clearing price p^*, expecting their value of holding θ_i^* to be obtained in the subsequent cheap talk game. The next proposition provides conditions under which competitive trading promotes perfect alignment.

Proposition 4 ***(competitive trading of θ promotes perfect alignment)*** *Suppose θ is traded competitively before the cheap-talk game. Fix any regular equilibrium correspondence Ω.*

1. *Suppose $q_i (\theta) = b_i + \theta$. Then at the perfect-alignment allocation all agents have the same inverse demand function of θ, if that is well-defined.*
2. *Suppose Assumption 6 holds. Then the perfect-alignment allocation is a competitive equilibrium given Ω.*

Proof. Denote by $\left\{\overline{\overline{\theta}}_i\right\}$ the perfect-alignment allocation such that $\sum_i \overline{\overline{\theta}}_i = 1$ and $q_i \left(\overline{\overline{\theta}}_i\right) = \overline{\overline{q}}$ for all i. Denote by $\overline{\overline{q}}_{-i}$ the allocation where all agents $-i$ hold $\overline{\overline{q}}$.

1. Since at the perfect alignment point $\overline{q}_R = \overline{q}_S = \overline{\overline{q}}$, Proposition 1 part 1 guarantees that if demand is well defined then, for all i:

$$D_i \left(\overline{\overline{q}}; \overline{\overline{q}}_{-i}, \Omega\right) = p.$$

   Therefore, agent i's marginal willingness to pay for θ at $\overline{\overline{\theta}}_i$ is:

$$p \cdot q_i' (\theta) = p.$$

   This proves that all agents have the same marginal willingness to pay for θ at the perfect-alignment allocation.

2. Since Assumption 6 holds, Corollary 3 together with Proposition 1 part 2 guarantee that the function

$$V_i^* \left(q_i (\theta), \overline{\overline{q}}_{-i}; \Omega\right) - p^* \cdot \theta,$$

   has a global maximum at $\theta = \overline{\overline{\theta}}_i$ if $p^* = 0$. Thus the perfect-alignment allocation obtains in a competitive equilibrium with $p^* = 0$.

■

Part 1 establishes that the first-order conditions for competitive equilibrium hold at a perfect-alignment allocation: every agent's inverse demand equals the putative equilibrium price p. Note that no assumptions on U are required for this result, but this result is dependent on a restrictive assumption on the form of the function $q_i(\cdot)$. Furthermore, part 1 does not ensure that the second-order conditions hold, i.e., that payoffs monotonically improve as alignment gets closer. Thus, the second-order conditions need to be checked on a case-by-case basis. Finally, depending on Ω, inverse demand might not be well-defined at the perfect alignment point.

In contrast, Part 2 proves the existence of a perfect-alignment equilibrium (including second-order conditions) without restrictions on $q_i(\cdot)$, but Assumption 6 is required. In either case, Proposition 4 states that perfect alignment among agents may be achieved, but *not necessarily perfect communication.* This is because the statements hold across many equilibrium correspondences, including those that do not feature perfect communication at perfect alignment.[13] The next result specializes to the case where perfect communication is achievable.

Corollary 4 ***(perfect communication)*** *Suppose $\omega \sim U[0,1]$. Then there exists a competitive equilibrium with perfect communication in Examples 1 and 2.*

Proof. The functional forms in Examples 1 and 2 satisfy Assumption 6, and if $\omega \sim U[0,1]$ then all regular equilibrium correspondences, including the most informative one, are comonotonic (see Lemma 1). Then Proposition 4 part 2 applies. ■

Proposition 4 does not explicitly address the uniqueness of the equilibrium allocation, but under the assumptions of Part 2, uniqueness follows directly from Proposition 1 part 2 provided that the "opposite sign of demand" property holds strictly,[14] which it does in most applications.

6.2. *Restrictions on Receiver's Ability to Trade*

We assume here that the receiver has an exogenously given share $\overline{\theta}_R$ of the enterprise, whereas all other agents, including the sender, acquire shares in the competitive market. This setup is like the competitive trading scenario in Section 6.1, except the receiver is not allowed to trade. For example, the receiver may be a government or company founder, who holds a controlling stake for non-financial reasons, but still seeks to maximize her monetary payoff given this constraint.

In this setting, full communication can be especially difficult to achieve. The next proposition illustrates this point vividly. The setting is comparable to Corollary 4 and, moreover, it assumes that all the agent's preferences are perfectly aligned, in the sense that $r_i = 1$ for all i. Despite this favorable setting, a competitive equilibrium with perfect communication does not exist if there are sufficiently many numbered agents.

Proposition 5 ***(impossibility of perfect communication if receiver cannot trade)*** *Consider Example 2 with $\omega \sim U[0,1]$, and moreover let preferences be perfectly aligned, i.e., set $r_i = 1$ for all i. If $\overline{\theta}_R > 0$ is fixed, then no equilibrium with perfect communication exists if there are more than $\frac{6}{5}\left(\frac{1}{\overline{\theta}_R} - 2\right)$ numbered agents.*

[13] For example, all correspondences that restrict the number of equilibrium partitions elements to not exceed a given ceiling do not permit perfect communication.

[14] To see this, observe that any allocation that satisfies the resource constraints and differs from $\left\{\overline{\overline{\theta}}_i\right\}$ must feature two agents k, ℓ with $q_k\left(\widetilde{\theta}_k\right) > \overline{\overline{q}} > q_\ell\left(\widetilde{\theta}_\ell\right)$. But then agent k's and ℓ's willingness to pay have strictly opposite signs. Therefore $\left\{\widetilde{\theta}_i\right\}$ cannot be a competitive equilibrium.

Proof. See Appendix A.2. ■

This result is a "counterexample" to Corollary 4. If demand for risk is sufficiently large, viz., when the receiver owns a large share of the enterprise, or when there are many numbered agents, then the equilibrium share price is too high for the sender to be willing to hold as much as needed for perfect alignment. Hence we cannot have full communication. In this setting, the numbered agents receive a positive benefit from the sender's share purchase which the sender does not internalize. Put differently, the numbered agents are free-riding on the sender's communicative activity.

This problem arises because the receiver is not allowed to sell some of her holdings. If she could, she would do so and, simultaneously, adopt a more aggressive investment strategy a, ultimately bringing her holdings down to the point of perfect alignment (Corollary 4).

In the setting of Proposition 5, intuitively, the sender should be buying more shares from a welfare perspective. To see this, consider any equilibrium where the equilibrium share price is too high, and so communication is imperfect. Now perturb the equilibrium allocation by taking one share away from a numbered agent and allocating it to the sender. The sender's gain from receiving one more share is p^* (this gain takes into account his private inframarginal benefit from improving communication with the receiver). The numbered agent's marginal loss from giving up one share *given the equilibrium investment strategy* is p^*; and in addition, every numbered agent experiences an inframarginal benefit from a *better investment strategy* which results from improved communication between the sender and the receiver.[15] Therefore, this perturbation results in a welfare gain for the pair involved in the trade, as well as for all other numbered agents. Finally, the receiver also benefits from better communication. Hence, the new allocation is welfare-improving. This observation suggests that the equilibrium price on the capital market fails to reward the positive informational externality that the sender provides by purchasing shares.

6.3. *Shares Are Allocated As Compensation in a Principal-Agent Relationship*

An agent's (sender) compensation contract is designed by a principal (receiver) in order to incentivize the agent to exert effort and to communicate transparently. We now show that, if the space of contracts includes a fixed salary plus shares in the enterprise, the optimal compensation scheme assigns the agent too much equity for her to be willing to communicate truthfully with the principal. Thus, in general the principal will face a trade-off between providing the agent with incentives to exert optimal effort and incentives to communicate transparently.

An agent (indexed by S, for sender) has private information ω about an investment and transmits this information to a principal (indexed by R, for receiver), who takes an action a. Exactly as in the rest of the paper, this game generates utility $U(a, q_i, \omega)$ for player i. We now add a moral hazard component: the agent also exerts effort e at a private cost $c(e)$. Effort e produces output e, which is distributed according to share ownership. If compensated with a salary T and shares $\theta \in [0, 1]$, the agent's equilibrium payoff conditional on effort is:

$$u_S(e; \theta, T) = \underbrace{V_S^*(q_S(\theta), q_R(1-\theta); \Omega)}_{\text{old preferences}} + \underbrace{\theta e - c(e)}_{\text{moral hazard part}} + T.$$

The principal chooses the agent's compensation scheme comprised of a fixed salary T plus a share θ_S in the enterprise. The principal owns the remaining shares $(1-\theta)$. After communicating with

[15] That this benefit is positive is intuitive, and is proved in Lemma A.4 in Antic and Persico (2017).

the agent, the principal chooses an investment strategy a and has the payoff function of the receiver in the previous sections. The principal's equilibrium payoff conditional on effort is then:

$$u_R(e;\theta,T) = \underbrace{V_R^*(q_R(1-\theta), q_S(\theta);\Omega)}_{\text{old preferences}} + \underbrace{(1-\theta)e}_{\text{moral hazard part}} - T.$$

Proposition 6 (Trade-off between moral hazard and imperfect communication) *Fix any regular equilibrium correspondence Ω and assume payoffs satisfy Assumption 6. Then the principal's choice of compensation scheme (salary T plus share θ) is such that the agent receives no fewer shares than the amount that he would receive absent the moral hazard problem.*

Proof. The principal's problem is:

$$\max_{\theta,T} \quad u_R(e^*(\theta);\theta,T)$$

$$\text{subject to} \quad : \quad e^*(\theta) \in \arg\max u_S(e;\theta,T) \text{ (incentive compatibility of effort)}$$
$$u_S(e^*(\theta);\theta,T) \geq 0 \text{ (individual rationality).}$$

By standard arguments, the optimal θ solves:

$$\max_\theta V_R^*(q_R(1-\theta), q_S(\theta);\Omega) + V_S^*(q_S(\theta), q_R(1-\theta);\Omega) + e^*(\theta) - c(e^*(\theta)).$$

Absent the moral hazard problem, the optimal θ maximizes the sum of two single-peaked functions V_R^* and V_S^* that both peak at perfect alignment (refer to Corollary 3), that is, when $q_S(\theta) = q_R(1-\theta)$. The term $e^*(\theta) - c(e^*(\theta))$ reflects the moral hazard problem and is an increasing function of θ. To see this, differentiate:

$$\frac{\partial}{\partial\theta}[e^*(\theta) - c(e^*(\theta))] = \frac{\partial}{\partial\theta}[(1-\theta)e^*(\theta)] + \frac{\partial}{\partial\theta}[\theta e^*(\theta) - c(e^*(\theta))]$$
$$= \frac{\partial}{\partial\theta}[(1-\theta)e^*(\theta)] + e^*(\theta) = (1-\theta)e^{*\prime}(\theta) > 0.$$

Therefore, taking moral hazard into consideration results in a (weakly) larger optimal θ than the perfect-alignment one. ■

The functional forms in Examples 1 and 2 both satisfy Assumption 6, and so in these examples Proposition 6 characterizes the optimal contract *for any regular equilibrium correspondence*. The optimal contract trades off providing the agent with incentives to exert optimal effort with providing incentives to communicate truthfully. As a result the agent is assigned too much equity to efficiently communicate with the principal, but less equity than would be optimal if only effort provision was at issue. In this setting, the agent will feel that the principal's investment strategy is too aggressive and this will lead the agent to misreport his signal.

6.4. *Covert Acquisition of θ*

So far we have assumed that the θ_i's become observable before the cheap talk game. In this section we briefly consider the case where players may not credibly communicate or otherwise signal their θ_i's to other players before the cheap talk game. We call this *covert acquisition of θ*.

In this case, the communication component of inverse demand vanishes for both sender and receiver. To see this, let $\overline{\theta}_{-i}$ denote agent i's expectation about agent $-i$'s θ-holdings, and let players play the cheap-talk equilibrium given $(\overline{\theta}_R, \overline{\theta}_S)$. If the sender covertly deviates from $\overline{\theta}_S$, the receiver cannot adjust her strategy $\{a_k^*\}$. The sender will adjust his strategy $\{\omega_k^*\}$, but a marginal adjustment has a vanishing effect on the sender's payoffs due to the envelope condition. Therefore, the only non-zero marginal effect of a marginal deviation from $\overline{\theta}_S$ is the direct effect on the sender's payoff, which is the non-communication component of inverse demand. The same argument applies to the receiver's covert deviation.

Intuitively, we expect the communication component of inverse demand to promote alignment (this is the economic content of the "opposite sign" property established in Lemma 2). Since the communication component of inverse demand is missing when θ is acquired covertly, we expect the incentives for alignment to be weaker in this case.

6.5. *Trading With Private Information*

In this section we explore the setting where the sender trades shares *after* receiving his private information. The setting is as in Example 2. There are: a receiver who holds exactly θ_C and has $r_C = 1$; a sender who is endowed with t^* shares; and many noise traders with a demand that is infinitely elastic at price p. After learning ω, the sender may acquire shares at a constant price p. The sender's final position after trading is denoted by $t(\omega)$. We assume that the sender's strategy $t(\omega)$ is bounded between ε (a very small but positive number) and $1 - \theta_C$. We assume that the receiver observes $t(\omega)$ before choosing a. The equilibrium construction is inspired by Section 4 in Kartik (2007), but our setting features significant technical challenges requiring a specific analysis. Online Appendix C provides the analysis.

We show that for low values of ω the equilibrium strategy $t(\omega)$ is monotonically increasing and thus fully separating, meaning that the receiver will be able to infer ω from $t(\omega)$ and so select the best possible a for herself. In this fully separating region the equilibrium strategy solves the following differential equation:

$$t'(\omega) = -\frac{\left(\frac{r_S}{\theta_C} t(\omega) - 1\right)\omega}{\left(\frac{r_S}{\theta_C} t(\omega) - 1\right)\omega^2 + \theta_C p} t(\omega).$$

For high values of ω the sender may choose to "max out" on shares at $t(\omega) = 1 - \theta_C$ and then rely on cheap talk to convey additional information to the receiver. By studying the equilibrium strategy as a function of p and r_S, we show that the sender will acquire more shares, and thus will more often have to rely on cheap talk, if the share price is low or if he is less risk-averse.

7. CONCLUSIONS

This paper has analyzed a cheap-talk setting where the conflict of interest between sender and receiver is determined endogenously prior to the cheap talk phase. The conflict of interest is determined by the choice of parameters θ_i, one for each agent.

We have provided sufficient conditions for determining the sign of agent i's inverse demand for θ *without assuming that the most informative equilibrium will necessarily be played in the cheap talk game.* In other words, our conditions characterize every inverse-demand function that arises for "regular" equilibrium selections in the cheap talk game. The sufficient conditions hold for two specific forms of agent i's utility functions that are commonly used in applications. For these two functional forms, we have derived analytically tractable approximations for agent i's demand for θ if the most-informative cheap talk equilibrium is played.

Turning to applications, under the sufficient conditions for signing the inverse demand functions, we have shown that if investors obtain shares in an enterprise by trading in an equity market, then the competitive equilibrium allocation promotes maximal information transmission. However, we have also shown that an inefficiency lurks in the background: a competitive market for shares fails to reward the positive externality that the sender provides to other agents by purchasing his shares. In another application, we have shown that in a principal-agent relationship, the optimal contract will not allocate equity in a way that achieves perfect communication.

We briefly pursued two extensions of the basic model. First, we have shown that when the θ's are acquired covertly rather than overtly, the incentives that lead toward alignment between sender and receiver are weaker. Second, we discussed the scenario in which the θ's are traded *after* the sender has received the information.

From a conceptual perspective, this paper has pointed out that the conflict of interest that is inherent in cheap talk may be reduced, or even completely eliminated, if it is made endogenous.

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APPENDIX A: APPENDIX

A.1. *Comonotonicity*

Let $[0, \overline{\omega}_1, ..., \overline{\omega}_{N-1}, 1]$ be an N-partition equilibrium in the cheap talk game given $(\overline{q}_R, \overline{q}_S)$. Henceforth, we restrict attention to environments where the function g is continuosly differentiable and the Jacobian J of the function $\mathbf{g}$ with respect to the variables $(\omega_1, ..., \omega_{N-1})$ is invertible in a neighborhood of $[\overline{\omega}_1, ..., \overline{\omega}_{N-1}; \overline{q}_R, \overline{q}_S]$. Then the implicit function theorem (Rudin 1976, Theorem 9.28) yields the following result.

Lemma A.1 *Differentiable functions $\mathfrak{w}_k(q_R, q_S)$ exist that solve the system (6) in a neighborhood of $(\overline{q}_R, \overline{q}_S)$, and that satisfy $\mathfrak{w}_k(\overline{q}_R, \overline{q}_S) = \overline{\omega}_k$.*

For example, set $(\overline{q}_R, \overline{q}_S) = (0, -1/20)$ in Example 1. Crawford and Sobel (1982, p. 1443) show that a 3-partition equilibrium exists with partition cutoffs $\left\{\frac{2}{15}, \frac{7}{15}\right\}$. We show in Lemma 1 below that the matrix J is invertible. Therefore, a 3-partition equilibrium correspondence $\Omega_3(\cdot, \cdot)$ exists such that $\Omega_3(0, -1/20) = \left\{\frac{2}{15}, \frac{7}{15}\right\}$ and, as (q_R, q_S) vary in a small neighborhood of $(0, -1/20)$, the image of $\Omega_3(\cdot, \cdot)$ varies smoothly.

The implicit function theorem says that if $\mathfrak{w}_k(q_R, q_S)$ solve the system (6), then for $i = R, S$:

$$\begin{bmatrix} \frac{\partial}{\partial q_i}\mathfrak{w}_1(q_R, q_S) \\ \vdots \\ \frac{\partial}{\partial q_i}\mathfrak{w}_{N-1}(q_R, q_S) \end{bmatrix} = -J^{-1} \cdot \begin{bmatrix} \frac{\partial}{\partial q_i} g(0, \omega_1, \omega_2; q_R, q_S) \\ \vdots \\ \frac{\partial}{\partial q_i} g(\omega_{N-2}, \omega_{N-1}, 1; q_R, q_S) \end{bmatrix}. \tag{13}$$

Therefore, the implicit function theorem yields the following lemma.

Lemma A.2 ***(sufficient conditions for comonotonicity)*** *If the right-hand side of (13) is negative for $i = R$ and positive for $i = S$, then $\Omega_N(\cdot, \cdot)$ is comonotonic at (q_R, q_S).*

The conditions of Lemma A.2 can sometimes be verified straightforwardly. The enabling result is provided by da Fonseca and Petronilho (2001), who show that the tridiagonal matrix:

$$J = \begin{bmatrix} a & -b & & & 0 \\ -b & a & -b & & \\ & -b & \ddots & \ddots & \\ & & \ddots & \ddots & -b \\ 0 & & & -b & a \end{bmatrix},$$

has a symmetric inverse matrix with generic term $k\ell$, for $k \leq \ell$:

$$\left(J^{-1}\right)_{k\ell} = \frac{1}{b} \frac{U_{k-1}(a/2b)\, U_{N-\ell-1}(a/2b)}{U_{N-1}(a/2b)}, \tag{14}$$

where $U_k(x)$ is the k-th order Chebyshev polynomial of the second kind.[16] This result establishes a bridge between comparative statics in parameterized cheap-talk games and the theory of Chebyshev polynomials. Lemma **1** obtains as a by-product.

Proof of Lemma 1

Proof. *Example 1***.** From Crawford and Sobel (1982), the equilibrium partition cutoffs are defined by:

$$-\omega_{k+1} + 2\omega_k - \omega_{k-1} + 4(b_R + \theta_R - b_S - \theta_S) = 0.$$

Therefore,

$$\frac{\partial g(\omega_{k-1}, \omega_k, \omega_{k+1}; q_R, q_S)}{\partial q_i} > 0 \text{ iff } i = R.$$

[16] Expression (14) slightly rewrites da Fonseca and Petronilho (2001), Corollary 4.2, using the property $U_k(-x) = (-1)^k U_k(x)$.

Moreover,

$$J_{CS}=\begin{bmatrix}2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2\end{bmatrix},$$

and since $U_{k-1}(1)=k$ and b is positive, (14) is positive for all (k,ℓ). Therefore J^{-1} is a positive matrix, whence the right-hand side of (13) is negative for $i=R$ and positive for $i=S$, proving that $\Omega_N(\cdot,\cdot)$ is comonotonic at any (q_R,q_S) and for any N.

Example 2. Proposition A.1 in Antic and Persico (2019) shows that the cheap talk equilibrium partition cutoffs are defined by:

$$-\omega_{k+1}+2x\omega_k-\omega_{k-1}=0, \tag{15}$$

where $x=2(q_R/q_S)-1$. Therefore,

$$\frac{\partial g(\omega_{k-1},\omega_k,\omega_{k+1};q_R,q_S)}{\partial q_i}>0 \text{ iff } i=R.$$

Moreover,

$$J_{MV}=\begin{bmatrix}2x & -1 & & & \\ -1 & 2x & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2x\end{bmatrix},$$

so (14) reads:

$$\left(J_{MV}^{-1}\right)_{k\ell}=\frac{U_{k-1}(x)\,U_{N-\ell-1}(x)}{U_{N-1}(x)}. \tag{16}$$

Now, for any $x>-1$ Chebyshev polynomials of the second kind solve the recursive equation:

$$-U_{k+1}(x)+2xU_k(x)-U_{k-1}(x)=0,$$

with initial conditions $U_{-1}(x)=0, U_0(x)=1$. It follows that the solution to (15) must have the form $\omega_k=\alpha U_{k-1}(x)$, and moreover, since $\omega_N=1$ then it must be $\alpha=1/U_{N-1}(x)$. The fact that each equilibrium partition cutoff ω_k is positive implies that $U_{k-1}(x)>0$ for all $k<N$, so the generic term (16) is positive for all (k,ℓ). Therefore J^{-1} is a positive matrix, whence the right-hand side of (13) is negative for $i=R$ and positive for $i=S$, proving that $\Omega_N(\cdot,\cdot)$ is comonotonic at any (q_R,q_S) and for any N.

Example 3. Denote $\alpha=\sqrt{a}$ and $\zeta=2q/\sqrt{Q}$. Then one may verify by substitution that:

$$\sqrt{Q}\frac{1}{\zeta}\left[\zeta\alpha\omega-\frac{1}{2}(\zeta\alpha)^2\right]+\sqrt{Q}\frac{\zeta}{2}=\omega\sqrt{aQ}+(1-a)\,q.$$

The RHS is the payoff in Example 3. Taking q (and hence ζ) as a fixed parameter in the cheap talk game, the LHS is a linear affine transformation of the payoff in Example 2 (the term in brackets, modulo the relabeling $\alpha=\sqrt{a}$). Hence the cheap talk game in Example 3 is strategically equivalent to that in Example 2, so when $\omega\sim U[0,1]$ the equilibrium partition cutoffs in Example 3 are generated by recurrence (15). ■

A.2. *Deferred proofs*

Proof of Proposition 2.

Proof. Let $\widehat{V}_i$ be an informational approximant of V_i^* on the interval $[q_i(\theta_L), q_i(\theta_H)]$. Fix any p. Denote agent i's approximate and actual surpluses by:

$$\widehat{S}_i(\theta) = \widehat{V}_i\left(q_i(\theta);\overline{q}_{-i},\Omega\right)-p\theta$$
$$S_i(\theta) = V_i^*\left(q_i(\theta),\overline{q}_{-i};\Omega_i\right)-p\theta.$$

1. Because i's approximate inverse demand function is decreasing, $\widehat{V}_i\left(q_i(\cdot);\overline{q}_{-i},\Omega\right)$ is strictly concave and so $\widehat{S}_i(\cdot)$ is strictly concave on $[\theta_L,\theta_H]$ and $\widehat{\theta}_i(p)$ is its unique maximum. Pick any point $\theta_i^*(p)$ defined in Definition 8 and suppose by contradiction that an interval $\widetilde{Q}=[c,d)$ exists defined in Definition 5 that is located between $q_i\left(\theta_i^*(p)\right)$ and $q_i\left(\widehat{\theta}_i(p)\right)$, and contains neither point. Denote $\widetilde{I}=q_i^{-1}\left(\widetilde{Q}\right)$. Let $\widehat{y}\in\widetilde{I}$ be such that $\widehat{S}_i(\widehat{y})=S_i(\widehat{y})$ (such a $\widehat{y}$ exists because $\widehat{V}_i$ is an informational approximant and $\widetilde{Q}$ must be entirely contained in the interval $[q_i(\theta_L),q_i(\theta_H)]$.) Because $\theta_i^*(p)$ and $\widehat{y}$ lie on the same side of $\widehat{S}_i$'s maximizer $\widehat{\theta}_i(p)$ and $\widehat{y}$ is closer to it, strict concavity implies:
$$\widehat{S}_i(\theta_i^*(p))<\widehat{S}_i(\widehat{y})=S_i(\widehat{y}).$$
Since $\widehat{S}_i\left(\theta_i^*(p)\right)\geq S_i\left(\theta_i^*(p)\right)$ by definition of $\widehat{S}_i(\cdot)$, the above equation implies:
$$S_i(\theta_i^*(p))<S_i(\widehat{y}).$$
The strict inequality contradicts the optimality of $\theta_i^*(p)$ (Definition 8). This proves that $\widehat{\theta}_i(p)$ and $\theta_i^*(p)$ belong to the same interval or to contiguous intervals.
2. $\widehat{S}_i(\cdot)$ is strictly convex because $\widehat{V}_i\left(q_i(\cdot);\overline{q}_{-i},\Omega\right)$ is strictly convex because i's approximate inverse demand is increasing. Denote by Q_L (resp., Q_H) the left-most (resp., right-most) among the half-open intervals defined in Definition 5 that are entirely contained in $[q_i(\theta_L),q_i(\theta_H)]$. If, by contradiction, $\theta_i^*(p)$ is not informationally close to θ_L or θ_H, then $q_i\left(\theta_i^*(p)\right)$ belongs to an interval that lies between Q_L and Q_H. Denote $I_L=q_i^{-1}(Q_L)$ and $I_H=q_i^{-1}(Q_L)$. Let $\widehat{y}_L\in I_L$ be such that $\widehat{S}_i(\widehat{y}_L)=S_i(\widehat{y}_L)$ (such a $\widehat{y}_L$ exists because $\widehat{V}_i$ is an informational approximant). Define $\widehat{y}_H$ analogously. Since $\widehat{y}_L<\theta_i^*(p)<\widehat{y}_H$ and $\widehat{S}_i(\cdot)$ is strictly convex,
$$\widehat{S}_i(\theta_i^*(p))<\max\left[\widehat{S}_i(\widehat{y}_L),\widehat{S}_i(\widehat{y}_H)\right].$$
By definition of $\widehat{S}_i(\cdot)$, the left hand side is no smaller than $S_i\left(\theta_i^*(p)\right)$. By definition of $\widehat{y}_L$ we have $\widehat{S}_i(\widehat{y}_L)=S_i(\widehat{y}_L)$, and the same is true for $\widehat{y}_H$. Therefore, we get:
$$S_i(\theta_i^*(p))<\max\left[S_i(\widehat{y}_L),S_i(\widehat{y}_H)\right].$$
This contradicts the optimality of $\theta_i^*(p)$.

■

Proof of Proposition 3

Proof. In both examples, the most-informative equilibrium correspondence is continuous by construction.

Part 1. Fix $\overline{q}_S$, and denote $b=q_R(\theta)-\overline{q}_S$. If an equilibrium with $N>1$ partition elements exists, Crawford and Sobel (1982) show that the receiver's payoff is:
$$V_R^*(b,N)=-\frac{1}{12N^2}-\frac{b^2}{3}\left(N^2-1\right).$$
The maximal number of partition elements $\mathcal{N}_{\max}(b)$ that is admissible in equilibrium is identified by conditions on the class of solutions to the recursion to (4). For $b>0$, the condition is that the class of solutions parameterized by ω_1 is such that $\omega_{\mathcal{N}_{\max}(b)}<1$ for some $\omega_1>0$. For $b<0$, that condition is that the class of solutions parameterized by ω_1 is such that $\omega_k>\omega_{k-1}$ for every $k\leq\mathcal{N}_{\max}(b)$. Writing out these conditions and solving yields:
$$\mathcal{N}_{\max}(b)=\left\lceil\frac{1}{2}\sqrt{1+\frac{2}{|b|}}-\frac{1}{2}\right\rceil,$$
where $\lceil z\rceil$ denotes the smallest integer greater than or equal to z, and $\mathcal{N}_{\max}(b)>1$ if and only if $|b|<\frac{1}{4}$. We seek to show that:
$$\widehat{V}_R(b)=\frac{-|b|+b^2}{3},$$
is an informational approximant for $V_R^*(b,\mathcal{N}_{\max}(b))$ for $|b|<\frac{1}{4}$, meaning that for any $N>1$:
$$\min_{b\in(\mathcal{N}_{\max})^{-1}(N)}\widehat{V}_R(b)-V_R^*(b,\mathcal{N}_{\max}(b))$$
$$=\min_{b\in(\mathcal{N}_{\max})^{-1}(N)}\left(\frac{1}{12N^2}+\frac{b^2}{3}N^2-\frac{|b|}{3}\right)=0. \tag{17}$$

The maximand is a continuous function of b with positive second derivative everywhere except $b = 0$. The first-order conditions:

$$2N^2 b - sgn\,[b] = 0,$$

identify the function's two unconstrained minima $b^*(N)$:

$$|b^*(N)| = \frac{1}{2N^2}.$$

The unconstrained minima happen to satisfy $b^*(N) \in (\mathcal{N}_{\max})^{-1}(N)$. To see this, observe that

$$2N \leq \sqrt{1+4N^2} \leq 2N+1,$$

so that indeed:

$$\mathcal{N}_{\max}(b^*(N)) = \left\lceil \frac{1}{2}\sqrt{1+4N^2} - \frac{1}{2} \right\rceil = N.$$

Thus $b^*(N)$ solves the program on the left-hand side of (17), and the minimand evaluated at $b^*(N)$ reads:

$$\left(\frac{1}{12N^2} + \frac{[b^*(N)]^2}{3} N^2 - \frac{|b^*(N)|}{3} \right) = 0.$$

This identity proves that $\widehat{V}_R(b)$ is an informational approximant of $V_R^*(b, \mathcal{N}_{\max}(b))$ for $|b| < \frac{1}{4}$
Now, differentiate the informational approximant to obtain the demand function. Since $q_R(\theta) = b_R + \theta$ we get:

$$\frac{\partial}{\partial \theta} \widehat{V}_R(q_R(\theta) - \overline{q}_S) = \frac{1}{3}\left[2(q_R(\theta) - \overline{q}_S) - sgn\,(q_R(\theta) - \overline{q}_S)\right].$$

This function is everywhere-increasing in θ except at the point $q_R(\theta) = \overline{q}_S$.
Let us now turn to the sender. Crawford and Sobel (1982) compute the sender's expected payoff as follows:

$$V_S^*(b, N) = V_R^*(b, N) - b^2.$$

We seek to show that:

$$\widehat{V}_S(b) = \frac{-|b| - 2b^2}{3},$$

is an informational approximant for $V_S^*(b, \mathcal{N}_{\max}(b))$ for $|b| < \frac{1}{4}$. The result follows immediately because:

$$\begin{aligned} \widehat{V}_S(b) - V_S^*(b, N) &= \frac{-|b| - 2b^2}{3} - V_R^*(b, N) + b^2 \\ &= \widehat{V}_R(b) - V_R^*(b, N), \end{aligned}$$

and so the previous argument applies verbatim.
Now, differentiate the informational approximant to obtain the demand function. Since $q_S(\theta) = b_S + \theta$ it follows that:

$$\frac{\partial}{\partial \theta} \widehat{V}_S(q_S(\theta) - \overline{q}_R) = \frac{1}{3}\left[sgn\,(\overline{q}_R - q_S(\theta)) + 4(\overline{q}_R - q_S(\theta))\right].$$

This function is decreasing in θ.
Part 2. From Antic and Persico (2019), set $E = S, C = R$, and replace θ_C with $r_C \theta_C$ in Lemma B.4. Given $\overline{q}_R$, the sender's (expert's) approximate inverse demand when $\frac{3}{4} \leq \rho < 1$ is:

$$-\frac{\overline{q}_R}{(q_S(\theta) - 4\overline{q}_R)^2}.$$

Because $\frac{3}{4} \leq \rho$ it follows that $q_S(\theta) < 4\overline{q}_R$, so increasing θ decreases approximate inverse demand. The case $\rho > 1$ is treated in Lemma B.4, and demand is shown to be decreasing in Corollary B.5 in Antic and Persico (2019). The case $\rho > 1$ can also be treated using results in Alonso (2009).
Corollary B.3 in Antic and Persico (2019) shows that the receiver's approximate inverse demand is increasing when $\frac{3}{4} \leq \rho < 1$. In the case $\rho > 1$, Lemma B.2 yields:

$$-\frac{1}{2}\frac{\overline{q}_S}{(4q_R(\theta) - \overline{q}_S)^2}.$$

Since $4q_R(\theta) > \overline{q}_S$, the denominator is increasing in $q_R(\theta)$, and as a result so is the entire expression. The case $\rho > 1$ can also be treated using results in Alonso (2009).

Part 3. The expression in part 1 has been shown to be an informational approximant above. The expressions in part 2 are exact in the case $\rho > 1$, and the expression for the case $\rho \in [3/4, 1]$ is shown to be informational approximant. Follows directly from Proposition 2. ■

Proof of Proposition 5

Proof. Perfect communication requires perfect alignment. For the sender to be willing to hold $\theta_S^* = \overline{\theta}_R$, the equilibrium price must not be too high. Proposition 3 part 2, case $\rho > 1$, indicates that it must be:

$$p^* \leq \frac{1}{18\overline{\theta}_R}. \tag{18}$$

Perfect communication implies that the receiver knows ω, and in this case the receiver's strategy is $a^*(\omega) = \omega/\overline{\theta}_R$. Substituting this strategy into numbered agent j's payoff yields:

$$\theta_j^*(p) = \arg\max_\theta \mathbb{E}\left(\theta\frac{\omega^2}{\overline{\theta}_R} - \frac{1}{2}\left(\theta\frac{\omega}{\overline{\theta}_R}\right)^2\right) - p\theta \quad \text{for } i \neq R, S.$$

Noting that $\mathbb{E}\left[\omega^2\right] = 1/3$, the first order conditions for the above problem read:

$$\frac{1}{3}\frac{\left(\overline{\theta}_R - \theta\right)}{\left(\overline{\theta}_R\right)^2} - p = 0.$$

Solving for θ yields:

$$\theta_j^*(p) = \overline{\theta}_R\left(1 - 3p\overline{\theta}_R\right) \text{ for } i \neq R, S.$$

Total demand at equilibrium cannot exceed 1, hence if there are J numbered agents, it must be:

$$\begin{aligned} 1 &\geq \overline{\theta}_R + \theta_S^* + J \cdot \theta_j^*(p) \\ &\geq 2\overline{\theta}_R + J \cdot \theta_j^*\left(\frac{1}{18\overline{\theta}_R}\right) \\ &= 2\overline{\theta}_R + J \cdot \frac{5}{6}\overline{\theta}_R, \end{aligned}$$

where the second inequality follows from condition (18) and the fact that $\theta_j^*(\cdot)$ is a decreasing function. Isolating J yields:

$$J \leq \frac{6}{5}\left(\frac{1}{\overline{\theta}_R} - 2\right).$$

If this condition is violated, there is no price that clears the market and leads to perfect communication. ■

APPENDIX B: ONLINE APPENDIX: PROOFS

Lemma B.1 *Assumptions 1 - 3 are satisfied in Examples 1 - 3. Assumptions 5 and 6 both hold in Example 1. In Example 2, Assumption 6 holds but Assumption 5 need not. In Example 3 Assumptions 4 and 5 hold, but Assumption 6 need not.*

Proof. *Example 1*

$$U(a,q,\omega) = -(a-\omega+q)^2.$$

The function $U(a,q,\omega)$ is strictly concave in q and in a. Complementarity holds because:

$$U_{a\omega}(a,q,\omega) = 2 > 0.$$

The single-crossing property holds because:

$$\begin{aligned} U(a,q,\omega) - U(a',q,\omega) &= (a'-\omega+q)^2 - (a-\omega+q)^2 \\ &= \left[(a'-\omega+q) + (a-\omega+q)\right]\left[(a'-\omega+q) - (a-\omega+q)\right] \\ &= (a'+a-2\omega+2q)(a'-a), \end{aligned}$$

which is increasing in q when $a' > a$.

Finally, Assumptions 5 and 6 both hold in Example 1 because when $U(a,q,\omega) = -(a-\omega+q)^2$ then $U_{aq} = -2$ and $U_a/U_q = 1$.

Example 2

$$U(a,q,\omega) = qa\omega - \frac{1}{2}(qa)^2.$$

The function $U(a,q,\omega)$ is strictly concave in q and in a. Complementarity holds because:

$$U_{a\omega}(a,q,\omega) = q > 0.$$

The single-crossing property holds because:

$$\begin{aligned} U(a,q,\omega) - U(a',q,\omega) &= q(a-a')\omega - \frac{1}{2}q^2\left[a^2 - (a')^2\right] \\ &= (a-a')q\left[\omega - \frac{1}{2}q(a+a')\right]. \end{aligned}$$

The expression in brackets is decreasing in q, and so when $a' > a$ the entire expression is increasing in q.

Finally, Assumption 6 holds because $U_a/U_q = q/a > 0$ for any $q > 0$.

Example 3

$$U(a,q,\omega) = \omega\sqrt{aQ} + (1-a)q.$$

The function $U(a,q,\omega)$ is concave (actually, linear) in q and strictly concave in a. Complementarity holds because:

$$U_{a\omega}(a,q,\omega) = \frac{1}{2}\frac{\sqrt{Q}}{\sqrt{a}} > 0.$$

The single-crossing property holds because:

$$\begin{aligned} U(a,q,\omega) - U(a',q,\omega) &= \left[\omega\sqrt{aQ} + (1-a)q\right] - \left[\omega\sqrt{a'Q} + (1-a')q\right] \\ &= \omega\sqrt{Q}\left(\sqrt{a} - \sqrt{a'}\right) + (a'-a)q \end{aligned}$$

When $a' > a$ this function is increasing in q and so the single-crossing property is satisfied.

Assumption 4 holds because $U_q = (1-a) \geq 0$.

Finally, Assumption 5 holds because $U_{aq} = -1$. ■

Lemma B.2 *If Assumptions 1 and 3 hold, the receiver's best response* a_k^* *defined in (3) is differentiable in* $(\omega_{k-1}, \omega_k, q_R)$, *and increasing in* ω_{k-1} *and in* ω_k.

Proof. The receiver's best response to a partition element (b, c) is the $a^*(b, c, q_R)$ that solves:

$$L(a; b, c, q_R) := \int_b^c U_a(a, q_R, \omega)\ dF(\omega) = 0.$$

Because of Assumption 1 $a^*(b, c, q_R)$ is unique. Since $U(\cdot,\cdot,\cdot)$ is twice continuously differentiable $L(a; b, c, q_R)$ is continuously differentiable in all its arguments, and then the implicit function theorem guarantees that $a^*(\cdot,\cdot,\cdot)$ is differentiable in its arguments. Using the implicit function theorem, and abbreviating $a^*(b, c) = a^*$, we get:

$$0 = \frac{d}{db} L(a^*; b, c, q_R) = -U_a(a^*, q_R, b) f(b) + \frac{\partial a^*}{\partial b} \cdot \int_b^c U_{aa}(a^*, q_R, \omega)\ dF(\omega),$$

whence

$$\frac{\partial a^*}{\partial b} = \frac{U_a(a^*, q_R, b) f(b)}{\int_b^c U_{aa}(a^*, q_R, \omega)\ dF(\omega)}.$$

The denominator is negative because of Assumption 1. The numerator is negative because, whenever $b < c$:

$$\begin{aligned} 0 &= L(a^*; b, c, q_R) \\ &> \left[\min_\omega U_a(a^*, q_R, \omega)\right] \cdot \int_b^c dF(\omega) \\ &= U_a(a^*, q_R, b) \cdot [F(c) - F(b)], \end{aligned}$$

where the first line comes from the definition of $a^*(b, c)$, and the strict inequality comes from Assumption 3 together with $b < c$. The proof in the case $a = b$ is trivial. Hence $\frac{\partial a^*}{\partial b} > 0$.

Following the same steps we get:

$$\frac{\partial a^*}{\partial c} = -\frac{U_a(a^*, q_R, c) f(c)}{\int_b^c U_{aa}(a^*, q_R, \omega)\ dF(\omega)}.$$

The denominator is negative because of Assumption 1. The numerator is positive because, whenever $b < c$:

$$\begin{aligned} 0 &= L(a^*; b, c, q_R) \\ &< \left[\max_\omega U_a(a^*, q_R, \omega)\right] \cdot \int_b^c dF(\omega) \\ &= U_a(a^*, q_R, c) \cdot [F(c) - F(b)]. \end{aligned}$$

Hence $\frac{\partial a^*}{\partial c} > 0$. ■

Lemma B.3 ***(well-definedness of inverse demand, and properties of expected payoffs)*** *Fix* q_{-i}. *Suppose the* N*-partition equilibrium correspondence* $\Omega_N(\cdot, q_{-i})$ *is defined on the interval* (c, d). *Then the function* $V_i^*(y, \overline{q}_{-i}; \Omega_N)$ *is differentiable at every* $y \in (c, d)$ *and the fundamental theorem of calculus applies. If, moreover, the correspondence* $\Omega_N(\cdot, q_{-i})$ *is extended continuously to* $[c, d]$ *then* $V_i^*(\cdot, \overline{q}_{-i}; \Omega_N)$ *is continuous on* $[c, d]$.

Proof. Fix $\overline{q}_S$. For any $y \in (c, d)$ we have:

$$V_R^*(y, \overline{q}_S; \Omega_N) = \sum_{k=1}^N \int_{\mathfrak{w}_{k-1}(y, \overline{q}_S)}^{\mathfrak{w}_k(y, \overline{q}_S)} U(\mathfrak{a}_k(y, \overline{q}_S), y, \omega)\ dF(\omega).$$

The functions $\mathfrak{w}_k(y, \overline{q}_S)$ are differentiable at y (refer to Section 2.7). By definition, $\mathfrak{a}_k(y, \overline{q}_S) = a(\mathfrak{w}_{k-1}(y, \overline{q}_S), \mathfrak{w}_k(y, \overline{q}_S), y)$ where the function $a(\cdot,\cdot,\cdot)$ was defined on page 6. This function was shown to be differentiable in all its arguments in Lemma B.2. Hence $\mathfrak{a}_k(y, q_S)$ is differentiable in y. The function U is differentiable by assumption, hence $U(\mathfrak{a}_k(y, \overline{q}_S), y, \omega)$ is differentiable in y. Therefore, the function $V_R^*(y, \overline{q}_S; \Omega_N)$ is differentiable at every $y \in (c, d)$. Therefore the fundamental theorem of calculus applies, i.e., for any $a, b \in (c, d)$ we have:

$$\begin{aligned} V_R^*(b, \overline{q}_S; \Omega_N) - V_R^*(a, \overline{q}_S; \Omega_N) &= \int_a^b \frac{\partial}{\partial y} V_R^*(y, \overline{q}_S; \Omega_N)\ dy \\ &= \int_a^b D_R(y; \overline{q}_S, \Omega_N)\ dy. \end{aligned}$$

Continuity of $V_R^*(\cdot, \overline{q}_S; \Omega_N)$ at the points c and d holds because if each $\mathfrak{w}_k(y, \overline{q}_S)$ is extended continuously to the closed interval $[c, d]$, then $a(\mathfrak{w}_{k-1}(y, \overline{q}_S), \mathfrak{w}_k(y, \overline{q}_S), y)$ also extends continuously, and since the function U is continuous by assumption, it follows that the function $V_R^*(y, \overline{q}_S; \Omega_N)$ also extends continuously.

The argument for the sender's expected payoff is identical. ■

Proof of Corollary 3

Proof. Suppose the interval (c, d) is in the support of $\Omega\left(\cdot, \overline{q}_{-i}\right)$. Consider all the intervals $[c_\tau, d_\tau)$ defined in in Definition 5 and number them such that $c \in [c_0, d_0)$ and $d \in [c_1, d_1)$. We have:

$$\begin{aligned} V_i^*\left(d, \overline{q}_{-i}; \Omega\right) - V_i^*\left(c, \overline{q}_{-i}; \Omega\right) \quad &= \left[V_i^*\left(d_0, \overline{q}_{-i}; \Omega_0\right) - V_i^*\left(c, \overline{q}_{-i}; \Omega_0\right)\right] \\ &+ \sum_{[c_\tau, d_\tau) \subset (c,d)} \left[V_i^*\left(d_\tau, \overline{q}_{-i}; \Omega_\tau\right) - V_i^*\left(c_\tau, \overline{q}_{-i}; \Omega_\tau\right)\right] \\ &+ \left[V_i^*\left(d, \overline{q}_{-i}; \Omega_1\right) - V_i^*\left(c_1, \overline{q}_{-i}; \Omega_1\right)\right]. \end{aligned}$$

where Ω_τ denotes the particular N-partition correspondence associated to the interval $[d_\tau, d_\tau)$ by the equilibrium correspondence Ω. Note that Ω evaluated at d_τ is set equal to Ω_τ, using continuity of Ω. By Proposition 1 part 4, the sign of every term in brackets is determined by the sign of D_i. Thus, if the interval (c, d) lies inside the region where D_i. is nonnegative (resp., non-positive), then each term in brackets has the same sign and the infinite summation converges. Then $V_i^*\left(\cdot, \overline{q}_{-i}; \Omega\right)$ is non-decreasing (resp., non-increasing). ■

APPENDIX C: ONLINE APPENDIX: TRADING WITH PRIVATE INFORMATION

There is an entrenched, controlling investor who owns θ_C share of the company. There is a continuum of non-controlling investors, who are essentially noise traders that have perfectly elastic demand at some price p. An expert investor starts out with a share endowment denoted by t^*, he observes ω and then he trades. The expert investor can sell all but ε of his endowment, or buy more shares up to a maximum holding of $(1-\theta_C)$, at the price set by the non-controlling investors, p. After observing the expert investor's trade, the two engage in cheap talk communication and the controlling investor chooses a.

C.1. *Model discussion*

In this model trading takes place under asymmetric information; therefore, trading serves as a signal of the expert's information. The assumption of constant price (i.e., noise traders with perfectly elastic demand) simplifies the analysis, but the analysis would not break down if we assumed a price function that is increasing in the expert's net trade. Finally, this model is most comparable to the entrenched shareholder setting discussed in Section 6.2, in that the controlling shareholder's holdings are fixed.

C.2. *Equilibrium characterization*

The equilibrium is characterized by two regions. A low-ω region where the expert investor acquires less than $(1-\theta_C)$; in this region the expert investor's trade fully reveals ω to the controlling investor. And, for some parameter values, a high-ω region where the expert investor acquires exactly $(1-\theta_C)$ shares and then relies on cheap talk to convey information.

The analysis is inspired by the signaling model in Section 4 of Kartik (2007), where in our case, the expert investor signals through trading shares. However, in the present paper the cost/benefit of signaling is endogenous because the share value is a function of the receiver's action. So his results cannot be applied directly.

C.3. *Fully separating region*

In the fully-separating region the controlling investor learns the exact state ω based on the expert investor's net trade after he learns the information. Denote the expert's ex post position after trading by $t(\omega)$. Suppose the expert investor retrades into $t(\widehat{\omega})$. If $t(\widehat{\omega})$ is fully separating, upon observing it the controlling investor correctly infers that the state is $\widehat{\omega}$ and she takes action $a^* = \widehat{\omega}/\theta_C$ (recall that $r_C = 1$). The expert investor's utility is then:

$$-\frac{\sigma_X^2}{2r_E}\left(r_E t(\widehat{\omega})\frac{\widehat{\omega}}{\theta_C} - \omega\right)^2 - \left(t(\widehat{\omega}) - t^*\right)p, \tag{19}$$

where p denotes the price of shares. This is the utility after the expert (and now informed) investor has re-traded his position using a separating trading strategy $t(\cdot)$. The term $(t(\widehat{\omega}) - t^*)\,p$ is the amount of money paid to achieve the new position $t(\widehat{\omega})$.

We want to show that the trading function described by the following differential equation:

$$t'(\omega) = -\frac{\left(\frac{r_E}{\theta_C}t(\omega) - 1\right)\omega}{\left(\frac{r_E}{\theta_C}t(\omega) - 1\right)\omega^2 + \frac{\theta_C}{\sigma_X^2}p}\,t(\omega) \ \text{ for } \omega > 0. \tag{20}$$

represents the separating region of an equilibrium.

Lemma C.1 ***(Features of the solution to the differential equation)*** *Let $t(\cdot)$ solve the differential equation (20) with initial condition $t(0) = \varepsilon$, where $0 < \varepsilon < \theta_C/r_E$. Then: $t'(0) = 0$; $t'(\omega) > 0$ for all $\omega > 0$; and the function $t(\omega)$ achieves θ_C/r_E asymptotically, but not for finite ω.*

Proof. Denote:

$$f(\omega) = \frac{r_E}{\theta_C}t(\omega) \quad \text{and} \quad k_0 = \frac{\theta_C}{\sigma_X^2}p,$$

so that equation (20) rewrites as:

$$f'(\omega) = \frac{(1 - f(\omega))\,\omega}{(f(\omega) - 1)\,\omega^2 + k_0} f(\omega). \tag{21}$$

The following equation is an implicit solution of equation (21):

$$2f(\omega) + 2\log(1 - f(\omega)) + \frac{\omega^2}{k_0} f(\omega)^2 + c_0 = 0, \tag{22}$$

where k_0 is fixed by the parameters of the problem, and $c_0 = -2\left[\log\left(1 - \frac{r_E}{\theta_C}\varepsilon\right) + \frac{r_E}{\theta_C}\varepsilon\right] > 0$ is chosen to satisfy our initial conditon $t(0) = \varepsilon$. This can be verified by differentiating (22) with respect to ω.
We now show that $f(\cdot)$ is strictly increasing. Fom (21), this is the case if:

$$-\frac{k_0}{\omega^2} < f(\omega) - 1 < 0. \tag{23}$$

Let's first focus on the right-hand inequality of (23). Since $t(0) = \varepsilon < \theta_C/r_E$, at $\omega = 0$ we have $f(\omega) - 1 < 0$. Furthermore, for any finite ω it must be $f(\omega) - 1 < 0$; indeed, at any $\widehat{\omega} < \infty$ such that $f(\widehat{\omega}) = 1$, equation (22) implies:

$$2 + 2\log(0) + \frac{(\widehat{\omega})^2}{k_0} + c_0 = 0,$$

which is a contradiction since the LHS is infinite. Let's now focus on the left-hand inequality of (23). This inequality holds at $\omega = 0$ because $f(\omega) - 1 > -\frac{k_0}{\omega^2} = -\infty$. Furthermore, this inequality holds for any finite ω; indeed, by contradiction, let $\widetilde{\omega}$ be the smallest value of ω at which the left-hand inequality of (23) fails:

$$(f(\widetilde{\omega}) - 1)\,\widetilde{\omega}^2 + k_0 = 0.$$

Then equation (22) reads:

$$2\left(1 - \frac{k_0}{\widetilde{\omega}^2}\right) + 2\log\left(\frac{k_0}{\widetilde{\omega}^2}\right) + \frac{\widetilde{\omega}^2}{k_0}\left(1 - \frac{k_0}{\widetilde{\omega}^2}\right)^2 + c_0 = 0.$$

Denoting $x = \frac{k_0}{\widetilde{\omega}^2} > 0$,this equation rewrites as:

$$2(1 - x) + 2\log(x) + \frac{1}{x}(1 - x)^2 + c_0 = 0.$$

The above function is decreasing in x (its derivative equals $-(1 - x)^2/x^2$), and it equals $c_0 > 0$ at $x = 1$. Therefore, the function is positive for all $x \in (0, 1]$. But we know that $x = 1 - f(\widetilde{\omega}) < 1$ because by definition of $\widetilde{\omega}$, equation (23) holds for all $\omega \in (0, \widetilde{\omega})$, whence $f(\widetilde{\omega}) > 0$. Therefore there is no $\widetilde{\omega}$ that solves equation (22). This establishes that $f(\cdot)$ is strictly increasing.
We now show that $\lim_{\omega\to\infty} f(\omega) = 1$, which proves that $t(\omega)$ achieves θ_C/r_E asymptotically. Sending ω to infinity in equation (22) causes the term involving ω^2 to approach $+\infty$ which, recalling that $f(\omega) > 0$, requires that the log term converges to $-\infty$. Thus $\lim_{\omega\to\infty} 1 - f(\omega) = 0$. ■

Corollary 5 *Let $t(\cdot)$ solve the differential equation (20) with initial condition $t(0) = \varepsilon$. Then $t(1) \le (\theta_C/r_E)$.*
As $t(\omega) \uparrow \frac{\theta_C}{r_E}$ the function $t'(\omega)$ goes to zero hence $t(\omega)$ becomes very flat. This means that slight variations in shares convey a lot of information about the expert's signal. If $\frac{\theta_C}{r_E} < (1 - \theta_C)$ this "efficient signaling" takes place at ownership levels below $1 - \theta_C$. In this case the signaling equilibrium can be perfectly separating (and revealing) for all ω. For the equilibrium to involve some pooling it must be that

$$\frac{\theta_C}{r_E} > (1 - \theta_C).$$

Corollary 6 *Since the $t(\cdot)$ that solves the differential equation (20) with initial condition $t(0) = \varepsilon$ is positive and nondecreasing in ω, the product $t(\omega)\,\omega$ is strictly increasing for all ω.*

Lemma C.2 ***(Best response property)*** *Let $t(\cdot)$ solve the differential equation (20) with initial condition $t(0) = \varepsilon$. Suppose the controlling investor expects type ω to play $t(\omega)$. Then any expert type ω prefers $t(\omega)$ to $t(\widehat{\omega})$ for any $\widehat{\omega} > 0$.*

Proof. Suppose the expert investor who knows the state ω acquires $t(\widehat{\omega})$; then his utility is:

$$u(\widehat{\omega};\omega) = -\frac{\sigma_X^2}{2r_E}\left(r_E t(\widehat{\omega})\frac{\widehat{\omega}}{\theta_C} - \omega\right)^2 - (t(\widehat{\omega}) - t^*)\,p. \tag{24}$$

Differentiating this utility function with respect to $\widehat{\omega}$ yields the following first-order condition:

$$u_1(\widehat{\omega};\omega) = -\left[\frac{\sigma_X^2}{\theta_C}\left(r_E t(\widehat{\omega})\frac{\widehat{\omega}}{\theta_C} - \omega\right)\left(\frac{\partial}{\partial\widehat{\omega}}t(\widehat{\omega})\,\widehat{\omega}\right)\right] - t'(\widehat{\omega})\,p = 0. \tag{25}$$

For $\widehat{\omega} = \omega$ to be a maximum, this first-order condition must hold at $\widehat{\omega} = \omega$. We now show that the condition holds if $t(\cdot)$ solves the differential equation (20). To see this, set $\widehat{\omega} = \omega$ and rewrite the first-order condition as follows:

$$\begin{aligned} u_1(\omega;\omega) &= -\left[\frac{\sigma_X^2}{\theta_C}\left(r_E t(\omega)\frac{\omega}{\theta_C} - \omega\right)\left(\frac{\partial}{\partial\omega}t(\omega)\,\omega\right)\right] - t'(\omega)\,p \\ &= -\left(\frac{r_E}{\theta_C}t(\omega)\,\omega - \omega\right)\left[t'(\omega)\,\omega + t(\omega)\right] - \frac{\theta_C}{\sigma_X^2}t'(\omega)\,p \\ &= -\left[\left(\frac{r_E}{\theta_C}t(\omega)\,\omega - \omega\right)\omega + \frac{\theta_C}{\sigma_X^2}p\right]t'(\omega) - \left(\frac{r_E}{\theta_C}t(\omega)\,\omega - \omega\right)t(\omega) = 0 \end{aligned} \tag{26}$$

which holds if $t(\cdot)$ solves the differential equation (20). In other words, $t(\omega)$ solves the differential equation (20) for $\omega \in [0,\infty)$ if and only if

$$u_1(\omega;\omega) = 0 \text{ for all } \omega \in [0,\infty). \tag{27}$$

Now suppose $t(\omega)$ solves the differential equation (20), and let's check the second-order conditions for a maximum. These conditions require that the expert's utility function (24) be single-peaked. Denoting this function by $u(\widehat{\omega},\omega)$, the first order conditions (25) can be written as:

$$u_1(\widehat{\omega},\omega)\big|_{\widehat{\omega}=\omega} = 0.$$

Using (25) we can write:

$$\begin{aligned} u_1(\widehat{\omega},\omega) &= u_1(\widehat{\omega},\widehat{\omega}) + \frac{\sigma_X^2}{\theta_C}(\omega - \widehat{\omega})\left(\frac{\partial}{\partial\widehat{\omega}}t(\widehat{\omega})\,\widehat{\omega}\right) \\ &= \frac{\sigma_X^2}{\theta_C}(\omega - \widehat{\omega})\left(\frac{\partial}{\partial\widehat{\omega}}t(\widehat{\omega})\,\widehat{\omega}\right), \end{aligned}$$

where the second equality holds because of (27). Since by Corollary 6 $\frac{\partial}{\partial\widehat{\omega}}t(\widehat{\omega})\,\widehat{\omega} > 0$, this expression shows that the first derivative is is positive for $\widehat{\omega} < \omega$ and negative otherwise. Hence $u(\widehat{\omega},\omega)$ is indeed single-peaked as a function of $\widehat{\omega}$ and it attains a maximum at $\widehat{\omega} = \omega$. ■

Lemma C.3 ***(Initial condition of equilibrium trading strategy)*** *There are potentially many signalling equilibria, each associated with a different value of $t(0)$. The controling investor is indifferent among them all. The one that is most preferred by the expert investor is the one with the smallest $t(0) = \varepsilon$.*

Proof. The family of strategies identified by differential equation (20) indexed by its initial conditions $t(0)$, give rise to a family of signaling equilibria. Higher initial conditions result in a pointwise-higher strategies $t(\cdot)$. Irrespective of the initial condition $t(0)$, all strategies identified by differential equation (20) induce the same fully informed controlling agent's action. Therefore the expert investor's preferred equilibrium within this family of signaling equilibria is the one where his holdings $t(\omega)$ are closest to the expert investor's *preferred holdings conditional on the controlling agent being fully informed*. We now show that the expert investor's preferred holdings conditional on the controlling agent being fully informed are lower than any equilibrium signaling strategy. To see this, notice that if the controlling investor knows ω the expert's problem is:

$$\max_t -\frac{\sigma_X^2}{2r_E}\left(r_E t\frac{\omega}{\theta_C} - \omega\right)^2 - (t - t^*)\,p.$$

The first order conditions read:

$$-\frac{\sigma_X^2}{\theta_C}\left(\frac{r_E}{\theta_C}t^{FI} - 1\right)\omega^2 - p = 0,$$

where t^{FI} denotes the full-information optimal holdings for the expert investor. A slight rearrangement yields:

$$\left(\frac{r_E}{\theta_C}t^{FI} - 1\right)\omega^2 + \frac{\theta_C}{\sigma_X^2}p = 0.$$

From (23), which must hold in equilibrium, we have:

$$\left(\frac{r_E}{\theta_C} t(\omega) - 1\right)\omega^2 + \frac{\theta_C}{\sigma_X^2} p > 0,$$

which in comparison to the previous equation verifies that for any ω, t^{FI} is smaller than the equilibrium trading level $t(\omega)$. Therefore the equilibrium that is most preferred by the expert investor is the one with the smallest $t(0) = \varepsilon$. ■

Proposition C.1 ***(Characterization of fully separating equilibrium)*** *Let $t(\cdot)$ solve the differential equation (20) with initial condition $t(0) \to 0$.*

1. *Suppose $t(1) \leq 1 - \theta_C$. Then there is a fully separating equilibrium where all expert types in $[0,1]$ trade according to $t(\cdot)$. The amount of shares acquired after retrading cannot exceed θ_C / r_E.*

C.4. *Comparative statics for the fully separating equilibrium strategy*

Denote by $t\left(\omega; r_E, \theta_C, \sigma_X^2, p\right)$ solve the differential equation (20) with initial condition $t(0) = \varepsilon$. We want to see whether, as the parameters $r_E, \theta_C, \sigma_X^2, p$ change, whether the effect on $t(\cdot)$ is monotonic. To do this, we will compare two solutions $t(\omega; p)$ and $t(\omega; p')$ with $p' > p$. If whenever $t(\omega; p) = t(\omega; p')$ we have $t'(\omega; p) > t'(\omega; p')$ then the two solutions never cross and we have our monotonicity result.

Lemma C.4 *$p' > p$ implies $t(\omega; p) > t(\omega; p')$.*

Proof. Suppose $t(\omega; p) = t(\omega; p') = t$. Then from differential equation (20) we have:

$$t'(\omega; p) = -\frac{\left(\frac{r_E}{\theta_C} t - 1\right)\omega}{\left(\frac{r_E}{\theta_C} t - 1\right)\omega^2 + \frac{\theta_C}{\sigma_X^2} p} t > -\frac{\left(\frac{r_E}{\theta_C} t - 1\right)\omega}{\left(\frac{r_E}{\theta_C} t - 1\right)\omega^2 + \frac{\theta_C}{\sigma_X^2} p'} t = t'\left(\omega; p'\right).$$

It follows that $t(\omega; p) > t(\omega; p')$. ■

Lemma C.5 *$r'_E > r_E$ implies $t(\omega; r_E) > t\left(\omega; r'_E\right)$.*

Proof. Since $r'_E > r_E$, the following inequalities hold for any $t > 0$:

$$\underbrace{\left(\frac{r_E}{\theta_C} t - 1\right)}_{-}\underbrace{\left[\left(\frac{r'_E}{\theta_C} t - 1\right)\omega^2 + \frac{\theta_C}{\sigma_X^2} p\right]}_{+} < \underbrace{\left(\frac{r'_E}{\theta_C} t - 1\right)}_{-}\underbrace{\left[\left(\frac{r_E}{\theta_C} t - 1\right)\omega^2 + \frac{\theta_C}{\sigma_X^2} p\right]}_{+}$$

$$\frac{\left(\frac{r_E}{\theta_C} t - 1\right)\omega}{\left(\frac{r_E}{\theta_C} t - 1\right)\omega^2 + \frac{\theta_C}{\sigma_X^2} p} < \frac{\left(\frac{r'_E}{\theta_C} t - 1\right)\omega}{\left(\frac{r'_E}{\theta_C} t - 1\right)\omega^2 + \frac{\theta_C}{\sigma_X^2} p}$$

$$-\frac{\left(\frac{r_E}{\theta_C} t - 1\right)\omega}{\left(\frac{r_E}{\theta_C} t - 1\right)\omega^2 + \frac{\theta_C}{\sigma_X^2} p} t > -\frac{\left(\frac{r'_E}{\theta_C} t - 1\right)\omega}{\left(\frac{r'_E}{\theta_C} t - 1\right)\omega^2 + \frac{\theta_C}{\sigma_X^2} p} t$$

If $t(\omega; r_E) = t\left(\omega; r'_E\right) = t$ then the above inequality reads:

$$t'(\omega; r_E) > t'\left(\omega; r'_E\right).$$

So $r'_E > r_E$ implies that if $t(\omega; r_E) = t\left(\omega; r'_E\right) = t$ then $t'(\omega; r_E) > t'\left(\omega; r'_E\right)$. It follows that $t(\omega; r_E) > t\left(\omega; r'_E\right)$. ■

Lemma C.6 *$\theta'_C > \theta_C$ implies $t(\omega; \theta_C) > t\left(\omega; \theta'_C\right)$ if and only if $t(\omega) < \frac{\theta_C}{2 r_E}$.*

Proof. Suppose $t(\omega; p) = t(\omega; p') = t$. Then in light of differential equation (20) the following inequalities are equivalent:

$$
\begin{aligned}
t'(\omega;\theta_C) &> t'(\omega;\theta'_C) \\
-\frac{\left(\frac{r_E}{\theta_C}t-1\right)\omega}{\left(\frac{r_E}{\theta_C}t-1\right)\omega^2+\frac{\theta_C}{\sigma_X^2}p}t &> -\frac{\left(\frac{r_E}{\theta'_C}t-1\right)\omega}{\left(\frac{r_E}{\theta'_C}t-1\right)\omega^2+\frac{\theta'_C}{\sigma_X^2}p}t \\
\frac{\left(\frac{r_E}{\theta_C}t-1\right)}{\left(\frac{r_E}{\theta_C}t-1\right)\omega^2+\frac{\theta_C}{\sigma_X^2}p} &< \frac{\left(\frac{r_E}{\theta'_C}t-1\right)}{\left(\frac{r_E}{\theta'_C}t-1\right)\omega^2+\frac{\theta'_C}{\sigma_X^2}p} \\
\left(\frac{r_E}{\theta_C}t-1\right)\left[\left(\frac{r_E}{\theta'_C}t-1\right)\omega^2+\frac{\theta'_C}{\sigma_X^2}p\right] &< \left(\frac{r_E}{\theta'_C}t-1\right)\left[\left(\frac{r_E}{\theta_C}t-1\right)\omega^2+\frac{\theta_C}{\sigma_X^2}p\right] \\
\underbrace{\left(\frac{r_E}{\theta_C}t-1\right)}_{-}\theta'_C &< \underbrace{\left(\frac{r_E}{\theta'_C}t-1\right)}_{-}\theta_C \\
\frac{1}{\theta_C}\underbrace{\left(\frac{r_E}{\theta_C}t-1\right)}_{-} &< \frac{1}{\theta'_C}\underbrace{\left(\frac{r_E}{\theta'_C}t-1\right)}_{-}
\end{aligned}
$$

The inequality holds if:

$$
\begin{aligned}
0 &< \frac{\partial}{\partial\theta}\left(\frac{r_E}{\theta^2}t-\frac{1}{\theta}\right) \\
&= -\frac{2}{\theta^3}r_Et+\frac{1}{\theta^2},
\end{aligned}
$$

which is equivalent to $t<\frac{\theta}{2r_E}$. ■

C.5. *Pooling region*

The pooling region has the form $[\overline{\omega},1]$. On that region all types purchase the maximum amount of available shares (in our case, $1-\theta_C$) and so no signal is conveyed by share position. Instead, communication takes place via cheap talk. Type $\overline{\omega}$ has to be indifferent between buying $t(\overline{\omega})$ and being perfectly revealed, or buying $1-\theta_C$ and pooling with the lowest interval in the cheap talk equilibrium partition. Given an equilibrium characterized by a threshold $\overline{\omega}$, a full-revelation trading function $t(\cdot)$, and a partition $\{\overline{\omega},\omega_1,\omega_2,...\}$ of the pooling region, type ω's payoff from buying $t(x)$ for any $x<\overline{\omega}$ is, by (24):

$$\underline{U}(x;\omega)=-\frac{\sigma_X^2}{2r_E}\left(r_Et(x)\frac{x}{\theta_C}-\omega\right)^2-(t(x)-t^*)p+\frac{\sigma_X^2}{2r_E}\omega^2 \text{ for any } x<\overline{\omega},$$

whereas type ω's payoff from pooling at $1-\theta_C$ and inducing any action $y\theta_C>\overline{\omega}\theta_C$ is:

$$\overline{U}(y;\omega)=-\frac{\sigma_X^2}{2r_E}\left(r_E(1-\theta_C)\frac{y}{\theta_C}-\omega\right)^2-(1-\theta_C-t^*)p+\frac{\sigma_X^2}{2r_E}\omega^2 \text{ for any } y>\overline{\omega}.$$

Lemma C.7 ***(Higher types are more inclined to pool)*** *Fix $x<\overline{\omega}<y$. Then $\overline{U}(y;\omega)-\underline{U}(x;\omega)$ is increasing in ω.*

Proof.

$$
\begin{aligned}
&\overline{U}(y;\omega)-\underline{U}(x;\omega) \\
=\ & -\frac{\sigma_X^2}{2r_E}\left(r_E(1-\theta_C)\frac{y}{\theta_C}-\omega\right)^2-(1-\theta_C-t^*)p+\frac{\sigma_X^2}{2r_E}\left(r_Et(x)\frac{x}{\theta_C}-\omega\right)^2+(t(x)-t^*)p \\
=\ & -\frac{\sigma_X^2}{2r_E}\left(r_E(1-\theta_C)\frac{y}{\theta_C}-\omega\right)^2+\frac{\sigma_X^2}{2r_E}\left(r_Et(x)\frac{x}{\theta_C}-\omega\right)^2+(t(x)-1+\theta_C)p
\end{aligned}
$$

The derivative with respect to ω is:

$$\begin{aligned} & \frac{\sigma_X^2}{r_E}\left(r_E\left(1-\theta_C\right)\frac{y}{\theta_C}-\omega\right)-\frac{\sigma_X^2}{r_E}\left(r_E t\left(x\right)\frac{x}{\theta_C}-\omega\right) \\ = & \frac{\sigma_X^2}{r_E}\left(r_E\left(1-\theta_C\right)\frac{y}{\theta_C}-r_E t\left(x\right)\frac{x}{\theta_C}\right), \end{aligned}$$

which is positive because $(1-\theta_C) > t(x)$ and $y > x$. ■

From this, we have that the equilibrium is characterized by a cutoff type, which is indifferent between separating and pooling, i.e., a type $\overline{\omega}$ which solves:

$$\begin{aligned} 0 &= -\frac{\sigma_X^2}{2r_E}\left(r_E t\left(\overline{\omega}\right)\frac{\overline{\omega}}{\theta_C}-\overline{\omega}\right)^2-\left(t\left(x\right)-t^*\right)p+\frac{\sigma_X^2}{2r_E}\left(r_E\left(1-\theta_C\right)\frac{y_1\left(\overline{\omega}\right)}{\theta_C}-\overline{\omega}\right)^2+\left(1-\theta_C-t^*\right)p \\ &= -\frac{\sigma_X^2}{2r_E}\left[\left(r_E t\left(\overline{\omega}\right)\frac{\overline{\omega}}{\theta_C}-\overline{\omega}\right)^2-\left(r_E\left(1-\theta_C\right)\frac{y_1\left(\overline{\omega}\right)}{\theta_C}-\overline{\omega}\right)^2\right]+\left(1-\theta_C-t\left(\overline{\omega}\right)\right)p. \end{aligned} \tag{28}$$