Testing Experts

Expository Talk and Recent Developments

– Overview of the problem
– Comparative testing
– Learning

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Experts: *How do we know that they know?*

- Unknown probability distribution $P$ generates observations
  - e.g. weather conditions, stock prices, or GDP levels,

- The true distribution $P$ is *unrestricted*

- In each period, an expert makes a probabilistic forecast that he claims is based on his knowledge of $P$

- *Can this claim be tested?*
Subjective beliefs: *How do we know that we know?*

Bayesian methodology says, roughly, that

\[
\text{"rationality} = \text{subjective beliefs} + \text{expected utility}\]

Bayesian methodology imposes no restrictions on how beliefs are formed.

But a smart Bayesian should be interested in testing his beliefs.

Is such testing possible?
The problem

Telling science from pseudo-science

Pseudo-sciences:
- Prophesies of Nostradamus
- Marxism-Leninism
- Wall Street technical analysis
- Psychoanalysis
- Management consulting
- Blue Ocean Strategies

Sciences:
- Double Helix theory
- Quantum mechanics
- Macroeconomic forecasting
- Behavioral Economics
- Equilibrium refinements
- Repeated games theory
Dawid  
*JASA '82*

**Foster & Vohra**  
*Biometrika '98*

**Sandroni**  
*Int. J. of Game Theory '03*

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**Calibration Literature:**
- Kalai, Lehrer, Smorodinsky, *MOR '99*
- Fudenberg & Levine, *GEB '99*
- Lehrer, *Ecta '01*
- Sandroni, Smorodinsky, Vohra, *MOR '03*

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**Possibility results:**
- Dekel & Feinberg, *RES '06*
- Olszewski & Sandroni, *'06*

**Impossibility results:**
- Olszewski & Sandroni, *'06*
- Shmaya, *'07*

**Multiple experts:**
- Al-Najjar & Weinstein, *'06*
- Feinberg & Stuart, *'06*

**Complexity:**
- Fortnow & Vohra, *'06*

**Restricting distributions:**
- *This paper*
1 Introduction

2 Basic model
   • Formal model
   • Properties and examples
   • Impossibility

3 Comparative testing

4 Learnable-predictive forecasts

5 Theorems

6 On truth..
A is a fixed finite set of outcomes

Assume $A = \{H, T\}$.

$\Delta(\cdot)$ is the set of probability measures on a space.

The horizon $n$ can be either $n < \infty$ or $n = \infty$.

Each period $t = 1, \ldots, n$ an expert submits a forecast

$$\alpha(t) \in \Delta(A)$$
A forecasting strategy is \( f \equiv \{f_t\}_{t=1}^n \) where

\[
f^t : H^{t-1} \rightarrow \Delta(A)
\]

- Note that \( f^t \) is pure strategy; it outputs a vector in \( \Delta(A) \)
- Identify each \( f \) with a probability measure \( Q \)
- \( F^n \) is the set of all forecasting strategies
- \( \Delta(F^n) \) is the set of expert’s mixed strategies
Formal model: *Tests*

- A *test* is any function

\[ T : A^n \times \Delta(A^n) \rightarrow \{0, 1\} \]

"\(T(a^n, P) = 1\)" means: \(P\) passes on the path \(a^n\)

- The probability that an expert passes when he reports a (pure) forecast \(Q\) when the truth is \(P\) is:

\[ P\{a^n : T(a^n, Q) = 1\} \]

- If this expert randomizes with a distribution \(\varphi\), his expected probability of passing is:

\[ z(P, \varphi) = \int_{\Delta(A^n)} P\{T(a^n, Q) = 1\} \, d\varphi(Q). \]
An expert’s payoff is his expected probability of passing

\[ z(P, \varphi) \]

\( T \) passes the truth with probability \( 1 - \epsilon \) if for any \( P \):

\[ z(P, P) > 1 - \epsilon. \]

\( T \) can be ignorantly passed with probability \( 1 - \epsilon \) if there is a mixed strategy \( \varphi \) such that for any \( P \):

\[ z(P, \varphi) > 1 - \epsilon. \]
Example: Calibration

$n \leq \infty$ and $\epsilon > 0$

**Naive Calibration:** Compare

- The average of the expert’s forecasts along the path
- The average of the path

The expert passes iff the two are within $\epsilon$ of each other.

**Sophisticated Calibration:** Fix a grid $0 = a_1 < \cdots < a_K = 1$.

The expert passes iff for each $k = 1, \ldots K - 1$ the average of the path in all periods in which the expert’s forecasts fall within an interval $[a_k, a_{k-1})$ also falls within the same interval.
Other examples of special tests

The results will cover:

**Continuous-valued tests:** Tests that take values in [0,1]

**Random tests:** A test is picked at random with known probability distribution

**Payoff threshold tests:** The tester takes an action each period maximizing expected payoff given the expert’s forecast; the expert passes if the tester’s average payoff exceeds some threshold

**Likelihood threshold tests:** The tester passes the expert if the likelihood of the path exceeds some threshold
Theorem (Sandroni, 03, IJGT):

If $T$ passes the truth with probability $1 - \epsilon$, then there is a mixed strategy $\varphi$ that ignorantly passes $T$ with probability $1 - \epsilon$.

- The only randomness stems from the randomization $\varphi$
- $\varphi$ passes on any sequence of outcomes $a^n$ with probability $1 - \epsilon$
A mathematical theory of ‘bullshit’

- There is no way to tell apart:
  - an expert who knows the truth $P$
  - from
  - a strategic “expert” who forecasts based on no knowledge whatsoever

- The strategic “expert” advice at time $n + 1$ is completely worthless

- This “expert” is not learning anything, but “bullshitting” by cleverly adjusting his predictions to pass the test
Proof of impossibility theorem

We want to show that:

$$\max_{\varphi \in \Delta(\Delta(A^n))} \min_{P \in \Delta(A^n)} z(P, \varphi) \geq 1 - \epsilon.$$ 

The minimax theorem says

$$\max_{\varphi \in \Delta(\Delta(A^n))} \min_{P \in \Delta(A^n)} z(P, \varphi) = \min_{P \in \Delta(A^n)} \max_{\varphi \in \Delta(\Delta(A^n))} z(P, \varphi).$$

But given any true distribution $P$, if the expert knew $P$, he can always report $P$ and get at least $1 - \epsilon$ so:

$$\min_{P \in \Delta(A^n)} \max_{\varphi \in \Delta(\Delta(A^n))} z(P, \varphi) \geq \min_{P \in \Delta(A^n)} z(P, P) > 1 - \epsilon.$$
This looks like magic!

This is a “hide-and-seek” game with a continuum of locations:

- the hider (nature) hides a distribution $P$ anywhere in $\Delta(A^n)$
- the seeker (expert) has to find $P$

How can the seeker beat the hider?
Key observation: *Nature has no meaningful randomizations*

The Minimax Theorem really says:

$$\max_{\varphi \in \Delta(\Delta(A^n))} \min_{\mu \in \Delta(\Delta(A^n))} z(\mu, \varphi) = \min_{\mu \in \Delta(\Delta(A^n))} \max_{\varphi \in \Delta(\Delta(A^n))} z(\mu, \varphi).$$

But any of Nature’s mixed strategy $\mu$ is payoff-equivalent to pure strategy $P_{\mu}$ obtained by reduction of compound lotteries:

$$z(\mu, \varphi) = z(P_{\mu}, \varphi), \quad \forall \mu, \varphi \in \Delta(\Delta(A^n)).$$
The crucial philosophical point is that in many areas of knowledge

“We never get to see Nature’s randomizations; only the path.”

This suggests ways to get around impossibility:

- Nature’s strategies are not reducible via compound lotteries:
  
  *Multiple experts, repeated trials*..

- The conditions of the Minimax theorem fail:
  
  *Nature’s strategies fail compactness or convexity*
Comparative Testing of Experts

Nabil Al-Najjar (MEDS)
Jonathan Weinstein (MEDS)
Now assume that there are two experts: 0 and 1.

An *n-period comparative test* is any (measurable) function

\[ T^n : A^n \times \Delta(A^n) \times \Delta(A^n) \rightarrow \{0, 0.5, 1\} \]

such that for every \( f, f' \in F^n \) and \( a^n \),

\[ T^n(a^n, f, f') = 1 - T^n(a^n, f', f). \]

\( i = T^n(h^n) \) means that the test picks expert \( i \) after observing the history of forecasts and Nature’s realizations.

If the test returns “0.5” then we interpret this to mean “inconclusive”.
Comparative test

Set $L_0(h^0) = 1$

For $t > 0$

$$L_t(h^t) = \frac{f_1^t(h^{t-1})(a(t))}{f_0^t(h^{t-1})(a(t))} L_{t-1}(h^{t-1})$$

Expert 1 is chosen if $L_n(h^n) > 1$

Expert 0 is chosen if $L_n(h^n) < 1$

The test returns “0.5” if $L_n(h^n) = 1$
Main possibility theorem

For every $\epsilon > 0$, there is an integer $K$ such that for all integers $n$, distributions $P$, and mixed forecasting strategies $\varphi_0$, $\varphi_1$ with at least one informed expert, there is $P$-probability at least $1 - \epsilon$ that either

(a) $T^n$ picks an informed expert; or

(b) The two experts’ forecasts are $\epsilon$-close in all but $K$ periods.

$K$ depends on $\epsilon$ but not on $n$, $P$ or the strategies.
**Intuition for the test**

- Identical to: assigning 50-50 prior to each expert announcing true probabilities, Bayesian updating.

- So, mathematically equivalent to updating concerning a player’s type in a reputation game.

- Realized states are like signals generated by actions of a player with two types, the probabilities announced are like the signal distribution conditional on each type.

- Remark: this is NOT a Bayesian game because the tester does not put a prior on (a) the true distribution and (b) the experts’ strategies.
Main result

Sketch of the proof 1

Assume Expert 0 is informed and that he reports the truth.

The stochastic process $\{L_t\}$ is a supermartingale under the distribution induced by the strategy of Expert 0.

As in Fudenberg & Levine (1992), define $\{\tilde{L}_t\}$ to be the faster process obtained from $\{L_t\}$ through a sequence of stopping times that contains all finite histories at which

$$|f^t_0(h^{t-1})(a(t)) - f^t_1(h^{t-1})(a(t))| > \epsilon$$
Fudenberg and Levine show that \( \{\tilde{L}_t\} \) is an active supermartingale with activity \( \epsilon \).

Their Theorem A.1 implies that for any \( \epsilon > 0 \) there is an integer \( K \) such that for any active supermartingale \( \{\tilde{L}_t\} \)

\[
P \left[ \sup_{k > K} \tilde{L}_k < 1 \right] > 1 - \epsilon.
\]

\( K \) depends only on \( \epsilon \) and not on the true stochastic process \( P \) or the forecasting strategy \( f_1 \).
How big a loophole is left?

The test potentially gives the false expert a loophole—if he can match the predictions of the true expert.

To win the reputation game the imposter must win the “matching game”—how likely is this?
Scope of strategic manipulations

Family of incomplete-information constant-sum games between Expert 0 and Expert 1, parametrized by $n = 1, 2 \ldots$ and $\mu \in \Delta(\Delta(A^n))$: 

- Nature chooses $P \in \Delta(A^n)$ according to $\mu$;
- Expert 0 is informed of $P$; Expert 1 only knows $\mu$;
- The two players simultaneously choose forecasts;
- Nature then chooses $a^n$ according to $P$;
- The payoff of Expert 1 is

$$T^n(a^n, f_0, f_1),$$

where $T^n$ is the test constructed in Theorem 1.

- The payoff of Expert 0 is $1 - T^n(a^n, f_0, f_1)$. 

A Bayesian game
Identify $\mu \in \Delta(\Delta(A^n))$ with its one-step-ahead conditionals,

$$\mu^t(\cdot | \alpha^{t-1}) \in \Delta(\Delta(A))$$

$\mathcal{M}(\epsilon, \delta, L) \subset \Delta(\Delta(A^n))$ consists of all $\mu$ such that there are at least $L$ periods $1 \leq t \leq n$ such that for $\mu$-a.e. $h^n$

$$\max_{p \in \Delta(A)} \mu^t(B_\epsilon(p) | \alpha^{t-1}) < 1 - \delta.$$

- In words, in each of at least $L$ periods $\mu$ does not concentrate its mass in some small ball
- Condition becomes less restrictive as $n$ becomes large.
A Bayesian game

**Possibility theorem: Bayesian game**

**Theorem**

For every $\epsilon$ and $\delta > 0$ there is an integer $L$ such that for every $\mu \in \mathcal{M}(\epsilon, \delta, L)$ the value of the game to Expert 1 is less than $\epsilon$.

If $\mu$ is sufficiently diffuse, player 0 is much better informed than player 1, and the value of player 1 is small.
Comparative test: *Infinite horizon*

\[ L^t \text{ is defined exactly as before} \]

Test picks Expert 0 if \( \lim_{n \to \infty} L_n(h^n) < 1 \)

Test picks Expert 1 if \( \lim_{n \to \infty} L_n(h^n) > 1 \)

Test picks at random if either \( \lim_{n \to \infty} L_n(h^n) = 1 \) or this sequence fails to converge.
Possibility theorem: *Infinite horizon*

Theorem

For any distribution $P$ and mixed forecasting strategies $\varphi_0, \varphi_1$ with at least one informed expert, with $P$-probability 1 either

(a) $T$ picks an informed expert; or

(b) $\lim_{t \to \infty} |f^t_0(h^{t-1}) - f^t_1(h^{t-1})| = 0$. 
Possibility theorem: *Infinite horizon*

\[ \mathcal{M}(\epsilon, \delta) \subset \Delta(\Delta(A^\infty)) \] consists of all \( \mu \) such that for \( \mu \)-a.e. infinite history \( h^\infty \), for infinitely many periods,

\[
\max_{\mu \in \Delta(A)} \mu^t(B_\epsilon(p) | \alpha^{t-1}) < 1 - \delta.
\]

**Theorem**

*For every \( \epsilon, \delta > 0 \) and \( \mu \in \mathcal{M}(\epsilon, \delta) \) the value of the game to Expert 1 is zero.*
Is there at least one informed expert?

Is there a way to determine if among two experts at least one is informed?

Consider the function:

\[ \tau : H^n \to \{0, 1\} \]

Interpret \( \tau(a^n, f_0, f_1) = 1 \) to mean that at least one expert is informed.
Impossibility of testing that someone knows

Impossibility theorem

Theorem

Suppose that \( \tau \) is such that for every \( P, f_0 \) and \( f_1 \)

\[
P\{ a^n : \tau(a^n, f_0, f_1) = 1 \} > 1 - \epsilon \quad \text{if either } f_0 = f_P \text{ or } f_1 = f_P.
\]

Then for every mixed strategy \( \varphi_0 \) of Expert 0 there is a mixed strategy \( \varphi_1 \) of Expert 1 such that for every \( a^n \)

\[
\varphi_0 \times \varphi_1 \{ (f_0, f_1) : \tau(a^n, f_0, f_1) = 1 \} > 1 - \epsilon.
\]
The tester allows only probability measures that belong to some class $\mathcal{Z}$.

Two interpretations

1. The tester believes the true process has some structure, reflected in the definition of $\mathcal{Z}$

2. The tester entertains the possibility that the true process can be anything, but fearing manipulability, restricts forecasts to be in $\mathcal{Z}$.

Bottom line: any forecast that does not belong to $\mathcal{Z}$ is rejected.
Representation of stochastic processes

A *representation* of a stochastic process \( \mu \) is, roughly, a decomposition of this process into a *compound lottery*.

Formally, a representation is an object:

\[
(\Theta, \lambda, (\mu_\theta)_{\theta \in \Theta})
\]

- \( \Theta \) is a set of parameters
- \( \mu_\theta \) is a stochastic process corresponding to \( \theta \)
- \( \lambda \) is a probability measure on \( \Theta \)

such that \( \mu \) is obtained by the reduction of the compound lottery, first \( \lambda \) then the \( \mu_\theta \)'s:

\[
\mu(S) = \int_{\Theta} \mu_\theta(S) \, d\lambda \quad \forall S.
\]
Any stochastic process has an obvious representation:

- Parameters $\Theta = H^\infty$
- each $\mu_{h^\infty}$ is dirac on the sample path $h^\infty$
- $\lambda = \mu$

such that

$$\mu(S) = \int_{\Theta} \mu_{h^\infty}(S) \, d\mu.$$ 

Intuitively, this is rather unhelpful:

*with this representation you never learn the true $\theta$, even approximately, no matter how much data you obtain*
The most famous of all representations

de Finetti’s Theorem:
\( \mu \) is exchangeable iff it has a representation with:

- Parameters \( \Theta = [0, 1] \)
- each \( \mu_\theta \) is i.i.d. with mean \( \theta \in [0, 1] \)
- distribution \( \lambda \) on \( \Theta \)

“Every exchangeable process is a convex combination of i.i.d.’s”

Here, by contrast, the parameters are, in a sense, learnable
Kalai and Lehrer (1994) introduced the following concept:

**Definition**

\( \mu \) **merges** with \( \mu_\theta \) if

- for every \( \epsilon > 0 \)
- every non-negative integer \( l \)
- and \( \mu_\theta \)-a.e. history \( h^\infty \)

there is \( T \) such that for all \( t \geq T \)

\[
\sup_{n \geq t} \sup_{A \in \mathcal{F}_n^{n+l}} |\mu(A|\mathcal{F}^t) - \mu_\theta(A|\mathcal{F}^t)| < \epsilon.
\]
Definition

The representation \((\Theta, \lambda, (\mu_\theta)_{\theta \in \Theta})\) is **learnable** if

\[
\mu \text{ merges with } \mu_\theta
\]

for \(\lambda\)-a.e. \(\theta\).

Roughly, whenever \(\theta\) is driving the observations, then \(\mu\)’s and \(\mu_\theta\)’s forecasts are eventually close.
Learnability by itself can be too weak

Here is a coarser representation of an exchangeable process:

\( \lambda \) is uniform distribution on i.i.d. coins

- \( \Theta = \{ \theta_1, \theta_2 \} \)
- \( \mu_{\theta_1}(S) = E(S \mid \text{coin } < 0.5) \)
- \( \mu_{\theta_2}(S) = E(S \mid \text{coin } \geq 0.5) \)
- \( \mu(\theta_1) = \mu(\theta_2) = 0.5 \)

such that

\[
\mu(S) = \mu(\theta_1) \mu_{\theta_1}(S) + \mu(\theta_2) \mu_{\theta_2}(S).
\]

Intuitively, there is something wrong here:

*with this representation you do not learn as much as you could*
Jackson, Kalai, and Smorodinsky (Econometrica, 1999) introduced the following concept:

**Definition**

A probability distribution $p$ is **sufficient for predictions** if for all $t$

$$\lim_{n} \sup_{A \in \mathcal{F}_n} |p(A|\mathcal{F}^t) - p(A)| = 0.$$ 

**Definition**

The representation $(\Theta, \lambda, (\mu_\theta)_{\theta \in \Theta})$ is **sufficient for predictions** if $\mu_\theta$ is sufficient for predictions for $\lambda$-a.e. $\theta$. 
What is special about de Finetti’s representation?

de Finetti’s representation is learnable and exhausts all learnable patterns.

Formally, it is the unique representation (of an exchangeable process) that is both

- **Learnable**

and

- **Sufficient for predictions**
A representation \((\Theta, \lambda, (\mu_\theta)_{\theta \in \Theta})\) of a stochastic process \(\mu\) is

**learnable predictive representation (LPR)**

if it is

- learnable; and
- sufficient for prediction.

Jackson, Kalai, and Smorodinsky (Econometrica, 1999):

- characterized LPR processes;
- showed the essential uniqueness of representations
1 Introduction

2 Basic model

3 Comparative testing

4 Learnable-predictive forecasts

5 Theorems
   - The test
   - Passing truthful LPR experts
   - Non-manipulability

6 On truth..
Here is an important implication of LPR:

If $\mu$ is LPR, then there is

1. a date $T$
2. a sequence of dates $\{m_1, m_2, \ldots\}$ (depending on $h^T$)
3. a number $\alpha \in [0, 1]$ (depending on $h^T$)

such that the random variables

$$\{a_{m_1}, a_{m_2}, \ldots\}$$

are “approximately i.i.d. with mean $\alpha$” given $h^T$. 
The test

The learnability test

Fix a sequence of infinite sets of integers $\{\mathcal{N}_T\}_{T=1}^{\infty}$ such that

The $\mathcal{N}_T$'s are disjoint & $n > T$ for every $n \in \mathcal{N}_T$

The test $T(\delta, L)$:

1. The forecaster submits a date $T$

2. Given history $h^T$, the forecaster submits dates $m_1, \ldots, m_L \subset \mathcal{N}_T$ & $\alpha \in [0, 1]$

3. The forecaster passes if and only if:

$$\left| \frac{a_{m_1}(h) + \cdots + a_{m_L}(h)}{L} - \alpha \right| < \delta.$$
The expert communicates very little information:

- The expert does not submit the entire probability measure as in many of the tests in the literature
- Thus this test does not use off-path or future forecasts
- Contrast this with
  - positive results of Dekel-Feinberg and Olszewski-Sandroni
  - impossibility results of Olszewski-Sandroni and Shmaya

The expert need not even learn probabilistically:

- During the learning phase, he may entertain a set of priors, may be a frequentist..etc.
Passing truthful LPR experts

For every $\epsilon, \delta > 0$
there exists $\bar{L}$
such that any expert who knows the truth to be LPR can pass the test $T(\delta, L)$ with probability at least $1 - \epsilon$ for every $L > \bar{L}$.

The argument relies on:

- Two uniformization lemmas
- A LLN for correlated random variables (due to Lehrer)
Uniformizing $T$

Given the representation $(\Theta, \lambda, (\mu_\theta)_{\theta \in \Theta})$,

- for every $\epsilon_1, \epsilon_2 > 0$ and integer $l$
- there is an integer $T = T(\epsilon_1, \epsilon_2, l)$
- and a set $\Theta_1 \subset \Theta$ with $\lambda(\Theta_1) > 1 - \epsilon_1$

such that for all $t \geq T$ and $\theta \in \Theta_1$

$$\sup_{n \geq t, A \in \mathcal{F}_n^{n+l}} \left| \mu(A|\mathcal{F}^t) - \mu_\theta(A|\mathcal{F}^t) \right| < \epsilon_2.$$
Given the representation \((\Theta, \lambda, (\mu_\theta)_{\theta \in \Theta})\) and

- For every \(\epsilon_3 > 0\)
- ...details...
- there is an increasing subsequence of dates \(\{m_1, m_2, \ldots\}\)
- and \(\Theta_2 \subset \Theta\) with \(\lambda(\Theta_2) > 1 - \epsilon_3\)

such that for every \(\theta \in \Theta_2\)

\[
\lim_{j \to \infty} \sup_{0 \leq i < j} \left| \text{cov} (a_{m_j}, a_{m_i} | \theta) \right| = 0.
\]
Let $g_1, g_2, \ldots$ be a sequence of bounded, 0-mean random variables such that

$$\lim_{j \to \infty} \sup_{0 \leq i < j} \text{cov}(g_j, g_i) = 0.$$ 

Then there is a subsequence $\{g_{i_l}\}$ such that the random variable

$$f_l = \frac{g_{i_1} + \cdots + g_{i_l}}{l}$$

converges to 0 a.s.

In particular, for every $\epsilon, \delta > 0$, there is $\bar{L}$ such that for every $L > \bar{L}$

$$P\{f_L \in [-\delta, \delta]\} > 1 - \epsilon.$$
Non-manipulability Theorem

For any $\epsilon > 0$
there is $L$ and $\delta$ such that:

the test $T(L, \delta)$ satisfies

- for any forecaster mixed strategy $\varphi$
- there is an LPR distribution $P$ at which the forecaster fails:

$$z(P, \varphi) < \epsilon.$$
Proof of non-manipulability 1

1. Find $\bar{T}$ such that $\varphi$ prescribes $T \leq \bar{T}$ with probability $1 - \epsilon_1$.

2. For each $T \leq \bar{T}$ select a number $\theta_T$ independently and uniformly from $[0, 1]$.

3. On $T \leq \bar{T}$ $\mu$ is i.i.d. on $\mathcal{N}_T$ with probability $\theta_T$.

4. On the remaining dates $\mu$ is the constant $0$. 
Proof of non-manipulability 2

We bound the forecaster’s payoff given any $T \leq \bar{T}$, $h^T$ and $\alpha$.

1. For every $\delta > 0$, probability of
   \[ \theta_T \in [\alpha - 2\delta, \alpha + 2\delta] \]
   is at most $4\delta$.

2. For every $\delta, \epsilon_2 > 0$ choose $L$ such that
   \[ f_L \not\in [\theta_T - \delta, \theta_T + \delta] \]
   with probability at most $\epsilon_2$.

3. Then $f_L \in [\alpha - \delta, \alpha + \delta]$ with probability at most $4\delta + \epsilon_2$. 
Proof of non-manipulability 3

1. Since $h^T$ and $\alpha$ were arbitrary, the unconditional probability that this expert passes is no more than

$$(4\delta + \epsilon_2)(1 - \epsilon_1) + \epsilon_1$$

2. The conclusion follows since $\delta$, $\epsilon_1$ and $\epsilon_2$ are arbitrary.
A Corollary

Impossibility of purely empirical knowledge:

- **Exchangeable case:** there are exchangeable beliefs that merge with the truth, for any exchangeable truth.

- **LPR case:** there is no LPR belief that merges with the truth, for any LRP truth.
  - This is a corollary to the non-manipulability theorem.

⇒ Under LPR, *knowledge of the true distribution cannot be picked up based on empirical observations alone.*
“It seems to me that the subjectivist forecaster is obliged to treat his own subjective distribution in the same tentative manner as he would an external statistical forecasting system.”

“All such system is applied only tentatively, [...] but] as soon as it is clear that there is a conflict between its predictions and reality [...] the system will be modified or discarded.”

“But such a process is intrinsically incoherent.”
Implications for subjectivists..

- **Impossibility literature**: suggests that the unrestricted use of subjectivity leads to untestable beliefs.

- **This paper**: suggests that a natural restriction on subjectivist beliefs can make them testable.

Here, I invoke the wisdom of my favorite philosopher, Prof. J. Weinstein:

"Anything that says something can be wrong."
“I’d rather be delusional than sacrifice coherence”

Dawid’s fundamental insight:
(25 years and dozens of papers later)

The potential to falsify subjective beliefs is deeply disturbing and problematic to a Bayesian

- The poor Bayesian has no mechanism to revise his prior beliefs when he discovers they were wrong
- Attempts at belief cause a brain meltdown:
  “Is life worth living if I violate de Finetti’s coherence?”
On truth and probability

Jonathan: If I cannot test probabilities, then I don’t see how they can be wrong.

Nabil: But if I cannot test probabilities, then I don’t see how they can be right.

..after a week’s reflection:

Jonathan: “Cannot be right” and “cannot be wrong” are really the same thing.