On the Relationship Between Pricing and Capacity Decisions in Inventory Systems with Stochastic Demand

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Abstract

We address the simultaneous determination of pricing and capacity investment strategies in a multi-period setting under demand uncertainty. In our model a monopolistic firm makes three decisions: capacity investment (or disinvestment), production (inventory), and price, all of which can be specified dynamically as a function of the state of the system. We analyze the optimal joint strategy and through that investigate the relationships between the main strategic decision variables: price and capacity. We consider models that allow for either bi-directional price changes or models with markdowns only, and in the latter case we prove that capacity and price are *strategic substitutes*.

Short Title: Relationship between pricing and capacity decisions

Keywords: capacity investment, pricing, inventory, stochastic demand, coordination.

1 Introduction

Recent years have witnessed an increased interest in the use of pricing in operations management practices. In this realm, a particular focal point has been the integration of inventory control and dynamic (state-dependent) pricing strategies. Concomitantly, studies focusing on the interface

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between capacity investment and replenishment strategies have led to further understanding of capacitated inventory systems and supply chains. A very useful qualitative insight in this context has been the understanding that capacity and inventory are in essence *strategic substitutes*; roughly speaking, decision variables are said to be strategic substitutes if increasing one variable decreases the return from increasing the other. (A more detailed definition will be advanced in section 4.)

One potential barrier in achieving coordination between capacity, inventory and pricing decisions is the fact that the main strategic decisions (determining price and capacity) are made by separate, often independent, divisions in the firm. Shapiro (1977) states that the conflict between capacity planning and long-range sales forecasting is one of the main areas in which there is a strong likelihood of conflict in managing the marketing/manufacturing interface in an industrial company. (In contrast, the conflict between production scheduling and short term sales forecasting is more operational or tactical.) To coordinate such actions one it is essential to understand the relationships between these decisions variables. However, as the literature review in Section 2 indicates, there have been relatively few papers to date that focus on the problem of joint capacity planning and pricing strategies, and even less that go on to explore the three-way relationship between capacity, inventory and pricing decisions.

In this paper we study a stylized problem in which a centralized monopolistic firm sells a product with a short life-cycle over a finite time horizon; the number of periods constituting this time horizon measure the time elapsed from the first introduction of the product to the market, up until the point where the firm terminates its production and sale. The firm reviews the state of the system periodically and at the beginning of each period makes three decisions: i.) invest or disinvest in capacity; ii.) replenish inventory (limited by production capacity); and iii.) fix a price for the produced good that will take effect in the following period. Subsequent to these decisions, demand is observed. We first allow the firm to carry inventory from one period to the next, and orders are allowed to be backlogged. Subsequently, we introduce a restriction that disallows carry-over of inventory from period to period; the firm must then either satisfy inventory shortage using emergency replenishment or by paying penalty fees. In the next stage, we also restrict the firm's pricing flexibility by only allowing markdowns, i.e., the price of the product can only decrease over its life cycle.

The main contribution of this paper is in studying the relationship between pricing and capacity decisions in the context of a dynamic optimization problem that has capacity, inventory and price as its variables. The analysis proceeds by first showing that the optimal capacity investment policy in the presence of pricing and inventory decisions is of a *target interval* form (see Theorem 1). Given a fixed capacity level, the optimal joint pricing-inventory decisions are seen to follow a modified base-stock list-price policy (see Theorem 2). We then study a model where no inventory carry-over is allowed, and where pricing is restricted to markdowns. In this important scenario we show that price and capacity are strategic substitutes both as decision variables and as state variables (see Theorem 3); an important implication of this property is the potential to use these levers in a complementary manner (see discussion in Section 6). Several numerical examples are used to illustrate these analytical results and demonstrate their significance.

The remainder of the paper. Section 2 summarizes related literature and known results. Section 3 describes the model and sets up the dynamic optimization problem. Section 4 provides the main results and numerical illustrations of the key findings. Section 5 discusses the cases where no carry-overs are allowed and when pricing is restricted to markdowns; both theoretical results and numerical illustrations are provided. Section 6 concludes with a discussion of the main qualitative insights that are revealed through the main results.

2 Literature Review

The literature review focuses only on models that incorporate at least two of the decision variables of interest, namely, pricing, inventory and capacity dis/investment. In addition, the review of models that consider only two decision variables focuses primarily on work that is close in nature to our model and that facilitates its understanding.

Inventory and pricing decisions. Federgruen and Heching (1999) study the coordination problem between price and inventory in an uncapacitated system with stochastic demand. They analyze a single item periodic review model with time dependent parameters (demand distribution and costs) and characterize the structure of the optimal price-inventory policy. Federgruen and Heching (1999) show that the the inventory level after ordering and the price charged are strategic substitutes. Further references in this stream of literature are surveyed in Elmaghraby and Keskinocak (2003); for a deterministic analysis, see, e.g. Thomas (1974) and Kunreuther and Schrage

(1973).

Inventory and capacity decisions. Angelus and Porteus (2002) study capacity decisions in cases where a firm can and cannot hold inventories in the presence of demand uncertainty. They show that a target interval policy is optimal, and when firms can hold inventory they establish that capacity and inventory are strategic substitutes. For example, if a firm experiences excess demand in a single period so that the initial inventory in the next period is "lower than expected," the optimal capacity for the next period will end up being "higher than expected." For an example of a model with simultaneous determination of production and capacity when there is no demand uncertainty, see Rao (1976). Van Mieghem and Eberley (1997) study a problem that can be viewed as a generalization of the "no-carry-over" version of Angelus and Porteus (2002). They characterize the optimal capacity policy when capacity is multi-dimensional and the cost of adjusting each dimension (or factor) is "kinked" linearly in a manner that makes it costly to reverse investments. The optimal policy in this case is of the type "Invest/Stay-Put/Disinvest," namely a target interval policy. Duenyas and Ye (2005) generalize Angelus and Porteus' (2002) no-carry-over model by allowing capacity adjustments to incur fixed costs in addition to the variables ones.

Pricing and capacity decisions. Duenyas and Ye (2003) consider the optimal joint management of capacity and pricing/production decisions with fixed and variable costs for adjusting capacity. They characterize the optimal policy structure which turns out to be quite complex (consisting of six regions).

The papers mentioned above analyze models in which firms are essentially capable of adjusting only two of the three variables (note that in Duenyas and Ye (2003) the production decision affects only the profits in the period in which the decision was made). We now turn to survey work on the simultaneous setting of price, production and capacity.

Joint pricing, production and capacity decisions. Maccini (1984) studies the effects of inventory dynamics and capital on pricing and capacity decisions. The goal of the paper is to explore the relationships between prices and production in the supply-side of a macroeconomic model. One of the main conclusions is that the price and output levels that firms set in the short-run will differ from the long-run depending on whether desired inventories and capital are above or below their long-run levels. In particular, excess capacity tends to cause prices to drop below their anticipated long-run level.

Gaimon (1987) studies the simultaneous setting of price, production level, and capacity over time with the objective of profit maximization. The main assumption introduced there is that the acquisition of new capacity reduces the per unit cost of production. By means of a numerical study, Gaimon (1987) indicates that upgrading capacity lowers the firm's per unit production cost and thus the prices it charges.

Li (1988) introduces a model of a production firm assuming that cumulative production and cumulative demand are two counting processes with random intensities parameterized by production, capacity and price, respectively. A distinction is made between static decision making (capacity levels are set at time zero), and dynamic operating decisions (pricing and production). Li (1988) highlights the fact that in the dynamic model, inventory "ties together" production and pricing decisions at different points in time.

Van Mieghem and Dada (1999) study different possible postponement strategies when firms make three decisions: capacity investment, production (inventory) quantity, and price. Their results suggest that managers can make optimal capacity decisions by deterministic reasoning when allowed some level of price flexibility, but the analysis is restricted to a single period problem and thus is unable to capture some of the key relations between the strategic variables and their long range effects. For a recent survey and further references on pricing, inventory and capacity decisions the reader is referred to Chan et. al. (2004, §4.2).

The present paper strives to contribute to the streams of literature surveyed above by focusing on the relationships between the key decision variables in the joint optimization problem, in particular, price and capacity. Synergetic effects yields several insights that, in turn, suggest guidelines according to which a firm can coordinate pricing, inventory and capacity decisions in a beneficial manner. As opposed to the recent study of Duenyas and Ye (2003), we do not concentrate on the optimal policy per se but rather focus on qualitative relations between the key decision variables.

3 Problem Formulation

We consider a monopolistic firm that produces a single product whose capacity, inventory, and price are reviewed periodically. At the beginning of each period the firm makes three decisions: (i) capacity investment (or disinvestment); (ii) production level; and (iii) the price it will charge for the product. We assume that capacity investments and produced goods become available instantaneously. The life cycle of the product, and therefore the time horizon, is set to be T periods. The sequence of events in each period, $t = 1, \ldots, T$, is as follows:

1. Investment or disinvestment in capacity, setting it to a level equal to z_t.
2. Production (if needed) to set the inventory level to y_t.
3. A price p_t is set and held fixed up until period $t+1$.
4. Demand is realized and satisfied if it is less than available inventory, or backlogged otherwise. Backlog and holding costs are incurred.

Demand in consecutive periods is independent and non-negative. Demand in period t, D_t, depends on the prevailing price which is given by a general stochastic demand function

$$D_t = d_t(p_t, \epsilon_t), \tag{1}$$

where

$$\begin{aligned} p_t &= \text{price charged in period } t, \\ \epsilon_t &= \text{random noise with known distribution.} \end{aligned}$$

Feasible price levels are confined to the interval $\left[\underline{p}, \overline{p}\right]$, where $\overline{p}$ and $\underline{p}$ are the highest and lowest prices, respectively. In what follows, for simplicity and ease of exposition, we consider a specific demand function that is linear in price and additive in the noise, i.e.,

$$d_t(p_t, \epsilon_t) = a_t - b_t p_t + \epsilon_t \tag{2}$$

where $a_t, b_t > 0$, $\overline{p} \leq \frac{a_t}{b_t}$, for all $t = 1, \ldots, T$. In §4 and §5 we indicate how the main results extend to more general demand functions. Let

$$\begin{aligned} x_t &= \text{inventory level at the beginning of period } t\text{, before ordering,} \\ y_t &= \text{inventory level at the beginning of period } t\text{, after ordering.} \end{aligned}$$

The firm incurs two types of production and inventory costs: the end-of-period inventory carrying (and backlogging) costs, and a variable production cost. Specifically,

$$\begin{aligned} h_t(I) &= \text{inventory (or backlogging) cost incurred in period } t \text{ with} \\ & \quad \text{terminal inventory level equals } I, \\ c_t &= \text{per unit purchasing or production cost in period } t. \end{aligned}$$

Let

$$G_t(y, p) = \mathbb{E}h_t(y - D_t) = \mathbb{E}h_t(y - d_t(p, \epsilon_t)), \tag{3}$$

denote the single-period expected inventory and backlogging costs for period t, for a given price p and an inventory level (after ordering) y, where the expectation here, as well as in the remainder of the paper, is taken with respect to the distribution of the random noise term. We assume that

(A1) $\mathbb{E}|\epsilon_t| < \infty$, for all $t = 1, \ldots, T$,

(A2) $h_t(\cdot)$ is convex for all $t = 1, \ldots, T$.

These assumptions ensure that the cost functions $G_t(y, p)$ are well defined, finite, and jointly convex in y and p for all $t = 1, \ldots, T$.

Let $\gamma_t(y, p)$ denote the expected contribution in profits in period t, if the firm has y units at the beginning of the period (i.e., post production) and it charges p per produced unit that is sold on the market. That is, in period t

$$\gamma_t(y, p) = p\mathbb{E}[d_t(p, \epsilon_t)] - c_t y - G_t(y, p). \tag{4}$$

Let

$$z_t = \text{the capacity level at the beginning of period } t\text{, after adjustment.}$$

We define three capacity related costs:

$$\begin{aligned} K &= \text{the cost of adding a unit of capacity,} \\ k &= \text{the return from selling a unit of capacity,} \\ h_c &= \text{the capacity overhead cost per unit.} \end{aligned}$$

Hence h_c amalgamates all costs that are associated with maintaining production, but are independent of the production volume. We assume that $K \geq k$ which reflects the fact that capacity is usually sold for less than the purchase price. Revenues and costs are discounted with a discount factor $\alpha \in (0,1]$. We note that all capacity-related costs are taken to be time-homogeneous for simplicity, and the analysis that follows can easily be adjusted to account for temporal dependency. We assume that a firm begins the life-cycle of the product with capacity level z_0 and inventory level x_0 (allowing for the possibility of $x_0 = 0, z_0 = 0$).

Let $f_t(z,x)$ be the maximum expected present value of the total net return that can be earned in months t and on, given that month t starts with capacity level z and inventory level x. That is,

$$f_t(x,z) = \max \Big\{ \gamma_t(y,p) + c_t x - C(z'-z) - h_c z' + \alpha \mathbb{E} f_{t+1}(y - d_t(p,\epsilon_t), z') : \\ z' \geq 0, x \leq y \leq x + z', \underline{p} \leq p \leq \overline{p} \Big\}, \quad (5)$$

for $t = 1, \ldots, T$, where

$$C(z) = \begin{cases} kz & \text{if} \quad z \leq 0 \\ Kz & \text{if} \quad z > 0. \end{cases} \quad (6)$$

At the terminal period we assume that demand is satisfied and the remaining capacity is sold immediately thereafter, that is, we set

$$f_{T+1}(x,z) = kz - h_{T+1}(x).$$

To recapitulate, at the beginning of each period $t = 1, \ldots, T$, the firm must determine a capacity investment level z', an inventory level y, and a price p based on the initial inventory and capacity, x and z. These decisions are held fixed throughout period t. The objective of the firm is to maximize the sum of discounted profits over the time horizon T with respect to the abovementioned decision variables; the maximum value of this dynamic optimization problem is given by $f_1(x,z)$.

For future purposes it will be convenient to rewrite $f_t(x,z)$ as follows (see Angelus and Porteus (2002))

$$f_t(x,z) = \max_{z' \geq 0} \left[c_t x - C(z'-z) - h_c z' + \Gamma_t(x,z') \right], \quad (7)$$

where

$$\begin{aligned} \Gamma_t(x,z) &= \max \Big\{ a_t(y,p,z) \ : \ y \in [x, x+z], p \in [\underline{p}, \overline{p}] \Big\}, \\ a_t(y,p,z) &= \gamma_t(y,p) + \alpha \mathbb{E} f_{t+1}(y - d_t(p,\epsilon_t), z), \end{aligned} \quad (8)$$

for all $t = 1, \ldots, T$. We define $\widehat{y}(x, z)$ and $\widehat{p}(x, z)$ as follows:

$$(\widehat{y}_t(x, z), \widehat{p}_t(x, z)) = \arg\max \left\{ a_t(y, p, z) \ : \ y \in [x, x+z], p \in [\underline{p}, \overline{p}] \right\}.$$

The existence and the uniqueness of $\widehat{y}_t(x, z)$ and $\widehat{y}_t(x, z)$ for given initial capacity and inventory levels, x and z, will be shown in the sequel.

4 The Optimal Policy and Key Relations

4.1 Main results

In this section we characterize the structure of a policy that maximizes the expected discounted profits; the maximum value of this objective is given by $f_1(\cdot, \cdot)$, where $f_t(\cdot, \cdot)$ is defined in (7) . We will begin by analyzing the optimal capacity investment policy. Then, given the optimal capacity at the beginning of a period, we will derive the optimal joint inventory-pricing policy. It is important to note that the three decisions are made simultaneously, and the optimal policy is described in a sequential manner to allow for a more transparent representation.

To characterize the optimal capacity investment policy, we first describe a family of ISD policies (Invest/Stay Put/Disinvest), often referred to as *target interval* policies.

Definition 1 *A sequence $\{z_t\}$ for period $t = 1, \ldots, T$, constitutes a target interval policy with respect to a sequence $\{L_t, U_t\}$ for $t = 1, \ldots, T$, if:*

(i) $L_t \leq U_t$

(ii) *L_t and U_t are independent of z_{t-1};*

(iii) $$z_t = \begin{cases} L_t & \text{if} \quad z_{t-1} < L_t, \\ z_{t-1} & \text{if} \quad L_t \leq z_{t-1} \leq U_t \\ U_t & \text{if} \quad z_{t-1} \geq U_t. \end{cases}$$

The upper and lower targets L_t and U_t can be functions of the state of the system, and the notation $L_t(\cdot)$ and $U_t(\cdot)$ will be used to indicate this dependence; in the following theorem, both are functions of the initial inventory x. We then have the following result.

Theorem 1 ***(Optimal capacity investment policy)*** *The optimal capacity investment decision follows a target interval policy in each period, with lower- and upper- capacity targets $L_t(x)$ and $U_t(x)$ for each $t = 1, \ldots, T$ and each initial inventory level $x \in \mathbb{R}$.*

Based on the optimal capacity investment, we will now show that the optimal joint production-pricing decision takes the form of a modified base-stock list-price policy (we use the term "modified" because of the capacity constraint on the production). This policy is characterized by a base-stock level and a list-price combination $(\widehat{y}_t(x,z), \widehat{p}_t(x,z))$ given as a function of the initial inventory and capacity (x, z). If the inventory level, x, is below the base-stock level, it is increased to that value and the list-price is charged. If the inventory level is above the base stock level, then nothing is ordered, and a price discount is offered, i.e., the price charged is below the list price. (The higher the excess in the initial inventory level, the larger the optimal discount offered.) If the sum of inventory and capacity is below the base-stock level, the maximum possible amount is produced (i.e., the production level equals the capacity level), and the price charged is higher than the list price. No discounts are offered unless the product is overstocked, and no higher-than-list-prices are charged unless the product is in shortage, i.e., the current capacity is not sufficient to support the "desired" inventory level. These observations are summarized in Theorem 2, for the purpose of which we introduce the following definition.

Definition 2 *Variables $u, v \in \mathbb{R}$ are said to be strategic substitutes with respect to a function $f(u,v) : \mathbb{R}^2 \to \mathbb{R}$, if $f(u,v)$ is submodular in u and v.*

For a definition of submodularity see, e.g., Topkis (1978), and for further discussion of economic implications and interpretation see, e.g., Milgrom and Roberts (1990). For brevity, we omit in what follows explicit reference to the function relative to which variables are said to be strategic substitutes when this function can be clearly identified from the context of the discussion or result.

Theorem 2 ***(Optimal pricing-inventory policy)***

(a) *The optimal inventory-pricing policy is a base-stock list-price with base-stock $\widehat{y}_t(x,z)$ and list-price $\widehat{p}_t(x,z)$, for $t = 1, \ldots, T$. At period $t \in \{1, \ldots, T\}$ and given a capacity level z: if $x \le \widehat{y}_t(x,z) \le x + z$, it is optimal to order up to the base-stock level $\widehat{y}_t(x,z)$ and to charge the list-price $\widehat{p}_t(x,z)$; if $x > \widehat{y}_t(x,z)$, it is optimal not to order and to charge $p_t \le \widehat{p}_t(x,z)$; and if $\widehat{y}_t(x,z) > x + z$, it is optimal to order z units and charge $p_t \ge \widehat{p}_t(x,z)$.*

(b) *For each period $t \in \{1, \dots, T\}$ and fixed capacity and inventory state values $x, z \in \mathbb{R}$, the price and inventory decision variables (p_t, y_t) are strategic substitutes with respect to the function $a_t(\cdot, \cdot, z)$ given in (8).*

Remark. The assumption of linear demand can be generalized to any demand function which is continuous and strictly decreasing in the price variable and for which the revenue rate $d\mathbb{E}(d_t^{-1}(d, \epsilon_t))$ is concave in d, where d_t^{-1} is the inverse function of d_t for fixed ϵ_t. This assumption is rather benign and quite standard in the revenue management literature; see Chen and Simchi-Levi (2002) for further discussion.

4.2 Illustrative examples

The theorems presented in the previous section and in the following sections do not describe how the optimal policy, throughout the life-cycle of a product, varies with changes in key parameters. We present the following example and those in the sequel with the goal of providing insights regarding these relationships. To facilitate visual interpretation of the figures in this section and throughout the paper, overlapping trajectories in the graphs have been slightly shifted. (There will be no difficulty distinguishing this artificial editing from cases where there is "real" separation between trajectories.) When reading the figures it is also useful to keep in mind that the optimal capacity, inventory and pricing levels are integer valued.

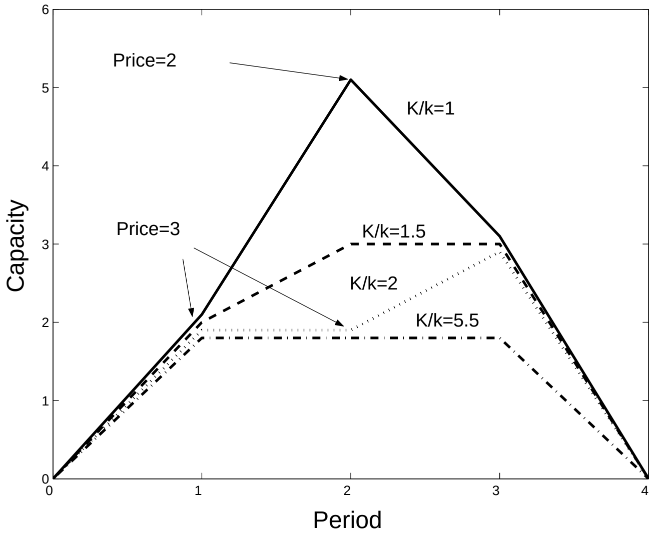
Example 1: The effects of capacity investment irreversibility ratio on the optimal policy. (Figure 1) Consider a firm that produces and sells a product during three periods; the fourth being the terminal period in which the firm must resolve backlogged demand and sell its capacity. The firm starts off with no capacity and zero inventory. Demand is anticipated to be low in the first period, increase during the middle period, and then return to its initial level in the final period. To encode this using our model parameters, we put $a_1 = a_3 = 5$ and $a_2 = 10$ in the demand function. The firm expects the customers' price sensitivity to vary in a similar manner. In particular, the initial sensitivity is low due to the novelty of the product, while in the subsequent period, with the increase in customer awareness, the firm experience a higher level of customers sensitivity to price. In the final period, the demand pattern matches the one in the first period. We set $b_1 = b_3 = 1$ and $b_2 = 2$, reflecting the fact that for any given price level the demand in the second period will be twice that in the first and third periods. For purposes of this example, we take the error term ϵ_t to follow a Poisson distribution with mean 1, independent for each period $t = 1, 2, 3$. The firm's variable cost of production is $c_1 = c_2 = c_3 = 1$. The backlogging and holding costs are proportional to the end-of-period backlog or inventory levels, $h_t(x) = h_t^+ \max(x, 0) - h_t^- \min(x, 0)$. However, we allow the cost-per-unit of backlogged demand, $h_1^- = h_2^- = h_3^- = 2$, to be slightly higher than the cost-per-unit held, $h_1^+ = h_2^+ = h_3^+ = 1.5$. We begin with a base scenario in which $k = 1$, $K = 2$ and $h_c = 0$, i.e., no capacity overhead costs, however, reversing capacity investment is costly: at most half of the amount invested in capacity is going to be returned when disinvesting. We take the discount factor to be $\alpha = 0.9$.

Varying the ratio between K and k (the cost of purchase and resale of capacity, respectively), we plot the optimal capacity investment level in Figure 1. The optimal policy for each combination of initial inventory and capacity is computed for each scenario. For simplicity, the graph depicts realizations in which the error term is set to zero. In the case corresponding to $K/k = 1$ (the upper most line), capacity investments are entirely reversible, therefore the firm can invest in the optimal unrestricted capacity level and charge the optimal corresponding prices ($p_1 = \$3, p_2 = \$2$, and $p_3 = \$2$ as indicated by the arrows in the graph). In the second period, the firm chooses to invest to a high level and lower the price simultaneously, even though the demand volume increases and hence one may expect an increase in price. We observe that in every period the optimal capacity level is non-increasing in the ratio of K/k. When the ratio $K/k = 1.5$ (the dashed line), the firm invests only up to the same level as in the last period (i.e., $z_2 = 3$). When $K/k = 2$ (the dotted line), the firm is inclined not to invest and employs only inventory-oriented strategies (backlogging) in order to deal with the increase in demand. While the firm decides not to invest in additional capacity during the peak-demand period, the higher backlog costs (compared to holding inventories) eventually drive it to invest in capacity in the final period. This strategy is discouraged once the ratio K/k is set to 5.5 (the dashed-dotted line), where the firm invests only up to a certain level in the first period, maintains this level in the subsequent periods, and disinvest when the sales period ends.

5 Joint Capacity Planning and Pricing

In this section we analyze a particular instance of the joint capacity planning and pricing problem when inventory cannot be carried over from period to period and prices can only be decreased

Figure 1: Optimal capacity levels for different K/k ratios.



throughout the life-time of the product. As a result of this no carry-over restriction, firms may not be able to use inventory produced in off-peak periods to absorb peak-demand. In cases where there are no inventory carry-overs we assume that stockouts should be satisfied at the end of the period in which they occur. (Federgruen and Heching (1999) describe such a mechanism as *emergency purchases* or production runs.)

5.1 Main results

Let $f_t^M(z,p)$ denote the maximum expected present value of the total net return that can be earned in periods t up until T, given that period t starts with capacity z and price p (and with no inventory carry-overs). The optimality equations for $t = 1, \ldots, T$ are given by

$$\begin{aligned} f_t^M(z,p) &= \max_{z' \geq 0} \max \left\{ \gamma_t(y,p) - C(z'-z) - h_c z' + \alpha \mathbb{E} f_{t+1}^M(p',z') \ : \ 0 \leq y \leq z', \underline{p} \leq p' \leq p \right\}, \\ f_{T+1}^M(z) &= kz. \end{aligned}$$

The decision variables in the above equation are the price (p') and capacity (z') set in period t. We then have the following result.

Theorem 3 *Assume a firm cannot carry inventories and increase prices from period to period. Then, the following properties hold for all* $t = 1, \ldots, T$*:*

(a) $f_t^M(p,z)$ *is submodular and jointly concave in the state variables* (p,z)*.*

(b) *The decision variables p' and z' are strategic substitutes.*

(c) *The optimal capacity policy is a target interval policy in each period. The Capacity targets $L_t(p)$ and $U_t(p)$ satisfy $L_t(p) \leq U_t(p)$ for each $t = 1, \ldots, T$, and each initial price p.*

(d) *$L_t(p)$ and $U_t(p)$ are non-increasing in p.*

Note that the upper and lower barriers $L_t(\cdot), U_t(\cdot)$ are now functions of the price in the beginning of the period, unlike the case with inventory carry-overs and bi-directional price changes where these barriers were functions of the inventory level in the beginning of the period.

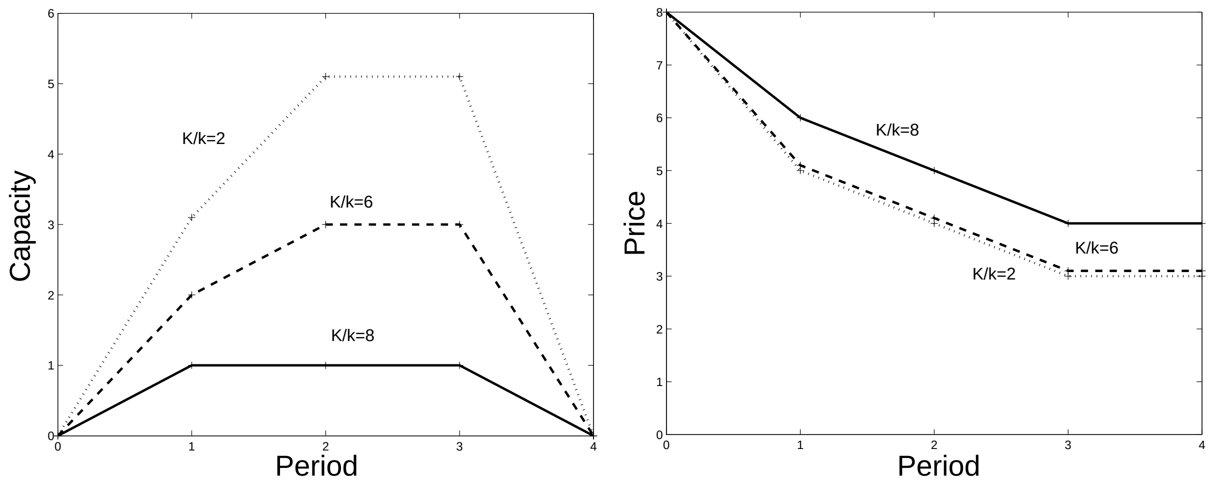
Remark. The model can be extended to treat non-linear demand functions by assuming that $p\mathbb{E}d_t(p, \epsilon_t)$ is concave in p and that $G_t(y, p)$ is jointly concave in (y, p), for all $t = 1, \ldots, T$. The first condition is easily satisfied for a broad family of demand functions. For a discussion of conditions that ensure the joint concavity of $G(\cdot, \cdot)$ see Federgruen and Heching (1999). Note that in contrast to the case with inventory carry-overs (see section 4, the remark following Theorem 2), here an additional condition on $G(\cdot, \cdot)$ is needed.

5.2 Numerical examples

As in Section 4.2, we consider a three period model, introducing the following change to the model parameters. To reflect the fact that the firm cannot carry inventory and is thus inclined to resolve any excess demand within the period, we set $h_t^- = 3, t = 1, 2, 3$. The price restriction is captured by assuming that demand is determined by the following parameters: $a_1 = a_3 = 8, a_2 = 10$ and $b_t = 1, t = 1, 2, 3$.

Example 2: An illustration of the relationship between price markdown and capacity investment decisions for varying values of the capacity investment irreversibility ratio (Figure 2). The figure depicts three capacity investment irreversibility values: $K/k = 2, 6, 8$ (dotted, dashed, and solid lines, respectively). We observe that as long as the ratio is lower than 6, the firm essentially uses the same pricing scheme, charging \$5, \$4 and \$3, and lowers the level of acquired capacity. However, once the ratio increases above 8, the firm utilizes a different pricing scheme, charging \$6, \$5 and \$4 while lowering the capacity level it purchases. Since the firm can foresee that it will not be able to absorb demand using a high level of production (and capacity), and since the firm cannot increase its price in the middle of the product life cycle, it elects to charge

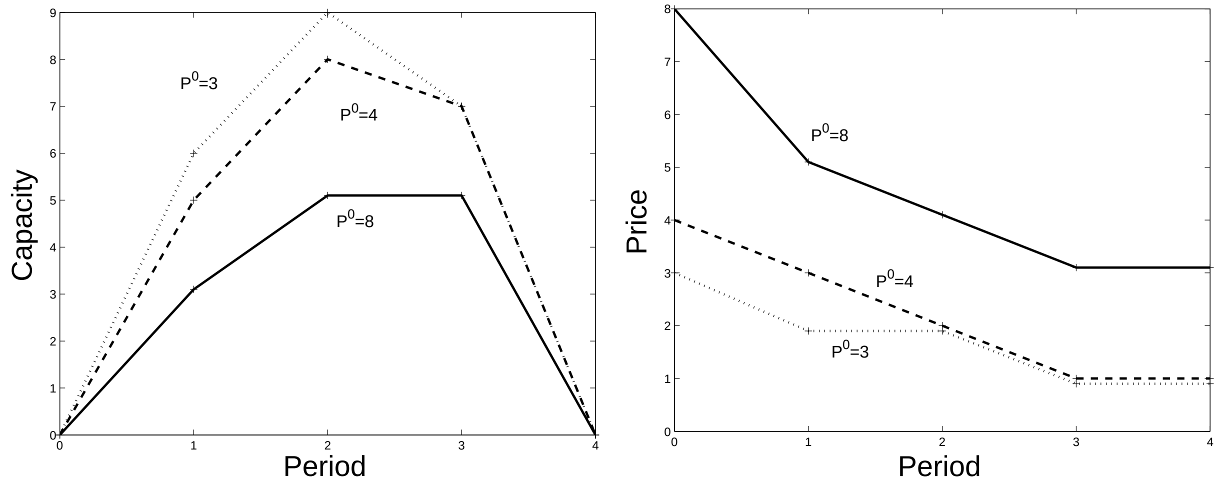
Figure 2: Optimal capacity and price levels for different K/k ratios.



a relatively high price in the first period even though the demand in this period is not greater than other periods. The firm then decreases prices in each subsequent period. In terms of capacity planning: the firm always invests in capacity in the first period, may invest in the second period (to accommodate the peak-demand anticipated in period 2), and "stays-put" in the third period (even tough demand is expected to be lower than in the second period). The above may be viewed as an illustration of complementarity between price and capacity. To wit, the first period commences with a relatively high price (as prices cannot increase over the rest of the time horizon), and a relatively low level of capacity, leading to a high utilization of this capacity. In the second period, the firm increases capacity level to its maximum, and lowers price to attract a higher number of customers. In the third period, since the firm already has acquired a significant level of capacity, it will again lower its price to allow for full utilization of the capacity, even though the expected demand is lower than that in the middle period.

Example 3: An illustration of the relationship between price markdown and capacity investment decisions for varying levels of the initial price (Figure 3). We modify the initial feasible price level in the previous example to reflect changes in the product market, and examine the effect this has on the behavior of the firm. In the three scenarios depicted in Figure 3, the initial prices are \$8, \$4 and \$3 (solid, dashed and dotted lines, respectively). We observe that as the initial price decreases, the firm is inclined to invest in higher levels of capacity to accommodate high demand levels in the middle period, and due to the lower price levels in

Figure 3: Optimal capacity and price levels for different initial price levels.



the middle period. (The dotted line dominates the dashed line which dominates the solid line in the Figure 3(left)). In Figure 3(right) we observe the same patterns of price changes and capacity investment/disinvestment for initial price levels of \$8 and \$4. When the initial price level is set to \$3 (dotted line), the firm does not employ this strictly monotone decreasing price scheme. In particular, during the middle period, in which the excess demand is experienced, the firm employs the same price as in the case of $p^0 = \$4$, but then compensates for this by maintaining the same level of capacity during the next period as well. Note that the resemblance of the graphs in Figures 2 and 3 stems from the demand pattern having a peak in the middle period.

6 Discussion and Qualitative Insights

Price and capacity as strategic substitutes. The fact that price and capacity are strategic substitutes is equivalent to a complementarity relation between the level of capacity investment and the level of price decrease (relative to the maximum price $\overline{p}$). The notion of complementarity that we are referring to is due to Edgeworth, according to which activities are considered *Edgeworth complements* if increasing the level of any one of them results in an increase in the return of engaging more in the other. Milgrom and Roberts (1990, 1995) summarize the principal results of the theory of supermodular optimization which underlies the notion of complementarity, and apply them to study the shift in “modern manufacturing.” Milgrom and Roberts (1990) describe supermodularity

Table 1: Expected Profits: The value of capacity flexibility

K/k	1	2	6	8
Fixed capacity	44.5365	40.5018	28.0212	26.0132
Flexible capacity	44.8183	40.7027	30.0208	29.70307
Percentage increase	0.63 %	0.50 %	7.14 %	14.18 %

as providing a way to formalize the intuitive idea of synergistic effects. In our example, a firm that coordinates sales planning and capacity investment can increase its profits on the basis of the observed complementarity.

Benefits of capacity flexibility in the presence of restrictions on price changes (Table 1). To explore further the importance of capacity flexibility we compare the expected profits of a firm in two configurations: (i) the firm sets its capacity level at the beginning of the life-cycle; and (ii) the firm is capable of adjusting its capacity periodically. In both cases, the firm is allowed only to markdown its price, and cannot carry inventory from period to period. We use the same setting as in Example 3.

We observe in Table 1 that when the cost of adjusting capacity (i.e., the ratio K/k) is low, the value added from capacity flexibility is negligible. The firm can sell the capacity at the end of the life-cycle without incurring any losses and thus will probably invest in the maximum required capacity. As this ratio increases, capacity flexibility becomes more valuable for the firm.

Benefits of coordination between pricing, inventory and capacity decisions (Figures 4 and 5, Table 2). To discuss some of the advantages of coordinating price, inventory, and capacity decisions, we use three simple yet illustrative examples. We consider, again, a firm that produces and sells a single product for three periods, as in Example 1 (see section 4.) The demand parameters are as in that example, while the the ratio of capacity investment cost to disinvestment is set to $K/k = 5.5$. In Figures 4 and 5 we depict the state of inventory in the beginning of each period (left panel), and the optimal capacity levels together with the upper and lower bounds of the target interval, depicted as dashed and dotted lines respectively (right panel). In Figure 4 the noise is taken to be identically zero ($\epsilon \equiv 0$) in each period, while Figure 5 depicts a realization of the same example focusing on a path with a moderate magnitude of noise ($\epsilon \equiv 2$). We note that the optimal policy is computed based on the distributional assumptions pertaining to the noise term.

When the noise is set to zero, the capacity and optimal inventory are set to be constant throughout the product life cycle. One may view this as the long-term plan for the firm, equivalent to the long-run policy derived by Maccini (1984). Even though there is a peak-demand period, the firm handles it via exclusive use of an inventory-oriented policy. In the noisy realization, due to the relatively high price of backlogging and despite the high price of capacity investment-disinvestment, the firm selects to invest in higher capacity in order to cope with both the peak-demand period and the high demand realization. As can be seen in the Figure 5 (left panel), the firm varies its optimal inventory positions hence employing a policy that is both capacity-oriented and inventory-oriented. This can be viewed as a manifestation of the short-term adjustments discussed by Maccini (1984), where due to a relatively extreme realization of demand the firm is inclined to deviate from its long-term plan. However, due to the ability to vary both inventory and capacity, the firm manages to avoid a significant reduction in its profits relative to the case where the error terms equal 0.

To conclude the discussion of the benefits of capacity, pricing and inventory coordination, we compare the following four settings: (i) The firm is capable of setting its price and capacity levels only in the beginning of the life-cycle (and selling the capacity at the end of the life-cycle) and is allowed to replenish its inventory periodically; (ii) The firm sets its price in the beginning of the life-cycle, adjusts its capacity periodically and replenishing its inventory periodically; (iii) The firm sets its capacity level in the beginning of the life-cycle, sets its price level periodically and replenishes its inventory periodically; and (iv) the three decisions: pricing, inventory and capacity are reviewed and adjusted periodically. We use the same parameter setting as in the numerical examples in §5.

In Table 2 we depict the expected profit values for a firm in the above four settings. We observe, again that when the ratio K/k is low, the value due to capacity and price flexibility is relatively low, and this grows with the increase in the capacity irreversibility ratio.

Figure 4: Inventory and capacity level: zero-noise realization.

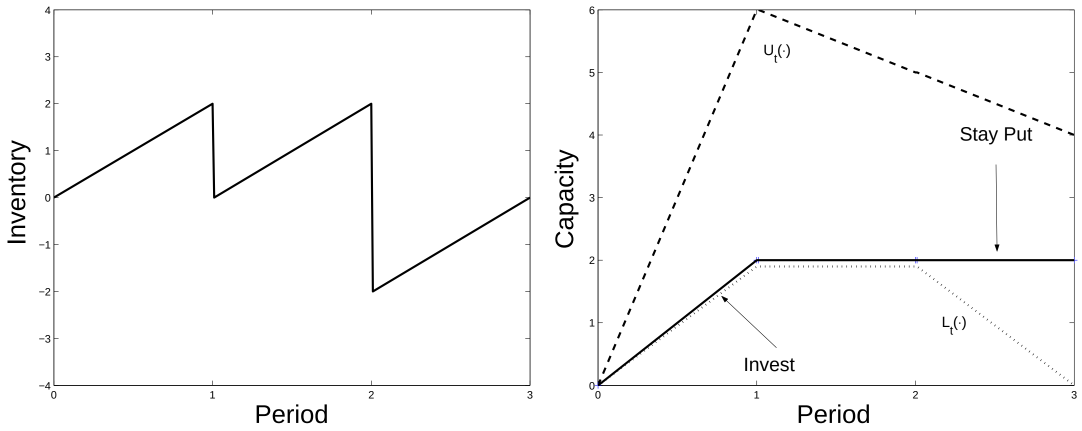


Figure 5: Inventory and capacity levels: noisy realizationf.

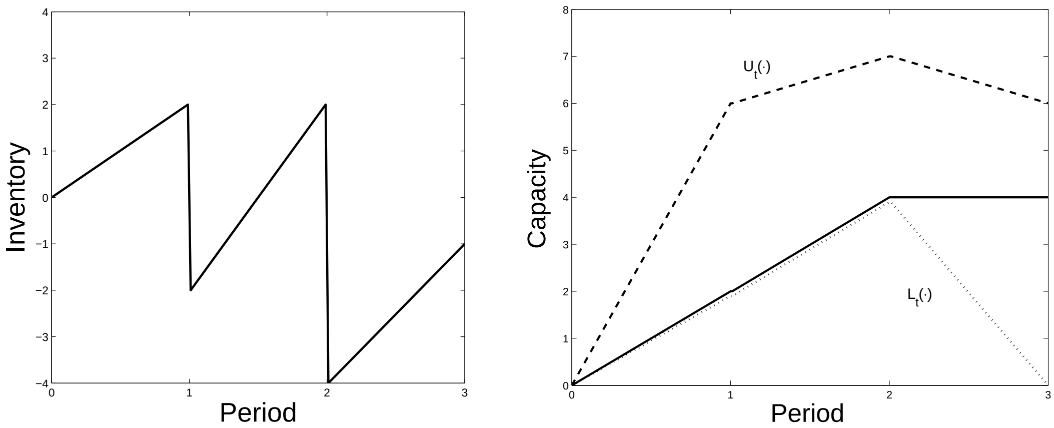


Table 2: Expected Profits: the value of capacity and pricing flexibility

K/k	1	1.5	2	5.5
Fixed price and capacity	14.284	11.9075	10.4436	0.7238
Fixed price	15.0856	13.055	11.0616	3.3933
Fixed capacity	14.3784	12.4096	10.9064	3.6258
Flexible price and capacity	15.2614	13.555	11.6616	3.7052

A Proofs of the Main Results

All notations in this appendix follows that in the paper. The proofs of auxiliary results are deferred to Appendix B.

Proof of Theorem 1: Fix x. We first show that the solution can be expressed in terms of two function $L_t(x)$ and $U_t(x)$ that satisfy the three conditions in Definition 1, and then solve

$$\max_{z'\geq 0} f(x,z) = \max_{z'\geq 0} \left\{\Gamma_t(x,z') - C(z'-z) - h_c z'\right\}.$$

To this end, we need the following result whose proof is deferred to Appendix B.

Lemma 1 *$f_t(x,z)$ is jointly concave for all $t = 1,\ldots,T$.*

Define

$$\begin{aligned} L_t(x) &= \arg\max_{z'\geq 0}\{\Gamma(x,z') - K(z'-z) - h_c z'\} \\ U_t(x) &= \arg\max_{z'\geq 0}\{\Gamma(x,z') - k(z'-z) - h_c z'\} \end{aligned}$$

Let $\nabla_z f_t(x,z)$ denote the subgradient of $f_t(x,z)$ at the point (x,z), i.e., $f_t(x,v) \leq f_t(x,z) + \nabla_z f_t(x,z)(v-z)$,

Lemma 2 *(Royden [38,p. 113]) For all $t = 1,\ldots,T$, $f_t(x,z)$ is continuous and has non-increasing left and right partial derivatives with respect to z which are equal almost everywhere.*

Thus, the subgradient $\nabla_z f_t(x,z)$ is unique and equal to the gradient of $f_t(x,z)$, except on a set of Lebesgue measure zero. Since $f_t(x,z)$ is concave, the first order (sub)differential conditions are sufficient. Thus, we can write

$$L_t(x) = \begin{cases} 0 & \text{if} \quad \nabla_z \Gamma_t(x,z')|_{z'=0} < K - h_c, \\ \sup\{z' : \nabla_z \Gamma(x,z') \geq K - h_c\} & \textit{otherwise}, \end{cases}$$

$$U_t(x) = \begin{cases} \infty & \text{if} \quad \nabla_z \Gamma_t(x,z') > K - h_c \forall z' > 0, \\ \inf\{z' : \nabla_z \Gamma(x,z') \leq K - h_c\} & \textit{otherwise}. \end{cases}$$

Since both $L_t(x)$ and $U_t(x)$ are independent of z, we can partition the space into the following three regions: (i) $z < L_t(x)$; (ii) $L_t(z) \leq z \leq U_t(x)$; and (iii) $z > U_t(x)$. In each of these regions we will compare the three possible decisions: investing, disinvesting and staying put.

Region (i): if the firm decides to invest, by the definition of $L_t(z)$, it is the optimal value. If the firm decides to disinvest, then, since $U_t(x) > L_t(x) > z$, it is better to stay put. However, staying put is inferior to investing in this region, since $L_t(x) > z$ (if staying put were better, then $L_t(x)$ would equal z).

Region (iii): if the firm decides to disinvest, by the definition of $U_t(z)$, it is be the optimal value. If the firm decides to invest, then, since $z > U_t(x) > L_t(x)$, it is is better to stay put. However, staying put is inferior to disinvesting in this region, since $U_t(x) < z$ (if staying put were better, then $U_t(x)$ would equal z.)

Region (ii): if the firm decides to invest, since $z \geq L_t(x)$, it is better to stay put. If the firm decides to disinvest, since $z \leq U_t(x)$, it is better off staying put as well. Therefore it is optimal in this region to stay put.

Thus, we have established the existence of two functions that satisfy the conditions of Definition 1, which completes the proof. ■

Proof of Theorem 2: Fix $t \in \{1, \ldots, T\}$. We will begin by analyzing the relationship between the optimal inventory level after ordering, y_t, and the starting inventory level x_t.

Lemma 3 *If $x \leq \widehat{y}_t(x, z) \leq x + z$, it is optimal to order up to the base-stock level $\widehat{y}_t(x, z)$ and to charge the list-price $\widehat{p}_t(x, z)$; if $x > \widehat{y}_t(x, z)$, it is optimal not to order, and if $\widehat{y}_t(x, z) > x + z$, it is optimal to order z units.*

To prove that a base-stock list-price policy is optimal, it suffices to show that the optimal price to be selected in any given period is non-increasing in the prevailing inventory level. In other words, under higher starting inventory levels, a price is selected that is no larger than the optimal price under a lower starting inventory. Monotonicity of the optimal price level, p_t, depends on the submodularity of the function $a_t(y, p, z)$.

We would like first to show that $a_t(y, p, z)$ is submodular in (y, p). Since the sum of submodular functions is submodular, it suffices to establish submodularity of each of the terms $\gamma_t(y, p)$ and $\mathbb{E} f_{t+1}(y - d_t(p, \epsilon_t), z)$. To show submodularity of $\gamma_t(y, p)$, it suffices, by definition, to show supermodularity of $G_t(y, p)$. Fix ϵ_t. Then, the function $h_t(y - d_t(p, \epsilon_t))$ is supermodular in y and p by the following lemma.

Lemma 4 *If $g(\cdot)$ is a convex function and $h(\cdot)$ in a non-decreasing function, then $g(u + h(v))$ is supermodular in u, v.*

The stated supermodularity therefore applies to the function $G_t(y, p) = \mathbb{E}(y - d_t(p, \epsilon_t))$, and thus to the function $\gamma_t(y, p)$. Since $f_t(x, z)$ is concave in x, by Lemma 4, for fixed ϵ_t, $f_{t+1}(y - d_t(p, \epsilon_t), z)$ is submodular in y and p. Taking expectation preserves this property and hence completes the proof of part (b), i.e., the submodularity of $a_t(y, p, z)$ with respect to y and p.

The decision problem in period t, given capacity z_t (after adjustments) can be viewed as consisting of two stages. In the first stage, the inventory after ordering y_t is chosen, and in the second stage the corresponding price p_t is set. The second stage thus has $\mathcal{S} = \mathbb{R}$ as its state space, and $\mathcal{A} = [\underline{p}, \overline{p}]$ as the set of feasible (price) actions. Since $a_t(y, p, z)$ is strictly concave in (y, p), and the feasible set is convex, the optimal price p_t is unique. Since $a_t(y, p, z)$ is submodular, it follows from Theorem 8-4 in Heyman and Sobel (1984) that the optimal price p_t is nonincreasing in the state y_t, and hence in x. The proof is complete. ■

Proof of Theorem 3: $f^M_{T+1}(z, p) = kz$ is clearly jointly concave and submodular in z, p. We assume that $f^M_{t+1}(z, p)$ is submodular and jointly concave in (z, p), and prove that this implies that for $t \in \{1, \ldots, T\}$, $f_t(z, p)$ is submodular and jointly concave in (z, p). Note that

$$\gamma_t(y, p) = p\mathbb{E}d_t(p, \epsilon_t) - c_t y - G_t(y, p)$$

was shown in the proof of Theorem 2 to be jointly concave and submodular in y, p, thus $\gamma_t(y, p)$ is supermodular and jointly concave in $(-y, p)$. Define

$$g_t(z, p) = \max\{\gamma_t(y, p) \ : \ y \leq z\} = \max\{\gamma_t(y, p) \ : \ -y \geq -z\}$$

We now use the following lemma.

Lemma 5 *if $g(y, v)$ is jointly concave and supermodular in (y, v), then $G(y, u) = \max\Big\{g(y, v) \ : \ v \geq u, \underline{v} \leq v \leq \overline{v}\Big\}$ is jointly concave and supermodular in (y, u).*

Consequently, $g_t(z, p)$ is jointly concave and supermodular in $(-z, p)$, and therefore submodular and jointly concave in (z, p). Let

$$\Gamma_t(z, p) := \alpha\mathbb{E}f^M_{t+1}(z, p) + g_t(z, p).$$

By the induction assumption, $\Gamma_t(z,p)$ is submodular and jointly concave in (z,p), and thus jointly concave and supermodular in $(z,-p)$. Define

$$F(z,p) = \max\left\{\Gamma_t(z,p') \ : \ p' < p, \underline{p} \leq p' \leq \overline{p}\right\} = \max\left\{\Gamma_t(z,p') \ : \ -p' > -p, -\underline{p} \geq -p' \geq -\overline{p}\right\}$$

By Lemma 5, $F(z,p)$ is supermodular and jointly concave in $(z,-p)$, and thus jointly concave and submodular in (z,p). Let

$$\widehat{F}_t(z^B, z^A, p) = F_t(z^A, p) - h_c z^A - C(z^A - z^B)$$

where z^A and z^B are the inventory levels *after* and *before* adjustment (investment / disinvestment), respectively. Now, $C(\cdot)$ is convex, therefore by Lemma 4, $C(z^A - z^B)$ is submodular in (z^A, z^B), and therefore $-C(z^A - z^B)$ is supermodular and jointly concave in (z^A, z^B). Since this function is independent of p, it is (trivially) jointly concave and supermodular in $(z^A, z^B, -p)$. $F(z^A, p)$ is supermodular and jointly concave in (z^A, p), and since it is independent of z^B, we can conclude using the same reasoning that $\widehat{F}(z^A, z^B, -p)$ is supermodular and jointly concave in $(z^A, z^B, -p)$. We then write

$$f_t^M(z,-p) = \max\left\{\widehat{F}_t(z, z', -p) \ : \ z' \geq 0\right\}.$$

Since $\{z' \geq 0\}$ is a lattice and a convex set, $f_t^M(z,-p)$ is supermodular in $(z,-p)$, and its joint concavity and supermodularity are immediate from the the preservation under maximization theorems (Theorem 4.3 of Topkis (1978), and Proposition B-4 of Heyman and Sobel (1984), respectively). Thus, f_t^M is jointly concave and submodular in (z,p), which completes the induction proof and the proofs of parts (a) and (b).

For the proof of part (c) define

$$L_t(p) = \begin{cases} 0 & \text{if} \quad \nabla_z F_t(p,z')|_{z'=0} < K - h_c, \\ \sup\{z' : \nabla_z F_t(p,z) \geq K - h_c\} & \textit{otherwise} \end{cases}$$

$$U_t(p) = \begin{cases} \infty & \text{if} \quad \nabla_z F_t(p,z') > K - h_c, \text{ for all } z' > 0, \\ \inf\{z' : \nabla_z F(p,z') \leq K - h_c\} & \textit{otherwise.} \end{cases}$$

Now repeat the arguments given in the proof of Theorem 1 to conclude that the optimal capacity policy is a target interval policy with $L_t(p)$ and $U_t(p)$ as its barrier functions.

Since $L_t(p)$ and $U_t(p)$ are maximizers of submodular functions in (z,p), it follows (again, from Theorem 8-4 in Heyman and Sobel (1984)) that both $L_t(p)$ and $U_t(p)$ are non-decreasing in p. This completes the proof. ■

B Proof of Auxiliary Results

Proof of Proposition 1: $f_{T+1}(x,z) = kz - h_{T+1}(x)$ is concave in (x,z) since h_{T+1} is convex in x. Fix $t \in \{1,\ldots,T\}$, and suppose that $f_{t+1}(x,z)$ is concave. We shall show that f_t is concave. We first prove that $a_t(y,p,z)$ is jointly concave in (y,p,z). We will prove the concavity in each of its two elements. Fix ϵ_t. Since $d_t(p_t,\epsilon_t)$ is linear in p, $y - d_t(p,\epsilon_t)$ is an affine function of (y,p). By the concavity assumption for $f_{t+1}(x,z)$, and since affine mappings preserve concavity (see Boyd and Vandenberghe (2004) section 3.2.2), $f_{t+1}(y - d_t(p,\epsilon_t))$ is jointly concave. (Note that concavity is preserved under expectation with respect to the random variable ϵ_t.) We now establish that $\gamma_t(y,p) = p\mathbb{E}d_t(p,\epsilon_t) - c_t y - G_t(y,p)$ is jointly concave. First, note that $G_t(y,p)$ is jointly convex. Thus, we are only required to show that $p\mathbb{E}d_t(p,\epsilon_t)$ is jointly concave in (y,p). Since $d_t(p,\epsilon_t)$ is linear and decreasing in p, it is straightforward that if we fix ϵ_t, $pd_t(p,\epsilon_t)$ is concave in p. Again, concavity is preserved under expectation with respect to ϵ_t. Now, note that the set

$$\{(y,p,z,x) \;:\; x \geq 0, z \geq 0, x \leq y \leq x+z, \underline{p} \leq p \leq \overline{p}\} \tag{9}$$

is convex. Thus, by the concavity preservation under maximization theorem (see Proposition B-4, Heyman and Sobel (1984)), $\Gamma(x,z)$ is jointly concave in (x,z). Since $C(\cdot)$ is convex using again, the concavity preservation under maximization theorem, $f_t(x,z)$ is jointly concave, which completes the induction proof. ■

Proof of Lemma 3: Fix $t \in \{1,\ldots,T\}$ and $z \in \mathbb{R}$. Since $a_t(y,p,z)$ is jointly concave in (y,p), $(\widehat{y}_t(x,z), \widehat{p}_t(x,z))$ is the optimal decision pair when $x \leq \widehat{y}_t(x,z) \leq x+z$, i.e., in this region it is optimal to order up to the base stock level $\widehat{y}_t(x,z)$ and to charge the list price $\widehat{p}_t(x,z)$ if $x \leq \widehat{y}_t(x,z) \leq x+z$. Similarly, it is optimal to choose $y_t = x$ if $x > \widehat{y}_t(x,z)$, i.e., not to produce. Now, if $x > \widehat{y}_t(x,z)$, and a decision pair (y,p') is chosen with $y > x$, then for the pair (x,p'') on the line segment connecting $(\widehat{y}_t(x,z), \widehat{p}_t(x,z))$ with (y,p'), $a_t(x,p'',z) \geq a_t(y,p',z)$, using the joint concavity of $a_t(y,p,z)$. Using the same logic, we can show that if $\widehat{y}_t(x,z) > x+z$, it is optimal to set $y_t = x+z$, namely, to produce the maximum possible amount. In particular, if $\widehat{y}_t(x,z) > x_t$ and a decision pair (y,p') is chosen with $y < x+z$, then for the pair $(x+z,p'')$ on the line segment connecting $(\widehat{y}_t(x,z), \widehat{p}_t(x,z))$ with (y,p') we have that, $a_t(x+z,p'',z) \geq a_t(y,p',z)$, using the joint concavity of $a_t(y,p,z)$. We conclude that y_t is nondecreasing in x. This completes the proof. ■

Proof of Lemma 4: Assume without loss of generality that $u_1 > u_2$, and $v_1 > v_2$. Then,

$$\begin{aligned} g(u_1 + h(v_1)) - g(u_2 + h(v_1)) &= g(u_2 + h(v1) + (u_1 - u_2)) - g(u_2 + h(v_1)) \\ &\geq g(u_2 + h(v_2) + (u_1 - u_2)) - g(u_2 + h(v_2)) \\ &= g(u_1 + h(v_2)) - g(u_2 + h(v_2)), \end{aligned}$$

where the inequality follows from the convexity of g and the fact that h is increasing. ■

Proof of Lemma 5: Let $v^*(y)$ denote the smallest maximizer of $g(y, \cdot)$ on $[\underline{v}, \overline{v}]$ (clearly the function has a maximizer on a bounded interval). Since $g(y, v)$ is concave in v, for a given y

$$G(y, u) = \begin{cases} g(y, v^*(y_1)) & \text{if} \quad u \leq v^*(y) \\ g(y, u) & \text{if} \quad v^*(y) \leq u. \end{cases} \tag{10}$$

Since $g(\cdot, \cdot)$ is supermodular, $v^*(y)$ is nondecreasing in y. Therefore, if $y_1 > y_2$, then $v^*(y_1) \geq v^*(y_2)$. Thus, we can write

$$G(y_1, u) - G(y_2, u) = \begin{cases} g(y_1, v^*(y_1)) - g(y_2, v^*(y_2)) & \text{if} \quad u < v^*(y_2) \leq v^*(y_1) \\ g(y_1, v^*(y_1)) - g(y_2, u) & \text{if} \quad v^*(y_2) \leq u \leq v^*(y_1) \\ g(y_1, u) - g(y_2, u) & \text{if} \quad v^*(y_1) \leq u. \end{cases} \tag{11}$$

If $u \leq v^*(y_2)$ then the function is constant. Since for all $u \geq v^*(y_2)$, $g(y_2, v^*(y_2)) \geq g(y_2, u)$ by concavity, thus the function $g(y_1, v^*(y_1)) - f(y_2, u)$ is non-decreasing. For $u > v^*(y_1)$, $G(y_1, u) - G(y_2, u) = g(y_1, u) - g(y_2, u)$ has increasing differences in view of g having increasing differences. Joint concavity of $G(\cdot, \cdot)$ is immediate from the concavity preservation under maximization theorem (see Proposition B-4, Heyman and Sobel (1984)). This completes the proof. ■

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