“Rigid Production Networks”
Online Appendix

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This online appendix contains the proofs and derivations omitted from the main body of the paper.

Proof of Proposition 1

We characterize the equilibrium via backward induction. Starting with the firms' decisions at $t=1$, recall that firms optimally choose their labor input and flexible intermediate input quantities to meet the realized demand. Taking the prices, its realized demand, and their rigid input demands as given, firm k in industry i faces the following cost-minimization problem:

$$\begin{aligned} \min_{l_{ik},\{x_{ij,k}\}_{j\in\mathcal{F}_i}} \quad & wl_{ik}+\sum_{j\in\mathcal{F}_i} p_j x_{ij,k} \\ \text{subject to} \quad & y_{ik}=z_i\zeta_i l_{ik}^{\alpha_i}\prod_{j=1}^{n} x_{ij,k}^{a_{ij}}. \end{aligned}$$

Solving this problem implies that the firm's expenditure on labor and flexible input demands are given by

$$wl_{ik}=\alpha_i(y_{ik}/Q_{ik})^{1/(1-\sum_{j\in\mathcal{R}_i}a_{ij})} \tag{A.1}$$

$$p_j x_{ij,k}=a_{ij}(y_{ik}/Q_{ik})^{1/(1-\sum_{j\in\mathcal{R}_i}a_{ij})} \qquad \text{for all } j\in\mathcal{F}_i \tag{A.2}$$

respectively, where Q_{ik} only depends on the firm's productivity, its input prices, the nominal wage, and the intermediate input decisions that are sunk by $t=1$:

$$Q_{ik}=z_i w^{-\alpha_i}\prod_{j\in\mathcal{F}_i}p_j^{-a_{ij}}\prod_{j\in\mathcal{R}_i}(x_{ij,k}/a_{ij})^{a_{ij}}. \tag{A.3}$$

Therefore, the firm faces the following problem when deciding on its nominal price at $t=1$:

$$\max_{p_{ik}} \quad (1-\tau_i)p_{ik}y_{ik}-wl_{ik}-\sum_{j\in\mathcal{F}_i}p_j x_{ij,k} \tag{A.4}$$

$$\text{subject to} \quad y_{ik}=(p_{ik}/p_i)^{-\theta_i}y_i \tag{A.5}$$

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as well as the labor and intermediate input demand constraints (A.1) and (A.2). The first-order conditions of this optimization implies that

$$(1-\tau_i)(1-\theta_i)(p_{ik}/p_i)^{-\theta_i}y_i - (y_{ik}/Q_{ik})^{1/(1-\sum_{j\in\mathcal{R}_i}a_{ij})}\,\frac{1}{y_{ik}}\frac{\mathrm{d}y_{ik}}{\mathrm{d}p_{ik}} = 0. \tag{A.6}$$

Solving this optimization problem implies that the nominal price set by firm k in industry i is given by

$$p_{ik} = \left((p_i^{\theta_i}y_i)^{\sum_{j\in\mathcal{R}_i}a_{ij}}Q_{ik}^{-1}\right)^{1/(1+(\theta_i-1)\sum_{j\in\mathcal{R}_i}a_{ij})}, \tag{A.7}$$

where Q_{ik} is given by (A.3) and we are using the assumption that $\tau_i = 1/(1-\theta_i)$. With the firm's price and quantity decisions at $t=1$ in hand, we can now turn to the rigid intermediate input decisions of the firm at $t=0$. Recall that firms choose their rigid intermediate inputs in order to maximize the expected real value of their profits given their information set. Therefore, firm k in industry i faces the following optimization problem at $t=0$:

$$\max_{\{x_{ij,k}\}_{j\in\mathcal{R}_i}} \quad \mathbb{E}_i\left[\frac{U'(C)}{P}\left((1-\tau_i)p_{ik}y_{ik} - wl_{ik} - \sum_{j=1}^{n}p_jx_{ij,k}\right)\right]$$

subject to constraints (A.1)–(A.2), (A.5), and (A.7), where $\mathbb{E}_i[\cdot]$ denotes the expectation operator with respect to the information set of firms in industry i, $U'(C)=1/C$ is the household's marginal utility, and P is the price of the consumption good bundle. Note that, $PC=m$. Therefore, the first-order condition of the firm's problem at $t=0$ is given by

$$\begin{aligned}\mathbb{E}_i&\left[\frac{1}{m}\left((1-\tau_i)(1-\theta_i)(p_{ik}/p_i)^{-\theta_i}y_i - (y_{ik}/Q_{ik})^{1/(1-\sum_{j\in\mathcal{R}_i}a_{ij})}\,\frac{1}{y_{ik}}\frac{\mathrm{d}y_{ik}}{\mathrm{d}p_{ik}}\right)\frac{\mathrm{d}p_{ik}}{\mathrm{d}x_{ij,k}}\right]\\ &\qquad\qquad +\mathbb{E}_i\left[\frac{1}{m}\left((y_{ik}/Q_{ik})^{1/(1-\sum_{j\in\mathcal{R}_i}a_{ij})}\,\frac{1}{Q_{ik}}\frac{\mathrm{d}Q_{ik}}{\mathrm{d}x_{ij,k}} - p_j\right)\right] = 0.\end{aligned}$$

Equation (A.6) implies that the first term on the right-hand side of the above equation is equal to zero. Furthermore, note that (A.3) implies that $\mathrm{d}Q_{ik}/\mathrm{d}x_{ij,k} = a_{ij}Q_{ik}/x_{ij,k}$. Therefore,

$$x_{ij,k} = \frac{a_{ij}}{\mathbb{E}_i[p_j/m]}\mathbb{E}_i\left[\frac{1}{m}\,(y_{ik}/Q_{ik})^{1/(1-\sum_{j\in\mathcal{R}_i}a_{ij})}\right] \qquad \text{for all } j\in\mathcal{R}_i. \tag{A.8}$$

To simplify the above, note that given that all firms within the same industry are symmetric, they all set the same prices and produce the same quantities, that is, $p_{ik}=p_i$ and $y_{ik}=y_i$. Therefore, we can drop the firm index k from (A.7) and solve for Q_{ik} in terms of the price of firms in industry i:

$$Q_{ik} = (p_iy_i)^{\sum_{j\in\mathcal{R}_i}a_{ij}}/p_i. \tag{A.9}$$

Plugging this expression back into (A.2) and (A.8), we obtain

$$x_{ij,k} = \begin{cases} a_{ij}\lambda_i m/p_j & \text{if } j\in\mathcal{F}_i \\ a_{ij}\mathbb{E}_i[\lambda_i]/\mathbb{E}_i[p_j/m] & \text{if } j\in\mathcal{R}_i, \end{cases} \tag{A.10}$$

where we are using the fact that the Domar weight of industry i is given by $\lambda_i = p_i y_i/m$. This expression together with the market-clearing condition (6) for sectoral good i implies that

$$y_i = c_i + \sum_{j\in\mathcal{F}_i} a_{ji}\frac{\lambda_j}{p_i/m} + \sum_{j\in\mathcal{R}_i} a_{ji}\frac{\mathbb{E}_j[\lambda_j]}{\mathbb{E}_j[p_i/m]}.$$

Multiplying both sides of the above equation by p_i/m and using the fact that $c_i = \beta_i m/p_i$—which is a consequence of the household's optimization problem—then establishes (11).

We next establish (10). To this end, note that equations (A.3) and (A.10) imply that

$$Q_{ik} = z_i w^{-\alpha_i} \prod_{j\in\mathcal{F}_i} p_j^{-a_{ij}} \prod_{j\in\mathcal{R}_i} (\mathbb{E}_i[\lambda_i]/\mathbb{E}_i[p_j/m])^{a_{ij}}.$$

Combining the above equation with the expression for Q_{ik} in (A.9) then establishes (10). □

Proof of Lemma 1

As a first observation, note that combining (A.1) with the expression for Q_{ik} in (A.9) implies that the labor demand of firm k in industry i is given by $l_{ik} = \alpha_i\lambda_i m/w$. Therefore, aggregate demand for labor in the economy is equal to

$$\sum_{i=1}^n \int_0^1 l_{ik}\mathrm{d}k = (m/w)\sum_{i=1}^n \alpha_i\lambda_i.$$

Furthermore, note that the first-order conditions of the household's problem imply that total labor supply is given by $L = (m\chi/w)^{-\eta}$. Combining the above two equations therefore implies that the labor market equilibrium condition (5) is given by

$$(w-\bar{w})\left((m\chi/w)^{-\eta} - (m/w)\sum_{i=1}^n \alpha_i\lambda_i\right) = 0, \qquad w \geq \bar{w}, \qquad \chi m/w \leq \left(\frac{1}{\chi}\sum_{i=1}^n \alpha_i\lambda_i\right)^{-1/(1+\eta)}.$$

We consider two separate cases. First, suppose that $w > \bar{w}$. The first condition above implies that $w = m\chi^{\eta/(1+\eta)}\left(\sum_{i=1}^n \alpha_i\lambda_i\right)^{1/(1+\eta)}$. This is consistent with the original conjecture as long as $\bar{w} < m\chi^{\eta/(1+\eta)}\left(\sum_{i=1}^n \alpha_i\lambda_i\right)^{1/(1+\eta)}$. As the second case, suppose $w = \bar{w}$. In that case, the last inequality above implies that $\bar{w} \geq m\chi^{\eta/(1+\eta)}\left(\sum_{i=1}^n \alpha_i\lambda_i\right)^{1/(1+\eta)}$. Putting the two cases together establishes (13). Finally, note that taking $\eta \to \infty$ in (13) implies that $w = \max\{\chi m, \bar{w}\}$. □

Proof of Proposition 2

We prove this result by establishing that the optimality conditions corresponding to the planner's problem coincide with the equilibrium conditions in equations (10)–(13). As a first observation, note that since all firms in the same industry have identical production technologies and information sets, we can drop the firm index k in the planner's problem.

To express the planner's problem, let

$$s = (z, m, (\omega_1, \ldots, \omega_n)) \in S = \mathbb{R}_+^{n+1} \times \Omega_1 \times \cdots \times \Omega_n$$

denote the aggregate state of the economy, consisting of all realized productivity and demand shocks, as well as the cross-sectional profile of signals, where $\omega_i \in \Omega_i$ denotes the component of the state observable to firms in industry i. To ensure that the planner is subject to the same information and quantity adjustment frictions as the firms, we impose the following measurability constraint on the quantities: if j is a rigid input of industry i (so that $j \in \mathcal{R}_i$), then x_{ij} can be contingent on ω_i, but not on the aggregate state s. We capture this measurability constraint by denoting corresponding input quantity by $x_{ij}(\omega_i)$. In contrast, if j is a flexible input for firms in industry i (so that $j \in \mathcal{F}_i$), then x_{ij} can be contingent on the economy's aggregate state, in which case we denote this quantity by $x_{ij}(s)$. Finally, note that since labor supply, labor demand, and consumption are not subject to informational frictions, they can depend on the economy's aggregate state. We therefore denote the corresponding quantities by $l_i(s)$, $L(s)$, and $c_i(s)$, respectively.

Using the above notation, we can now express the planner's problem as follows. The planner maximizes the household's expected utility

$$\int_{s\in S}\left(\sum_{i=1}^{n}\beta_i \log c_i(s) - \chi\frac{L^{1+1/\eta}(s)}{1+1/\eta}\right)\mathrm{d}G(s) \tag{A.11}$$

subject to the following resource and technology constraints:

$$y_i(s) = c_i(s) + \sum_{j:i\in\mathcal{R}_j} x_{ji}(\omega_j) + \sum_{j:i\in\mathcal{F}_j} x_{ji}(s) \tag{A.12}$$

$$L(s) = \sum_{i=1}^{n} l_i(s) \tag{A.13}$$

$$y_i(s) = z_i F_i(l_i(s), \{x_{ij}(s)\}_{j\in\mathcal{F}_i}, \{x_{ij}(\omega_i)\}_{j\in\mathcal{R}_i}),, \tag{A.14}$$

where $G(s)$ denotes the probability distribution of the economy's aggregate state and F_i denotes the production function of firms in industry i and is given by (1). The Lagrangian corresponding to the above problem is thus given by

$$\begin{aligned}\mathcal{L} = &\int_{s\in S}\left(\sum_{i=1}^{n}\beta_i \log c_i(s) - \chi\frac{L^{1+1/\eta}(s)}{1+1/\eta}\right)\mathrm{d}G(s) + \int_{s\in S}\nu_0(s)\left(L(s) - \sum_{i=1}^{n} l_i(s)\right)\mathrm{d}G(s)\\ &+\sum_{i=1}^{n}\int_{s\in S}\psi_i(s)\Big(y_i(s) - c_i(s) - \sum_{j:i\in\mathcal{R}_j} x_{ji}(\omega_j) - \sum_{j:i\in\mathcal{F}_j} x_{ji}(s)\Big)\mathrm{d}G(s)\\ &+\sum_{i=1}^{n}\int_{s\in S}\nu_i(s)\Big(z_i F_i(l_i(s), \{x_{ij}(s)\}_{j\in\mathcal{F}_i}, \{x_{ij}(\omega_i)\}_{j\in\mathcal{R}_i}) - y_i(s)\Big)\mathrm{d}G(s).\end{aligned}$$

where $\nu_i(s)\mathrm{d}G(s)$ is the Lagrange multiplier corresponding to good i's resource constraint (A.12), $\nu_0(s)\mathrm{d}G(s)$ is the multiplier corresponding to labor resource constraint (A.13), and $\psi_i(s)\mathrm{d}G(s)$ is the multiplier for industry i's technology constraint, (A.14). Therefore, the first-order conditions with respect to $c_i(s)$, $L(s)$, and $y_i(s)$ are given by

$$\beta_i/c_i(s) = \psi_i(s), \qquad \chi L^{1/\eta}(s) = \nu_0(s), \qquad \psi_i(s) = \nu_i(s), \tag{A.15}$$

respectively, whereas the first-order conditions with respect to $l_i(s)$ and $x_{ij}(s)$ for $j \in \mathcal{F}_i$ are given by

$$\nu_0(s) = \nu_i(s) z_i \frac{\partial F_i}{\partial l_i}(s) = \alpha_i \nu_i(s) y_i(s)/l_i(s) \tag{A.16}$$

$$\psi_j(s) = \nu_i(s) z_i \frac{\partial F_i}{\partial x_{ij}}(s) = a_{ij} \nu_i(s) y_i(s)/x_{ij}(s), \tag{A.17}$$

respectively. Finally, the first-order condition with respect to the rigid input $x_{ij}(\omega_i)$ is given by

$$\int_{s\in\Omega_i} \psi_j(s) \mathrm{d}G(s) = a_{ij} \int_{s\in\Omega_i} \nu_i(s) y_i(s)/x_{ij}(\omega_i) \mathrm{d}G(s),$$

where $\Omega_i \subseteq S$ denotes the subset of states with corresponding element ω_i. Note that dividing both sides of the above equation by $G(\Omega_i)$ leads to

$$\mathbb{E}_i[\psi_j(s)] = a_{ij} \mathbb{E}_i[\nu_i(s) y_i(s)]/x_{ij}(\omega_i). \tag{A.18}$$

Plugging in the expressions for $c_i(s)$, $x_{ij}(s)$, and $x_{ij}(\omega_i)$ in (A.15), (A.17), and (A.18) into the resource constraint (A.12) implies that

$$\psi_i(s) y_i(s) = \beta_i + \sum_{j:i\in\mathcal{R}_j} a_{ji} \psi_i(s) \mathbb{E}_j[\psi_j(s) y_j(s)]/\mathbb{E}_j[\psi_i(s)] + \sum_{j:i\in\mathcal{F}_j} a_{ji} \psi_j(s) y_j(s), \tag{A.19}$$

where we are using the fact that $\nu_i(s) = \psi_i(s)$, established in (A.15). Next, note that plugging the same expressions and the expression for $l_i(s)$ in (A.16) into the technology constraint in (A.14) leads to

$$y_i(s) = z_i \left(\psi_i(s) y_i(s)/\nu_0(s)\right)^{\alpha_i} \prod_{j\in\mathcal{F}_i} \left(\psi_i(s) y_i(s)/\psi_j(s)\right)^{a_{ij}} \prod_{j\in\mathcal{R}_i} \left(\mathbb{E}_i[\psi_i(s) y_i(s)]/\mathbb{E}_i[\psi_j(s)]\right)^{a_{ij}}. \tag{A.20}$$

Finally, plugging the expressions for $L(s)$ and $l_i(s)$ in (A.15) and (A.16) into the resource constraint for labor (A.13) implies that

$$\sum_{i=1}^{n} \alpha_i \psi_i(s) y_i(s) = v_0^{1+\eta}(s)/\chi^{\eta}. \tag{A.21}$$

The proof is complete once we verify that equations (A.19)–(A.21) coincide with equilibrium conditions (10)–(13). We do so by a simple change of variables. Let

$$\lambda_i(s) = \psi_i(s) y_i(s), \qquad p_i(s) = \psi_i(s) m(s), \qquad w(s) = \nu_0(s) m(s),$$

where $m(s)$ is an arbitrary function. Using this change of variables, it is then immediate to verify that, as long as the downward nominal wage rigidity constraint does not bind (that is $w > \bar{w}$), then equations (A.19)–(A.21) reduce to (10)–(13). □

An Auxiliary Result

We now state and prove a result that provides an exact expression for aggregate output in terms of model primitives and the nominal wage when there is only a single rigid industry. We will use this result in proving Propositions 3 and 4.

Proposition A.1. *If r is the only industry that is subject to frictions and Assumption 1 is satisfied, then,*

$$\log C = \sum_{j=1}^{n} \lambda_j^{\mathrm{ss}} \log z_j - \log(w/m) - \lambda_r^{\mathrm{ss}} \sum_{j\in\mathcal{R}_r} a_{rj}\mathbb{K}_r\left(\log(w/m) - \sum_{s=1}^{n} \ell_{js}\log z_s\right), \tag{A.22}$$

where $\mathbb{K}_r(x) = \log \mathbb{E}_r[\exp(x)] - x$.

Proof. We first show that $\lambda_r = \lambda_r^{\mathrm{ss}}$. Since industry r is the only industry subject to informational frictions, equation (11) implies that

$$\lambda_i = \beta_i + \sum_{j=1}^{n} a_{ji}\lambda_j + a_{ri}\left(\mathbb{E}_r[\lambda_r]\frac{p_i/m}{\mathbb{E}_r[p_i/m]} - \lambda_r\right)\mathbb{I}_{\{i\in\mathcal{R}_r\}}. \tag{A.23}$$

Taking expectations from both sides of the above equation with respect to the information set of industry r implies that $\mathbb{E}_r[\lambda_i] = \beta_i + \sum_{j=1}^{n} a_{ji}\mathbb{E}_r[\lambda_j]$ for all i. Solving this system of equations for $\mathbb{E}_r[\lambda_i]$ implies that $\mathbb{E}_r[\lambda_i] = \lambda_i^{\mathrm{ss}}$, where is the steady-state Domar weight of industry i. Consequently, we can rewrite equation (A.23) as follows:

$$\lambda_i = \beta_i + \sum_{j=1}^{n} a_{ji}\lambda_j + a_{ri}\left(\lambda_r^{\mathrm{ss}}\frac{p_i/m}{\mathbb{E}_r[p_i/m]} - \lambda_r\right)\mathbb{I}_{\{i\in\mathcal{R}_r\}},$$

Furthermore, note that the steady-state Domar weights satisfy the following system of equations: $\lambda_i^{\mathrm{ss}} = \beta_i + \sum_{j=1}^{n} a_{ji}\lambda_j^{\mathrm{ss}}$ for all i. Subtracting this equation from the previous one therefore implies that

$$\Delta_i = \sum_{j=1}^{n} a_{ji}\Delta_j + a_{ri}\left(\lambda_r^{\mathrm{ss}}\frac{p_i/m}{\mathbb{E}_r[p_i/m]} - \lambda_r\right)\mathbb{I}_{\{i\in\mathcal{R}_r\}}.$$

where $\Delta_i = \lambda_i - \lambda_i^{\mathrm{ss}}$. Solving the above system of equations for Δ_i implies that

$$\Delta_i = \sum_{j=1}^{n} \ell_{ji}a_{rj}\left(\lambda_r^{\mathrm{ss}}\frac{p_j/m}{\mathbb{E}_r[p_j/m]} - \lambda_r\right)\mathbb{I}_{\{j\in\mathcal{R}_r\}}, \tag{A.24}$$

where ℓ_{ji} denotes the (j,i) element of the economy's Leontief inverse $\mathbf{L} = (\mathbf{I}-\mathbf{A})^{-1}$. Setting $i = r$ in the above equation and using Assumption 1 then implies that the right-hand side of the above equation is equal to zero, thus establishing that $\lambda_r = \lambda_r^{\mathrm{ss}}$.

Next, note that since industry r is the only industry that is subject to frictions, equation (10) implies that the (log) nominal price of industry $i \neq r$ is given by

$$\log p_i = -\log z_i + \alpha_i \log w + \sum_{j=1}^{n} a_{ij}\log p_j.$$

Let $\tilde{p} \in \mathbb{R}^{n-1}$ denote the vector of nominal prices for all industries $i \neq r$ and let $\tilde{\mathbf{A}} \in \mathbb{R}^{(n-1)\times(n-1)}$ denote the sub-block of the input-output matrix $\mathbf{A}$ corresponding to all industries except for r. Writing the above equation in vector form therefore implies that $\log\tilde{p} = -\log\tilde{z} + \tilde{\alpha}\log w + \tilde{\mathbf{A}}\log\tilde{p} + \tilde{a}_r\log p_r$, where $\tilde{\alpha}$ and $\tilde{z}$ denote the vectors of labor shares and productivity shocks for all $i \neq r$ and $\tilde{a}_r \in \mathbb{R}^{n-1}$ is a vector with elements a_{is} for all $i \neq r$. Consequently,

$$\log\tilde{p} = \tilde{\mathbf{L}}\tilde{\alpha}\log w - \tilde{\mathbf{L}}\log\tilde{z} + \tilde{\mathbf{L}}\tilde{a}_r\log p_r,$$

where $\tilde{\mathbf{L}} = (\mathbf{I} - \tilde{\mathbf{A}})^{-1}$. Under Assumption 1, the elements of $\tilde{\mathbf{L}}$ can be expressed in terms of the elements of the economy's Leontief inverse $\mathbf{L}$. In particular, $\tilde{\ell}_{ij} = \ell_{ij} - \ell_{ir}\ell_{rj}$ for all $i, j \neq r$. Hence,

$$\log p_i = \log w \sum_{j\neq r}(\ell_{ij} - \ell_{ir}\ell_{rj})\alpha_j - \sum_{j\neq r}(\ell_{ij} - \ell_{ir}\ell_{rj}) \log z_j + \log p_r \sum_{j\neq r}(\ell_{ij} - \ell_{ir}\ell_{rj})a_{jr}$$

for all $i \neq r$. Consequently,

$$\log p_i = (1 - \ell_{ir}) \log w + \ell_{ir} \log p_r - \sum_{j\neq r}(\ell_{ij} - \ell_{ir}\ell_{rj}) \log z_j \tag{A.25}$$

for all $i \neq r$, where we are using the fact that $\sum_{j=1}^{n} \ell_{ij}\alpha_j = 1$ for all i and $\ell_{rr} = 1$, the latter of which is a consequence of Assumption 1. The above equation expresses all prices in terms of the price of industry r and the nominal wage. With the above in hand, we can therefore obtain an expression for aggregate output in terms of the nominal price of industry r. In particular, the fact that $\log C = \log m - \sum_{i=1}^{n} \beta_i \log p_i$ together with (A.25) implies that

$$\log C = \log(m/w) - \lambda_r^{\text{ss}} \log(p_r/w) + \sum_{j\neq r}(\lambda_j^{\text{ss}} - \lambda_r^{\text{ss}}\ell_{rj}) \log z_j, \tag{A.26}$$

where λ_j^{ss} denotes the steady-state Domar weight of industry j. Therefore, to obtain the expression for aggregate output is sufficient to characterize $\log p_r$. To this end, note that setting $i = r$ in equation (10) implies that

$$\log p_r = -\log z_r + \alpha_r \log w + \sum_{j\in\mathcal{F}_r} a_{rj} \log p_j + \log m \sum_{j\in\mathcal{R}_r} a_{rj} + \sum_{j\in\mathcal{R}_r} a_{rj} \log \mathbb{E}_r[p_j/m],$$

where we are also using the fact that $\lambda_r = \lambda_r^{\text{ss}}$. Replacing for $\log p_j$ from (A.25) for all $j \neq r$ into the above equation and using the implication of Assumption 1 that $a_{rj}\ell_{jr} = 0$ for all $j \in \mathcal{F}_r$ implies that

$$\begin{aligned}\log(p_r/m) = -\log z_r + \left(\alpha_r + \sum_{j\in\mathcal{F}_r} a_{rj}\right) \log(w/m) - \sum_{j\in\mathcal{F}_r}\sum_{s=1}^{n} a_{rj}\ell_{js} \log z_s \\ + \sum_{j\in\mathcal{R}_r} a_{rj} \log \mathbb{E}_r \left[\exp\left(\log(w/m) - \sum_{s=1}^{n} \ell_{js} \log z_s\right)\right].\end{aligned} \tag{A.27}$$

Plugging the above into the expression for $\log C$ in (A.26) and using Assumption 1 then establishes (A.22). □

Proof of Proposition 3

Proof of part (a) Recall from the proof of Proposition A.1 that $\Delta_i = \lambda_i - \lambda_i^{\text{ss}}$ satisfies (A.24). As a result,

$$\sum_{i=1}^{n} \alpha_i\lambda_i = \sum_{i=1}^{n} \alpha_i(\lambda_i^{\text{ss}} + \Delta_i) = 1 + \lambda_r^{\text{ss}} \sum_{j\in\mathcal{R}_r} a_{rj} \left(\frac{p_j/m}{\mathbb{E}_r[p_j/m]} - 1\right),$$

where we are using $\sum_{i=1}^{n} \alpha_i\lambda_i^{\text{ss}} = \sum_{i=1}^{n} \alpha_i\ell_{ji} = 1$ and the fact that $\lambda_r = \lambda_r^{\text{ss}}$, established in the proof of Proposition A.1. Therefore, to a first-order approximation

$$\log\left(\sum_{i=1}^{n} \alpha_i\lambda_i\right) = \lambda_r^{\text{ss}} \sum_{j\in\mathcal{R}_r} a_{rj} \left(\frac{p_j/m}{\mathbb{E}_r[p_j/m]} - 1\right) = \lambda_r^{\text{ss}} \sum_{j\in\mathcal{R}_r} a_{rj} \left(\log(p_j/m) - \mathbb{E}_r[\log(p_j/m)]\right).$$

Combining the above with equation (13), together with the assumption that the downward constraint on nominal wage does not bind, implies that

$$\log(w/m) = \frac{\eta}{1+\eta}\log\chi + \frac{1}{1+\eta}\lambda_r^{\text{ss}}\sum_{j\in\mathcal{R}_r} a_{rj}\Big(\log(p_j/m) - \mathbb{E}_r[\log(p_j/m)]\Big). \tag{A.28}$$

Next, recall from the proof of Proposition A.1 that $\log(p_r/m)$ is given by (A.27). Thus, to a first-order approximation,

$$\begin{aligned}\log(p_r/m) = &-\log z_r + \left(\alpha_r + \sum_{j\in\mathcal{F}_r} a_{rj}\right)\log(w/m) - \sum_{j\in\mathcal{F}_r}\sum_{s=1}^{n} a_{rj}\ell_{js}\log z_s\\ &+ \sum_{j\in\mathcal{R}_r} a_{rj}\mathbb{E}_r\left[\log(w/m) - \sum_{s=1}^{n}\ell_{js}\log z_s\right].\end{aligned}$$

Plugging this back into the expression for $\log p_i$ in (A.25) we get

$$\begin{aligned}\log(p_i/m) = \log(w/m) &- \sum_{j=1}^{n}\ell_{ij}\log z_j - \ell_{ir}\left(\sum_{j\in\mathcal{R}_r} a_{rj}\right)\Big(\log(w/m) - \mathbb{E}_r[\log(w/m)]\Big)\\ &- \ell_{ir}\sum_{j\in\mathcal{R}_r} a_{rj}\sum_{p=1}^{n}\ell_{jp}\mathbb{E}_r\left([\log z_p] - \log z_p\right)\end{aligned}$$

for all $i \neq r$. Taking expectations from both sides of the above equation and subtracting it from both sides therefore implies that

$$\begin{aligned}\sum_{i\in\mathcal{R}_r} a_{ri}(\log(p_i/m) - \mathbb{E}_r[\log(p_i/m)]) = &\sum_{i\in\mathcal{R}_r} a_{ri}\Big(\log(w/m) - \mathbb{E}_r[\log(w/m)]\Big)\\ &- \sum_{i\in\mathcal{R}_r} a_{ri}\sum_{j=1}^{n}\ell_{ij}(\log z_j - \mathbb{E}_r[\log z_j]),\end{aligned}$$

Note that (A.28) implies that $\mathbb{E}_r[\log(w/m)] = \frac{\eta}{1+\eta}\log\chi$. Therefore, we can rewrite the above equation as follows:

$$\begin{aligned}\sum_{i\in\mathcal{R}_r} a_{ri}(\log(p_i/m) - \mathbb{E}_r[\log(p_i/m)]) = &\sum_{i\in\mathcal{R}_r} a_{ri}\Big(\log(w/m) - \frac{1}{1+1/\eta}\log\chi\Big)\\ &- \sum_{i\in\mathcal{R}_r}\sum_{j=1}^{n} a_{ri}\ell_{ij}(\log z_j - \mathbb{E}_r[\log z_j]),\end{aligned}$$

Combining the above equation with (A.28) and solving for $\log(w/m)$ we obtain,

$$\log(w/m) = \frac{1}{1+1/\eta}\log\chi - \frac{\lambda_r^{\text{ss}}}{1+\eta-\lambda_r^{\text{ss}}\sum_{i\in\mathcal{R}_r} a_{ri}}\sum_{i\in\mathcal{R}_r}\sum_{j=1}^{n} a_{ri}\ell_{ij}(\log z_j - \mathbb{E}_r[\log z_j]).$$

Now, plugging the above expression into the expression for $\log C$ in (A.22) and performing a first-order approximation establishes (14). □

Proof of part (b) Recall from Proposition A.1 that log aggregate output is given by (A.22). Furthermore, note that by Lemma 1, when labor supply is fully elastic and the downward constraint on the nominal wage does not bind, $\log(w/m) = \log\chi$. Therefore, the expression in (A.22) simplifies as follows:

$$\log C = \sum_{j=1}^{n} \lambda_j^{\text{ss}} \log z_j - \log\chi - \lambda_r^{\text{ss}} \sum_{j\in\mathcal{R}_r} a_{rj}\mathbb{K}_r\left(-\sum_{s=1}^{n} \ell_{js}\log z_s\right),$$

where we are using the fact that $\mathbb{K}_r(x+a) = \mathbb{K}_r(x)$ for any constant a. Noting that $\log C^* = \sum_{j=1}^{n}\lambda_j^{\text{ss}}\log z_j - \log\chi$ then establishes (15). □

Proof of Proposition 4

Recall from Proposition A.1 that log aggregate output is given by (A.22). Therefore, when the downward constraint on the nominal wage binds (that is, $w = \bar{w}$) and the absence of productivity shocks, the expression for log aggregate output reduces to

$$\log C = \log m - \log\bar{w} - \mathbb{K}_r\left(-\log m\right)\lambda_r^{\text{ss}} \sum_{j\in\mathcal{R}_r} a_{rj},$$

which coincides with the expression in (19). Also note that (20) follows immediately from the observation that $\log P = \log m - \log C$.

Proof of Propositions 5

Let $\mathbb{E}_\omega[\cdot]$ denote the expectation operator conditional on the public signal, ω. Taking conditional expectations from both sides of (11) implies that $\mathbb{E}_\omega[\lambda_i] = \beta_i + \sum_{j=1}^{n} a_{ji}\mathbb{E}_\omega[\lambda_j]$ for all i. On the other hand, note that the steady-state Domar weights of all industries also satisfy the following system of equations: $\lambda_i^{\text{ss}} = \beta_i + \sum_{j=1}^{n} a_{ji}\lambda_j^{\text{ss}}$. Comparing the two equations then implies that

$$\mathbb{E}_\omega[\lambda_i] = \lambda_i^{\text{ss}} \qquad \text{for all } i.$$

Plugging this into equation (10) and taking logarithms from both sides then implies that

$$\log(p_i/m) = -\log z_i + \alpha_i\log(w/m) + \sum_{j\in\mathcal{F}_i} a_{ij}\log(p_j/m) + \sum_{j\in\mathcal{R}_i} a_{ij}\left(\log\mathbb{E}_\omega[p_j/m] + \log(\lambda_i/\lambda_i^{\text{ss}})\right)$$

To simplify notation, define $\hat{p}_i = p_i/m$ and $\hat{w} = w/m$. Writing the above equation in matrix form, we get

$$\log\hat{p} = -\log z + \alpha\log\hat{w} + \mathbf{A}_f\log\hat{p} + \mathbf{A}_r\log\mathbb{E}_\omega[\hat{p}] + \text{diag}(\mathbf{A}_r\mathbf{1})\log(\lambda/\lambda^{\text{ss}}),$$

where $\mathbf{A}_f$ is the matrix whose (i,j) element is equal to a_{ij} if $j\in\mathcal{F}_i$ and is equal to zero otherwise and $\mathbf{A}_r = \mathbf{A} - \mathbf{A}_f$. Consequently,

$$\log\hat{p} = \xi + \mathbf{L}_f\alpha\log\hat{w} + \mathbf{L}_f\mathbf{A}_r\log\mathbb{E}_\omega[\hat{p}], \tag{A.29}$$

where $\mathbf{L}_f = (\mathbf{I} - \mathbf{A}_f)^{-1}$ and

$$\xi = -\mathbf{L}_f \log z + \mathbf{L}_f \operatorname{diag}(\mathbf{A}_r \mathbf{1}) \log(\lambda/\lambda^{\text{ss}}). \tag{A.30}$$

Exponentiating both sides of (A.29), taking conditional expectations, and then taking logarithms implies that

$$\log \mathbb{E}_\omega[\hat{p}] = \log \mathbb{E}_\omega[e^\xi] + \mathbf{L}_f \alpha \log \hat{w} + \mathbf{L}_f \mathbf{A}_r \log \mathbb{E}_\omega[\hat{p}],$$

where note that since $\eta \to \infty$, Lemma 1 implies that $\hat{w} = w/m = \chi$, which is deterministic and hence is measurable with the respect to the firms' common information structure. Solving for $\log \mathbb{E}_\omega[\hat{p}]$ and using the observation that $(\mathbf{I} - \mathbf{L}_f \mathbf{A}_r)^{-1} = (\mathbf{I} - \mathbf{L}_f(\mathbf{A} - \mathbf{A}_f))^{-1} = \mathbf{L}(\mathbf{I} - \mathbf{A}_f)$, we can rewrite the above equation as follows:

$$\log \mathbb{E}_\omega[\hat{p}] = \mathbf{L}(\mathbf{I} - \mathbf{A}_f) \log \mathbb{E}_\omega[e^\xi] + \mathbf{1} \log \chi,$$

Plugging the above expression back into (A.29) leads to the following expression for log prices in terms of vector ξ defined in (A.30):

$$\log \hat{p} = \mathbf{1} \log \chi + \xi + \mathbf{L}\mathbf{A}_r \log \mathbb{E}_\omega[e^\xi].$$

Combining this equation with the observation that $\log C = \log m - \sum_{i=1}^n \beta_i \log p_i$ we get the following expression for log aggregate output in terms of vector ξ:

$$\log C = -\log \chi - \lambda^{\text{ss}\prime}(\mathbf{I} - \mathbf{A})\xi - \lambda^{\text{ss}\prime}\mathbf{A}_r \log \mathbb{E}_\omega[e^\xi],$$

which to a second-order approximation is equal to

$$\log C = -\log \chi - \lambda^{\text{ss}\prime}(\mathbf{I} - \mathbf{A})\xi - \lambda^{\text{ss}\prime}\mathbf{A}_r \left(\mathbb{E}_\omega[\xi] + \frac{1}{2}\text{var}_\omega(\xi) \right). \tag{A.31}$$

To express log output in (A.31) in terms of model primitives, we next need to solve for ξ and its first two conditional moments. We thus turn to (11), which can be rewritten as follows:

$$\lambda = \beta + \mathbf{A}_f' \lambda + \operatorname{diag}(\mathbf{A}_r' \lambda^{\text{ss}}) \frac{\hat{p}}{\mathbb{E}_\omega[\hat{p}]}.$$

Solving for the vector of Domar weights and using (A.29), we get

$$\lambda = \mathbf{L}_f' \beta + \mathbf{L}_f' \operatorname{diag}(\mathbf{A}_r' \lambda^{\text{ss}}) \frac{e^\xi}{\mathbb{E}_\omega[e^\xi]} = \lambda^{\text{ss}} + \mathbf{L}_f' \operatorname{diag}(\mathbf{A}_r' \lambda^{\text{ss}}) \left(\frac{e^\xi}{\mathbb{E}_\omega[e^\xi]} - \mathbf{1} \right),$$

and as a result,

$$\lambda/\lambda^{\text{ss}} = \mathbf{1} + \operatorname{diag}^{-1}(\lambda^{\text{ss}}) \mathbf{L}_f' \operatorname{diag}(\mathbf{A}_r' \lambda^{\text{ss}}) \left(e^{\xi - \log \mathbb{E}_\omega[e^\xi]} - \mathbf{1} \right). \tag{A.32}$$

Therefore, to a second-order approximation,

$$\begin{aligned}
\lambda/\lambda^{\text{ss}} &= \mathbf{1} + \mathbf{H}' \left(\xi - \log \mathbb{E}_\omega[e^\xi] + \frac{1}{2} \operatorname{diag}\left((\xi - \log \mathbb{E}_\omega[e^\xi])(\xi - \log \mathbb{E}_\omega[e^\xi])' \right) \right) \\
&= \mathbf{1} + \mathbf{H}' \left(\xi - \mathbb{E}_\omega[\xi] - \frac{1}{2}\text{var}_\omega(\xi) + \frac{1}{2} \operatorname{diag}\left((\xi - \mathbb{E}_\omega[\xi])(\xi - \mathbb{E}_\omega[\xi])' \right) \right),
\end{aligned}$$

where $\mathbf{H}' = \operatorname{diag}^{-1}(\lambda^{\text{ss}})\mathbf{L}_f' \operatorname{diag}(\mathbf{A}_r'\lambda^{\text{ss}})$. Plugging the above expression into equation (A.30) and performing a second-order approximation, we get

$$\begin{aligned}\xi = &-\mathbf{L}_f \log z + \mathbf{L}_f \operatorname{diag}(\mathbf{A}_r\mathbf{1})\mathbf{H}'\left(\xi - \mathbb{E}_\omega[\xi] - \frac{1}{2}\text{var}_\omega(\xi) + \frac{1}{2}\operatorname{diag}((\xi - \mathbb{E}_\omega[\xi])(\xi - \mathbb{E}_\omega[\xi])')\right)\\ &- \frac{1}{2}\mathbf{L}_f \operatorname{diag}(\mathbf{A}_r\mathbf{1}) \operatorname{diag}(\mathbf{H}'(\xi - \mathbb{E}_\omega[\xi])(\xi - \mathbb{E}_\omega[\xi])'\mathbf{H}).\end{aligned} \tag{A.33}$$

Taking conditional expectations from both sides of the above equation implies that

$$\mathbb{E}_\omega[\xi] = -\mathbf{L}_f\mathbb{E}_\omega[\log z] - \frac{1}{2}\mathbf{L}_f \operatorname{diag}(\mathbf{A}_r\mathbf{1}) \operatorname{diag}(\mathbf{H}'\text{var}_\omega(\xi)\mathbf{H}), \tag{A.34}$$

where $\text{var}_\omega(\xi)$ denotes the variance-covariance matrix of ξ conditional on the common signal ω. Subtracting the above equation from (A.33) leads to

$$\begin{aligned}\xi - \mathbb{E}_\omega[\xi] = &-\mathbf{L}_f(\log z - \mathbb{E}_\omega[\log z])\\ &+ \mathbf{L}_f \operatorname{diag}(\mathbf{A}_r\mathbf{1})\mathbf{H}'\left(\xi - \mathbb{E}_\omega[\xi] - \frac{1}{2}\text{var}_\omega(\xi) + \frac{1}{2}\operatorname{diag}((\xi - \mathbb{E}_\omega[\xi])(\xi - \mathbb{E}_\omega[\xi])')\right)\\ &- \frac{1}{2}\mathbf{L}_f \operatorname{diag}(\mathbf{A}_r\mathbf{1}) \operatorname{diag}(\mathbf{H}'(\xi - \mathbb{E}_\omega[\xi])(\xi - \mathbb{E}_\omega[\xi])'\mathbf{H}) + \frac{1}{2}\mathbf{L}_f \operatorname{diag}(\mathbf{A}_r\mathbf{1}) \operatorname{diag}(\mathbf{H}'\text{var}_\omega(\xi)\mathbf{H}).\end{aligned}$$

As a result,

$$\text{var}_\omega(\xi) = \text{var}_\omega(\mathbf{L}_f \log z) + \mathbf{L}_f \operatorname{diag}(\mathbf{A}_r\mathbf{1})\mathbf{H}'\text{var}_\omega(\xi)\mathbf{H} \operatorname{diag}(\mathbf{A}_r\mathbf{1})\mathbf{L}_f'.$$

Solving for $\text{var}_\omega(\xi)$ from the above equation, we get

$$\text{var}_\omega(\xi) = \sum_{k=0}^{\infty}(\mathbf{L}_f \operatorname{diag}(\mathbf{A}_r\mathbf{1})\mathbf{H}')^k\text{var}_\omega(\mathbf{L}_f \log z)(\mathbf{H} \operatorname{diag}(\mathbf{A}_r\mathbf{1})\mathbf{L}_f')^k. \tag{A.35}$$

Plugging the expressions in (A.34) and (A.35) into (A.31) and taking conditional expectations then implies that

$$\mathbb{E}_\omega[\log C] = \mathbb{E}_\omega[\log C^*] + \frac{1}{2}\lambda^{\text{ss}\prime} \operatorname{diag}(\mathbf{A}_r\mathbf{1}) \operatorname{diag}(\mathbf{H}'\text{var}_\omega(\xi)\mathbf{H}) - \frac{1}{2}\lambda^{\text{ss}\prime}\mathbf{A}_r\text{var}_\omega(\xi).$$

Taking unconditional expectations from both side of the above equation and letting $\mathbf{Q} = \mathbb{E}[\text{var}_\omega(\xi)]$ then establishes the result. □

Proof of Propositions 6

Recall from the proof of Proposition 5 that $\log \hat{p}_i = \log p_i - \log m$ satisfies equation (A.29), where vector ξ is given by (A.30). In the absence of productivity shocks, and when the downward constraint on the nominal wage binds, this implies that

$$\log \hat{p} = \theta + \mathbf{L}_f\alpha \log \bar{w} + \mathbf{L}_f\mathbf{A}_r \log \mathbb{E}_\omega[\hat{p}],$$

where

$$\theta = -\mathbf{L}_f\alpha \log m + \mathbf{L}_f \operatorname{diag}(\mathbf{A}_r\mathbf{1}) \log(\lambda/\lambda^{\text{ss}}). \tag{A.36}$$

Exponentiating both sides of the above equation, taking conditional expectations, and then taking logarithms implies that

$$\log \mathbb{E}_\omega[\hat{p}] = \log \mathbb{E}_\omega[e^\theta] + \mathbf{L}_f \alpha \log \bar{w} + \mathbf{L}_f \mathbf{A}_r \log \mathbb{E}_\omega[\hat{p}].$$

Solving for $\log \mathbb{E}_\omega[\hat{p}]$ and using the observation that $(\mathbf{I} - \mathbf{L}_f \mathbf{A}_r)^{-1} = (\mathbf{I} - \mathbf{L}_f(\mathbf{A} - \mathbf{A}_f))^{-1} = \mathbf{L}(\mathbf{I} - \mathbf{A}_f)$, we can rewrite the above equation as follows:

$$\log \mathbb{E}_\omega[\hat{p}] = \mathbf{L}(\mathbf{I} - \mathbf{A}_f) \log \mathbb{E}_\omega[e^\theta] + \mathbf{1} \log \bar{w}.$$

Plugging the above back into the expression for $\log \hat{p}$ leads to the following expression for log prices:

$$\log \hat{p} = \mathbf{1} \log \bar{w} + \theta + \mathbf{L}\mathbf{A}_r \log \mathbb{E}_\omega[e^\theta].$$

Combining this equation with the observation that $\log C = -\sum_{i=1}^n \beta_i \log \hat{p}_i$ we get the following expression for log aggregate output:

$$\log C = -\log \bar{w} - \lambda^{\mathrm{ss}\prime}(\mathbf{I} - \mathbf{A})\theta - \lambda^{\mathrm{ss}\prime} \mathbf{A}_r \log \mathbb{E}_\omega[e^\theta].$$

Next, note that steps identical to those in the proof of Proposition 5 imply that we can write the ratio of Domar weights to their steady-state values as follows:

$$\lambda/\lambda^{\mathrm{ss}} = \mathbf{1} + \operatorname{diag}^{-1}(\lambda^{\mathrm{ss}}) \mathbf{L}_f' \operatorname{diag}(\mathbf{A}_r' \lambda^{\mathrm{ss}}) \left(e^{\theta - \log \mathbb{E}_\omega[e^\theta]} - \mathbf{1} \right),$$

and hence, to a second-order approximation,

$$\lambda/\lambda^{\mathrm{ss}} = \mathbf{1} + \mathbf{H}' \left(\theta - \mathbb{E}_\omega[\theta] - \frac{1}{2}\mathrm{var}_\omega(\theta) + \frac{1}{2} \operatorname{diag}\left((\theta - \mathbb{E}_\omega[\theta])(\theta - \mathbb{E}_\omega[\theta])' \right) \right),$$

Plugging the above expression into equation (A.36) and performing a second-order approximation, we get

$$\begin{aligned} \theta = &-\mathbf{L}_f \alpha \log m + \mathbf{L}_f \operatorname{diag}(\mathbf{A}_r \mathbf{1}) \mathbf{H}' \left(\theta - \mathbb{E}_\omega[\theta] - \frac{1}{2}\mathrm{var}_\omega(\theta) + \frac{1}{2} \operatorname{diag}((\theta - \mathbb{E}_\omega[\theta])(\theta - \mathbb{E}_\omega[\theta])') \right) \\ &- \frac{1}{2} \mathbf{L}_f \operatorname{diag}(\mathbf{A}_r \mathbf{1}) \operatorname{diag}(\mathbf{H}'(\theta - \mathbb{E}_\omega[\theta])(\theta - \mathbb{E}_\omega[\theta])' \mathbf{H}). \end{aligned} \tag{A.37}$$

Taking conditional expectations from both sides of the above equation implies that

$$\mathbb{E}_\omega[\theta] = -\mathbf{L}_f \alpha \mathbb{E}_\omega[\log m] - \frac{1}{2} \mathbf{L}_f \operatorname{diag}(\mathbf{A}_r \mathbf{1}) \operatorname{diag}(\mathbf{H}' \mathrm{var}_\omega(\theta) \mathbf{H}). \tag{A.38}$$

Subtracting the above equation from (A.37) leads to

$$\begin{aligned} \theta - \mathbb{E}_\omega[\theta] = &-\mathbf{L}_f \alpha (\log m - \mathbb{E}_\omega[\log m]) \\ &+ \mathbf{L}_f \operatorname{diag}(\mathbf{A}_r \mathbf{1}) \mathbf{H}' \left(\theta - \mathbb{E}_\omega[\theta] - \frac{1}{2}\mathrm{var}_\omega(\theta) + \frac{1}{2} \operatorname{diag}((\theta - \mathbb{E}_\omega[\theta])(\theta - \mathbb{E}_\omega[\theta])') \right) \\ &- \frac{1}{2} \mathbf{L}_f \operatorname{diag}(\mathbf{A}_r \mathbf{1}) \operatorname{diag}(\mathbf{H}'(\theta - \mathbb{E}_\omega[\theta])(\theta - \mathbb{E}_\omega[\theta])' \mathbf{H}) + \frac{1}{2} \mathbf{L}_f \operatorname{diag}(\mathbf{A}_r \mathbf{1}) \operatorname{diag}(\mathbf{H}' \mathrm{var}_\omega(\theta) \mathbf{H}). \end{aligned}$$

As a result,

$$\mathrm{var}_\omega(\theta) = \mathrm{var}_\omega(\mathbf{L}_f \alpha \log m) + \mathbf{L}_f \,\mathrm{diag}(\mathbf{A}_r \mathbf{1})\mathbf{H}' \mathrm{var}_\omega(\theta) \mathbf{H} \,\mathrm{diag}(\mathbf{A}_r \mathbf{1}) \mathbf{L}_f'.$$

Solving for $\mathrm{var}_\omega(\theta)$ from the above equation, we get

$$\mathrm{var}_\omega(\theta) = \sum_{k=0}^{\infty} (\mathbf{L}_f \,\mathrm{diag}(\mathbf{A}_r \mathbf{1})\mathbf{H}')^k \mathrm{var}_\omega(\mathbf{L}_f \alpha \log m)(\mathbf{H} \,\mathrm{diag}(\mathbf{A}_r \mathbf{1})\mathbf{L}_f')^k. \tag{A.39}$$

Plugging the expressions in (A.38) and (A.39) into the expression for aggregate output and taking conditional expectations then implies that

$$\mathbb{E}_\omega[\log C] = \mathbb{E}_\omega[\log C^*] + \frac{1}{2}\lambda^{\mathrm{ss}\prime} \,\mathrm{diag}(\mathbf{A}_r \mathbf{1}) \,\mathrm{diag}(\mathbf{H}' \mathrm{var}_\omega(\theta)\mathbf{H}) - \frac{1}{2}\lambda^{\mathrm{ss}\prime} \mathbf{A}_r \mathrm{var}_\omega(\theta).$$

Taking unconditional expectations from both side then establishes the result. □

Proof of Proposition 7

We consider a more general vertical production network with industries 1 through n arranged on a chain, with industry 1 as the final good producer. We then specialize this economy to the case of $n = 3$ in Proposition 7.

Recall from Proposition 1 that nominal prices and Domar weights satisfy the system of equations in (10) and (11). Applying these equations to the vertical production network economy, we obtain

$$p_i = \frac{1}{z_i} w^{1-a_i} \left(m \frac{\mathbb{E}_i[p_{i+1}/m]}{\mathbb{E}_i[\lambda_i]/\lambda_i} \right)^{a_i} \qquad \text{for } 1 \le i \le n, \tag{A.40}$$

$$\lambda_{i+1} = a_i \mathbb{E}_i[\lambda_i] \frac{p_{i+1}/m}{\mathbb{E}_i[p_{i+1}/m]} \qquad \text{for } 1 \le i \le n-1 \tag{A.41}$$

with the initial conditions that $\lambda_1 = 1$ and the convention that $a_n = 0$. Solving for $\mathbb{E}_i[\lambda_i]/\mathbb{E}_i[p_{i+1}/m]$ from (A.41) and plugging it back into (A.40) implies that $p_i = \frac{1}{z_i} a_i^{a_i} (\lambda_i/\lambda_{i+1})^{a_i} w^{1-a_i} p_{i+1}^{a_i}$. Hence,

$$\log p_i = a_i(\varphi_i - \varphi_{i+1}) - \log z_i + a_i \log p_{i+1} + (1-a_i)\log w,$$

for $1 \le i \le n$, where $\varphi_i = \log \lambda_i - \log \lambda_i^{\mathrm{ss}}$ and we are using the fact that $\lambda_{i+1}^{\mathrm{ss}} = a_i \lambda_i^{\mathrm{ss}}$. Solving the above recursion, we can express nominal prices in terms of Domar weights:

$$\log p_i = \log w - \sum_{j=i}^{n} (a_i a_{i+1} \dots a_{j-1}) \log z_j + \sum_{j=i}^{n-1} (a_i a_{i+1} \dots a_j)(\varphi_j - \varphi_{j+1}). \tag{A.42}$$

Next, note that, to a first-order approximation, (A.41) can be expressed as

$$\log \lambda_{i+1} = \log a_i + \log(p_{i+1}/m) + \mathbb{E}_i[\log \lambda_i] - \mathbb{E}_i[\log p_{i+1}/m],$$

and as a result,

$$\varphi_{i+1} = \mathbb{E}_i[\varphi_i] + \log(p_{i+1}/m) - \mathbb{E}_i[\log p_{i+1}/m]. \tag{A.43}$$

We now have a system of linear expectations (A.42) and (A.43) that fully pins down equilibrium nominal prices and Domar weights in terms of the productivity shocks, nominal aggregate demand, and the nominal wage.

Specializing these equations to the case of $n = 3$ and shutting off all productivity shocks, it is immediate that $\log p_3 = \log w$, and hence, we get the following equations:

$$\begin{aligned}\log p_2 &= \log w + a_2(\varphi_2 - \varphi_3)\\ \varphi_2 &= \log p_2 - \mathbb{E}_1[\log p_2] - \log m + \mathbb{E}_1[\log m]\\ \varphi_3 &= \mathbb{E}_2[\varphi_2] + \log(w/m) - \mathbb{E}_2[\log(w/m)].\end{aligned}$$

Replacing for φ_3 into the expression for $\log p_2$, we get

$$\log p_2 = a_2(\varphi_2 - \mathbb{E}_2[\varphi_2]) + Q + \log m, \tag{A.44}$$

where

$$Q = (1 - a_2)\log(w/m) + a_2\mathbb{E}_2[\log(w/m)]. \tag{A.45}$$

Consequently, we get the following equation for φ_2:

$$\varphi_2 = a_2(\varphi_2 - \mathbb{E}_2[\varphi_2]) - a_2(\mathbb{E}_1[\varphi_2] - \mathbb{E}_1\mathbb{E}_2[\varphi_2]) + Q - \mathbb{E}_1[Q]$$

Noting that $\mathbb{E}_1[\varphi_2] = 0$, we get

$$\varphi_2 = -\frac{a_2}{1-a_2}\mathbb{E}_2[\varphi_2] + \frac{a_2}{1-a_2}\mathbb{E}_1\mathbb{E}_2[\varphi_2] + \frac{1}{1-a_2}(Q - \mathbb{E}_1[Q]). \tag{A.46}$$

Taking expectations with respect to the information set of firms in industry 2 from both sides of (A.46) and solving for $\mathbb{E}_2[\varphi_2]$ implies that

$$\mathbb{E}_2[\varphi_2] = a_2\mathbb{E}_2\mathbb{E}_1\mathbb{E}_2[\varphi_2] + \mathbb{E}_2[Q] - \mathbb{E}_2\mathbb{E}_1[Q].$$

We can thus solve for $\mathbb{E}_2[\varphi_2]$ in terms of the infinite regress of expectations as follows:

$$\mathbb{E}_2[\varphi_2] = \sum_{s=0}^{\infty} a_2^s(\mathbb{E}_2\mathbb{E}_1)^s(\mathbb{E}_2[Q] - \mathbb{E}_2\mathbb{E}_1[Q]) = \sum_{s=0}^{\infty} a_2^s(\mathbb{E}_2\mathbb{E}_1)^s\mathbb{E}_2[Q] - \sum_{s=0}^{\infty} a_2^s(\mathbb{E}_2\mathbb{E}_1)^{s+1}[Q].$$

Plugging the above expression into (A.46), we get

$$\varphi_2 = \frac{1}{1-a_2}\left(\sum_{s=0}^{\infty} a_2^{s+1}(\mathbb{E}_1\mathbb{E}_2)^{s+1}[Q] - \sum_{s=0}^{\infty} a_2^{s+1}(\mathbb{E}_2\mathbb{E}_1)^s\mathbb{E}_2[Q] + \sum_{s=0}^{\infty} a_2^s(\mathbb{E}_2\mathbb{E}_1)^s[Q] - \sum_{s=0}^{\infty} a_2^s\mathbb{E}_1(\mathbb{E}_2\mathbb{E}_1)^s[Q]\right).$$

Hence, combining this equation with (A.44) and using the observations that $\log p_1 = a_1(\varphi_1 - \varphi_2) + a_1\log p_2 + (1-a_1)\log w$ and $\varphi_1 = 0$, we get the following expression for $\log p_1$:

$$\log(p_1/m) = a_1\left(\sum_{s=0}^{\infty} a_2^s\mathbb{E}_1(\mathbb{E}_2\mathbb{E}_1)^s[Q] - \sum_{s=0}^{\infty} a_2^{s+1}(\mathbb{E}_1\mathbb{E}_2)^{s+1}[Q]\right) + (1-a_1)\log(w/m).$$

Rearranging terms, we get

$$\log(p_1/m) = a_1 \sum_{s=0}^{\infty} a_2^s (\mathbb{E}_1\mathbb{E}_2)^s \mathbb{E}_1 \Big[Q - a_2 \mathbb{E}_2[Q] \Big] + (1-a_1)\log(w/m).$$

On the other hand, note that (A.45) implies that $Q - a_2\mathbb{E}_2[Q] = (1-a_2)\log(w/m)$. Therefore,

$$\log(p_1/m) = a_1(1-a_2) \sum_{s=0}^{\infty} a_2^s (\mathbb{E}_1\mathbb{E}_2)^s \mathbb{E}_1[\log(w/m)] + (1-a_1)\log(w/m).$$

By Lemma 1, the assumption that $m < \bar{w}/\chi$ implies that $w = \bar{w}$, in which case the above equation immediately reduces to (28). Furthermore, noting that $\log m = \log(PC)$ then establishes (27). $\square$