The Macroeconomics of Supply Chain Disruptions – Online Appendix

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This online appendix contains the proofs of Lemmas 1 and A.3 and Propositions 1–3, A.1, and A.2.

Proof of Lemma 1

Let $\mathbf{G}_1$ and $\mathbf{G}_2$ denote two efficient production networks. Let $\mathbf{G}_1 \cup \mathbf{G}_2$ be the production network that contains all supplier-customer relationships that are present in either $\mathbf{G}_1$ or $\mathbf{G}_2$. Market clearing of the final good implies that

$$\begin{aligned} Y(\mathbf{G}_1 \cup \mathbf{G}_2) - Y(\mathbf{G}_1) &= (\mathcal{A}(\mathbf{G}_1 \cup \mathbf{G}_2) - \mathcal{A}(\mathbf{G}_1))L - \sum_{ij \in \mathbf{G}_2 \setminus \mathbf{G}_1} (c_{ij} + s_{ij}) \\ &\geq (\mathcal{A}(\mathbf{G}_2) - \mathcal{A}(\mathbf{G}_1 \cap \mathbf{G}_2))L - \sum_{ij \in \mathbf{G}_2 \setminus \mathbf{G}_1} (c_{ij} + s_{ij}) \\ &= Y(\mathbf{G}_2) - Y(\mathbf{G}_1 \cap \mathbf{G}_2), \end{aligned}$$

where the inequality is a consequence of Assumption 1(a). Since both $\mathbf{G}_1$ and $\mathbf{G}_2$ are efficient, the left-hand side of the above inequality is nonpositive, whereas the right-hand side is nonnegative. Therefore, both expressions have to be equal to zero, implying that $Y(\mathbf{G}_1 \cup \mathbf{G}_2) = Y(\mathbf{G}_1) = Y(\mathbf{G}_2)$. Thus, $\mathbf{G}_1 \cup \mathbf{G}_2$ is also efficient. An inductive argument then implies that the union of all efficient production networks is efficient. □

Proof of Lemma A.3

Proof of part (a). Fix a firm j and let $b = (b_j, b_{-j})$ and $\hat{b} = (\hat{b}_j, \hat{b}_{-j})$ denote two strategy profiles such that $\hat{b}_j \geq b_j$ and $\hat{b}_{-j} \geq b_{-j}$ element-wise. Equation (A.28) implies that

$$\begin{aligned} \phi_j(\hat{b}) - \phi_j(b_j, \hat{b}_{-j}) - \phi_j(\hat{b}_j, b_{-j}) + \phi_j(b) = \pi_j(\mathbf{G}(\hat{b})) &- \pi_j(\mathbf{G}(b_j, \hat{b}_{-j})) \\ &- \pi_j(\mathbf{G}(\hat{b}_j, b_{-j})) + \pi_j(\mathbf{G}(b)). \end{aligned} \tag{B.1}$$

Furthermore, the expression for firms' gross profits in (6) implies that

$$\pi_j(\mathbf{G}(\hat{b}))-\pi_j(\mathbf{G}(b_j,\hat{b}_{-j}))=\theta_j\sum_{T\ni j}\psi_j(T\setminus\{j\})\Big[\mathcal{A}(\mathbf{G}(\hat{b})|_T)-\mathcal{A}(\mathbf{G}(b_j,\hat{b}_{-j})|_T)\Big]L \tag{B.2}$$

$$\pi_j(\mathbf{G}(\hat{b}_j,b_{-j}))-\pi_j(\mathbf{G}(b))=\theta_j\sum_{T\ni j}\psi_j(T\setminus\{j\})\Big[\mathcal{A}(\mathbf{G}(\hat{b}_j,b_{-j})|_T)-\mathcal{A}(\mathbf{G}(b)|_T)\Big]L, \tag{B.3}$$

where we are using the fact that $\mathcal{A}(\mathbf{G}(b_j,b_{-j})|_T)$ is independent of b_j whenever $j\notin T$. Assumption 1(a) together with $\psi_j(T\setminus\{j\})\geq 0$ guarantees that the right-hand side of (B.2) is greater than equal to the right-hand right of (B.3). Thus, the right-hand side of (B.1) is nonnegative, implying that $\phi_j(b_j,b_{-j})$ has increasing differences in (b_j,b_{-j}). □

Proof of part (b). Let $b=(b_j,b_{-j})$ and $\hat{b}=(\hat{b}_j,\hat{b}_{-j})$ be two arbitrary strategy profiles. By (A.28),

$$\begin{aligned}&\phi_j(b_j\vee\hat{b}_j,b_{-j})+\phi_j(b_j\wedge\hat{b}_j,b_{-j})-\phi_j(\hat{b}_j,b_{-j})-\phi_j(b_j,b_{-j})\\&\quad=\pi_j(\mathbf{G}(b_j\vee\hat{b}_j,b_{-j}))+\pi_j(\mathbf{G}(b_j\wedge\hat{b}_j,b_{-j}))-\pi_j(\mathbf{G}(\hat{b}_j,b_{-j}))-\pi_j(\mathbf{G}(b_j,b_{-j})).\end{aligned} \tag{B.4}$$

Using the expression for firm j's gross profit in (6) to replace for the first and last terms on the right-hand side of (B.4) leads to

$$\begin{aligned}\pi_j(\mathbf{G}(b_j\vee\hat{b}_j,b_{-j}))-\pi_j(\mathbf{G}(b_j,b_{-j}))=\theta_j\sum_{T\subseteq N\setminus\{j\}}\psi_j(T)\Big[&\mathcal{A}(\mathbf{G}(b_j\vee\hat{b}_j,b_{-j})|_{T\cup\{j\}})\\&-\mathcal{A}(\mathbf{G}(b_j,b_{-j})|_{T\cup\{j\}})\Big]L,\end{aligned} \tag{B.5}$$

where we are using the fact that the value of $\mathcal{A}(\mathbf{G}(b_j,b_{-j})|_T)$ is independent of b_j whenever $j\notin T$. Similarly, writing the second and third terms on the right-hand side of (B.4) in terms of (6), we obtain

$$\begin{aligned}\pi_j(\mathbf{G}(\hat{b}_j,b_{-j}))-\pi_j(\mathbf{G}(b_j\wedge\hat{b}_j,b_{-j}))=\theta_j\sum_{T\subseteq N\setminus\{j\}}\psi_j(T)\Big[&\mathcal{A}(\mathbf{G}(\hat{b}_j,b_{-j})|_{T\cup\{j\}})\\&-\mathcal{A}(\mathbf{G}(b_j\wedge\hat{b}_j,b_{-j})|_{T\cup\{j\}})\Big]L.\end{aligned} \tag{B.6}$$

Assumption 1(a) together the fact that $\psi_j(T)\geq 0$ guarantees that the right-hand side of (B.5) is greater than or equal to that of (B.6). Consequently, the right-hand side of (B.4) is nonnegative, which means $\phi_j(b_j,b_{-j})$ is supermodular in b_j given an arbitrary b_{-j}. □

Proof of Proposition 1

Let $s^*_{\text{frg}}=\pi_j(\mathbf{G}\cup\{ik,kj\})-\pi_j(\mathbf{G}\cup\{ik\})$ be the threshold beyond which firm j drops its relationship with firm k in the fragmented economy. Aggregate output in the fragmented architecture is thus given by

$$Y_{\text{frg}} = \begin{cases} \mathcal{A}(\mathbf{G} \cup \{ik, kj\}) - s - \sum_{rl \in \mathbf{G}} s_{rl} & \text{if } s \leq s^*_{\text{frg}} \\ \mathcal{A}(\mathbf{G}) - \sum_{rl \in \mathbf{G}} s_{rl} & \text{if } s > s^*_{\text{frg}}, \end{cases} \tag{B.7}$$

where we are using the assumption that $\mathcal{A}(\mathbf{G} \cup \{ik\}) = \mathcal{A}(\mathbf{G})$. A similar argument implies that aggregate output in the integrated architecture is

$$Y_{\text{int}} = \begin{cases} \mathcal{A}(\mathbf{G} \cup \{ij\}) - s - \sum_{rl \in \mathbf{G}} s_{rl} & \text{if } s \leq s^*_{\text{int}} \\ \mathcal{A}(\mathbf{G}) - \sum_{rl \in \mathbf{G}} s_{rl} & \text{if } s > s^*_{\text{int}}, \end{cases} \tag{B.8}$$

where $s^*_{\text{int}} = \pi_j(\mathbf{G} \cup \{ij\}) - \pi_j(\mathbf{G})$ is the threshold beyond which j drops its relationship with i in the integrated economy. Comparing (B.8) to (B.7), it is immediate that if (13) is satisfied, and as long $s^*_{\text{frg}} < s^*_{\text{int}}$, then $Y_{\text{frg}} > Y_{\text{int}}$ for $s \leq s^*_{\text{frg}}$ and $Y_{\text{frg}} \leq Y_{\text{int}}$ for $s > s^*_{\text{frg}}$. The proof is therefore complete once we show that $s^*_{\text{frg}} < s^*_{\text{int}}$.

Recall that $s^*_{\text{frg}} = \pi_j(\mathbf{G} \cup \{ik, kj\}) - \pi_j(\mathbf{G} \cup \{ik\})$ and $s^*_{\text{int}} = \pi_j(\mathbf{G} \cup \{ij\}) - \pi_j(\mathbf{G})$. The expression for firms' profits in (6) therefore implies that

$$s^*_{\text{frg}} = \theta_j \sum_{T \subseteq N \setminus \{j\}} \psi_j(T \cup \{k\}) \Big[\mathcal{A}(\mathbf{G} \cup \{ik, kj\}|_{T \cup \{j,k\}}) - \mathcal{A}(\mathbf{G}|_{T \cup \{j,k\}}) \Big] L$$

$$s^*_{\text{int}} = \theta_j \sum_{T \subseteq N \setminus \{j\}} \psi_j(T) \Big[\mathcal{A}(\mathbf{G} \cup \{ij\}|_{T \cup \{j\}}) - \mathcal{A}(\mathbf{G}|_{T \cup \{j\}}) \Big] L,$$

where N denotes the set of firms in the integrated architecture and thus excludes firm k. Note that s^*_{int} does not depend on θ_k. Therefore, to show that $s^*_{\text{frg}} < s^*_{\text{int}}$ for large enough values of θ_k, it is sufficient to show that $\lim_{\theta_k \to \infty} \psi_j(T \cup \{k\}) = 0$ for all $T \subseteq N \setminus \{j\}$, which implies that $\lim_{\theta_k \to \infty} s^*_{\text{frg}} = 0$. To this end, recall from the proof of Theorem 1(c) that the weights ψ_j satisfy equations (A.20) and (A.21). The recursion in (A.20) and a simple inductive argument on set T implies that $\lim_{\theta_k \to \infty} \psi_j(T \cup \{k\}) = \lim_{\theta_k \to \infty} q(T \cup \{j,k\}) = 0$ for all $T \subseteq N \setminus \{j\}$. □

Proof of Proposition 2

Let k^* denote the depth of the supply chain in the economy's greatest equilibrium. Given that each supplier-customer relationship generates a productivity gain of $A \geq 1$ but requires a fixed cost of s, equilibrium aggregate output is $Y^* = A^{k^*-1} L - (k^* - 1)s$. We next derive the expression for k^*.

For any $k \leq n$, let $\mathbf{G}_k$ denote the production network of depth $k+1$, consisting of firms 0 through k. We make two observations. First,

$$\pi_k(\mathbf{G}_k) \leq \pi_i(\mathbf{G}_k) \quad \text{for all } i \leq k. \tag{B.9}$$

To see this, note that $\mathcal{A}(\mathbf{G}_k|_{T\cup\{i\}})=\mathcal{A}(\mathbf{G}_k|_T)$ unless $T\supseteq\{0,\ldots,i-1\}$. Therefore, by (6),

$$\pi_i(\mathbf{G}_k)=\theta_i\sum_{T\supseteq\{0,\ldots,i-1\}}\psi_i(T)[\mathcal{A}(\mathbf{G}_k|_{T\cup\{i\}})-\mathcal{A}(\mathbf{G}_k|_T)]L.$$

Since all summands on the right-hand side of the above equation are nonnegative, it is immediate that $\pi_i(\mathbf{G}_k)\geq\pi_{i+1}(\mathbf{G}_k)$, which establishes (B.9).

Second, we note that

$$\pi_n(\mathbf{G}_n)\geq\pi_k(\mathbf{G}_k)\quad\text{for all } k\leq n. \tag{B.10}$$

To show this, note that, according to (6),

$$\begin{aligned}\pi_k(\mathbf{G}_k)&=\theta_k\sum_{T\supseteq\{0,\ldots,k-1\},T\not\ni k}\psi_k(T)[\mathcal{A}(\mathbf{G}_k|_{T\cup\{k\}})-\mathcal{A}(\mathbf{G}_k|_T)]L\\&=\theta_k\left(\sum_{T\supseteq\{0,\ldots,k-1\},T\not\ni k}\psi_k(T)\right)[\mathcal{A}(\mathbf{G}_k)-\mathcal{A}(\mathbf{G}_{k-1})]L\\&=\theta_k\left(\sum_{T\supseteq\{0,\ldots,k-1\},T\not\ni k}\psi_k(T)\right)A^{k-1}(A-1)L,\end{aligned}$$

where the first equality is a consequence of the fact that $\mathcal{A}(\mathbf{G}_k|_{T\cup\{k\}})=\mathcal{A}(\mathbf{G}_k|_T)$ for any $T\not\supseteq\{0,\ldots,k-1\}$, the second equality follows from $\mathbf{G}_k|_{T\cup\{k\}}=\mathbf{G}_k$ and $\mathbf{G}_k|_T=\mathbf{G}_{k-1}$ for any set T that contains $\{0,\ldots,k-1\}$ but not k, and the last equality follows from $\mathcal{A}(\mathbf{G}_k)=A^k$. Since all firms have identical bargaining powers,

$$\theta_k\psi_k(T)=\frac{|T|!(n-|T|)!}{(n+1)!}.$$

As a result,

$$\pi_k(\mathbf{G}_k)=\sum_{r=k}^{n}\binom{n-k}{r-k}\frac{r!(n-r)!}{(n+1)!}A^{k-1}(A-1)L=\frac{1}{k+1}A^{k-1}(A-1)L.$$

Therefore, as long as $\log A>1/2$, then $\pi_k(\mathbf{G}_k)$ is increasing in k, thus establishing (B.10).

With (B.9) and (B.10) at hand, we now prove the result. Let $s^*=\pi_n(\mathbf{G}_n)$. As long as $s\leq s^*$, inequality (B.9) implies that $\phi_i(\mathbf{G}_n)=\pi_i(\mathbf{G}_n)-s\geq 0$, thus guaranteeing that all firms make nonnegative profits. This means that, in the economy's greatest equilibrium, all firms pay the fixed cost of establishing a relationship with their customer, and hence, $k^*=n+1$. If on the other hand, $s>s^*$, then (B.10) implies that $\phi_k(\mathbf{G}_k)=\pi_k(\mathbf{G}_k)-s<0$, which means no firm $k\neq 0$ is willing to pay the fixed cost s to serve as the most upstream firm in the supply chain. Thus, it must be the case that the equilibrium supply chain only consists of firm 0, i.e., $k^*=1$. □

Proof of Proposition 3

Let $\mathbf{G}$ denote a production network that contains customer-supplier relations between firms i and k and their designated suppliers $j=n+i$ and $l=n+k$, respectively. According to (6), the marginal (gross) benefit to firm i of maintaining a relationship with its supplier is given by

$$\pi_i(\mathbf{G})-\pi_i(\mathbf{G}\backslash\{ij\})=\theta_i\sum_{T\subseteq N\backslash\{i\}}\psi_i(T)\left[\mathcal{A}(\mathbf{G}|_{T\cup\{i\}})-\mathcal{A}(\mathbf{G}\backslash\{ij\}|_{T\cup\{i\}})\right]L.$$

Recall from the proof of Theorem 1 that the weights ψ_i in the above expression satisfy (A.21). Therefore,

$$\begin{aligned}\pi_i(\mathbf{G})-\pi_i(\mathbf{G}\backslash\{ij\})&=\theta_i\sum_{T\supseteq\{i\}}q(T)[\mathcal{A}(\mathbf{G}|_T)-\mathcal{A}(\mathbf{G}\backslash\{ij\}|_T)]L\\&=\theta_i\sum_{T\subseteq N}q(T)[\mathcal{A}(\mathbf{G}|_T)-\mathcal{A}(\mathbf{G}\backslash\{ij\}|_T)]L,\end{aligned}$$

where $q(\cdot)$ satisfies the recursion in (A.20) and the second equality follows from the fact that $\mathcal{A}(\mathbf{G}|_T)=\mathcal{A}(\mathbf{G}\backslash\{ij\}|_T)$ for any set T that does not contain i. The fact that each firm i in the bottom layer has only a single potential supplier in the top layer, together with the assumption that the input varieties used by the final good producer have no productivity advantage over the latter's in-house production technology, implies that $\mathcal{A}(\mathbf{G}\backslash\{ij\}|T)=\mathcal{A}(\mathbf{G}|_{T\backslash\{i\}})$. As a result,

$$\pi_i(\mathbf{G})-\pi_i(\mathbf{G}\backslash\{ij\})=\theta_i\sum_{T\subseteq N}q(T)[\mathcal{A}(\mathbf{G}|_T)-\mathcal{A}(\mathbf{G}|_{T\backslash\{i\}})]L.$$

A similar argument implies that

$$\pi_k(\mathbf{G})-\pi_k(\mathbf{G}\backslash\{kl\})=\theta_k\sum_{T\subseteq N}q(T)[\mathcal{A}(\mathbf{G}|_T)-\mathcal{A}(\mathbf{G}|_{T\backslash\{k\}})]L.$$

Since all firms in the bottom layer have identical bargaining powers, θ, subtracting the last equation from the previous one leads to

$$\Delta_{ik}=\theta\sum_{T\subseteq N}q(T)[\mathcal{A}(\mathbf{G}|_{T\backslash\{k\}})-\mathcal{A}(\mathbf{G}|_{T\backslash\{i\}})]L,$$

where $\Delta_{ik}=[\pi_i(\mathbf{G})-\pi_i(\mathbf{G}\backslash\{ij\})]-[\pi_k(\mathbf{G})-\pi_k(\mathbf{G}\backslash\{kl\})]$. If set T contains neither i nor k, then $\mathcal{A}(\mathbf{G}|_{T\backslash\{k\}})=\mathcal{A}(\mathbf{G}|_{T\backslash\{i\}})$. As a result,

$$\begin{aligned}\Delta_{ik}=\theta&\sum_{T\ni i,k}q(T)\left[\mathcal{A}(\mathbf{G}|_{T\backslash\{k\}})-\mathcal{A}(\mathbf{G}|_{T\backslash\{i\}})\right]L\\&+\theta\sum_{T\ni i,T\not\ni k}q(T)\left[\mathcal{A}(\mathbf{G}|_T)-\mathcal{A}(\mathbf{G}|_{T\backslash\{i\}})\right]L\\&+\theta\sum_{T\not\ni i,T\ni k}q(T)\left[\mathcal{A}(\mathbf{G}|_{T\backslash\{k\}})-\mathcal{A}(\mathbf{G}|_T)\right]L.\end{aligned}\tag{B.11}$$

Applying the change of variable $\tilde{T} = T \cup \{i\} \setminus \{k\}$ to the last summation on the right-hand side of (B.11) leads to

$$\Delta_{ik} = \theta \sum_{T \ni i,k} q(T) \Big[\mathcal{A}(\mathbf{G}|_{T \setminus \{k\}}) - \mathcal{A}(\mathbf{G}|_{T \setminus \{i\}}) \Big] L + \theta \sum_{T \ni i, T \not\ni k} q(T) \Big[\mathcal{A}(\mathbf{G}|_T) - \mathcal{A}(\mathbf{G}|_{T \setminus \{i\}}) \Big] L$$
$$+ \theta \sum_{\tilde{T} \ni i, \tilde{T} \not\ni k} q(\tilde{T} \cup \{k\} \setminus \{i\}) \Big[\mathcal{A}(\mathbf{G}|_{\tilde{T} \setminus \{i\}}) - \mathcal{A}(\mathbf{G}|_{\tilde{T} \cup \{k\} \setminus \{i\}}) \Big] L.$$

Equation (A.20) and the assumption that all bottom-tier firms have identical bargaining powers imply that $q(\tilde{T} \cup \{k\} \setminus \{i\}) = q(\tilde{T})$ for any set $\tilde{T}$ such that $\tilde{T} \ni i$ and $\tilde{T} \not\ni k$. Hence,

$$\Delta_{ik} = \theta \sum_{T \ni i,k} q(T) \Big[\mathcal{A}(\mathbf{G}|_{T \setminus \{k\}}) - \mathcal{A}(\mathbf{G}|_{T \setminus \{i\}}) \Big] L$$
$$+ \theta \sum_{T \ni i,k} q(T \setminus \{k\}) \Big[\mathcal{A}(\mathbf{G}|_{T \setminus \{k\}}) - \mathcal{A}(\mathbf{G}|_{T \setminus \{i\}}) \Big] L.$$

When $i,k \in T$, then $\mathcal{A}(\mathbf{G}|_{T \setminus \{k\}}) > \mathcal{A}(\mathbf{G}|_{T \setminus \{i\}})$ if and only if $A_i > A_k$. Therefore, $\Delta_{ik} > 0$ whenever $A_i > A_k$. This means that if $A_i > A_k$, then for any given level of aggregate TFP, $\bar{A}$, the marginal benefit to firm k of keeping its supplier is smaller than that to firm i. Hence, as $\bar{A}$ declines, firm k drops its supplier before firm i whenever $A_i > A_k$. □

Proof of Proposition A.1

To establish supermodularity at the extensive margin (Assumption 1(a)) we start by deriving an expression for $\mathcal{A}(\mathbf{G})$ in terms of the production network $\mathbf{G}$. Recall that $\mathcal{A}(\mathbf{G})$ denotes the economy's aggregate productivity when a social planner chooses firms' technologies I_i and the corresponding quantities to maximize aggregate output. Theorem 3(a) of Acemoglu and Azar (2020) establishes that to solve for the efficient allocation, one can simply focus on the competitive equilibrium, in which all firms price at marginal cost. We thus derive the expression for $\mathcal{A}(\mathbf{G})$ by first solving for prices in the competitive equilibrium of the economy with production network $\mathbf{G}$ and then using the fact that $\mathcal{A}(\mathbf{G}) = w / \mathrm{mc}_0(\mathbf{G})$, where $\mathrm{mc}_0(\mathbf{G})$ is the marginal cost of the final good producer.

Under marginal cost pricing, firm i sets the same price irrespective of the identity of the customer it is selling to. Let p_i denote the price set by firm i. The CES production function, together with marginal cost pricing, implies that

$$p_i = \min_{I_i : \substack{ik \in \mathbf{G} \\ k \in I_i}} \left\{ \left(\alpha_i w^{1-\sigma} + \sum_{j \in I_i} \gamma_{ij} (\min\{p_j / A_{ij}, w\})^{1-\sigma} + \sum_{j \notin I_i} \gamma_{ij} w^{1-\sigma} \right)^{1/(1-\sigma)} \right\}$$
$$= \left(\alpha_i w^{1-\sigma} + \sum_{j : ij \in \mathbf{G}} \gamma_{ij} (\min\{p_j / A_{ij}, w\})^{1-\sigma} + \sum_{j : ij \notin \mathbf{G}} \gamma_{ij} w^{1-\sigma} \right)^{1/(1-\sigma)}.$$

Since $\alpha_i + \sum_{j=1}^{n} \gamma_{ij} = 1$, we can bound the right-hand side of the above equation from above by w, thus establishing that $p_i \leq w$ for all firms i. When paired with the assumption that $A_{ij} \geq 1$, this implies that

$$p_i = \left(\alpha_i w^{1-\sigma} + \sum_{j:ij \in \mathbf{G}} \gamma_{ij} (p_j / A_{ij})^{1-\sigma} + \sum_{j:ij \notin \mathbf{G}} \gamma_{ij} w^{1-\sigma} \right)^{1/(1-\sigma)}$$

for all $i = \{1, \dots, n\}$. When $\sigma \neq 1$, we can rewrite this system of equations in vector form as follows:

$$p^{\circ(1-\sigma)} = \alpha w^{1-\sigma} + (\mathbf{G} \circ \mathbf{\Gamma} \circ \mathbf{A}^{\circ(\sigma-1)}) p^{\circ(1-\sigma)} + ((\mathbf{1}\mathbf{1}' - \mathbf{G}) \circ \mathbf{\Gamma}) \mathbf{1} w^{1-\sigma}.$$

In the above expression, $p = (p_1, \dots, p_n)'$ denotes the vector of input prices, $\circ$ denotes the Hadamard (i.e., element-wise) matrix product and matrix power, $\mathbf{A} = [A_{ij}] \in \mathbb{R}^{n \times n}$ is a square matrix of pairwise productivities, $\mathbf{\Gamma} = [\gamma_{ij}] \in \mathbb{R}^{n \times n}$, and with some abuse of notation, we use $\mathbf{G}$ to denote a square binary matrix that captures pairwise supplier-customer relationships in network $\mathbf{G}$. Solving the above system of equations, we have

$$p^{\circ(1-\sigma)} = \mathbf{Q}(\alpha + ((\mathbf{1}\mathbf{1}' - \mathbf{G}) \circ \mathbf{\Gamma})\mathbf{1}) w^{1-\sigma} = \mathbf{Q}(\mathbf{I} - \mathbf{G} \circ \mathbf{\Gamma}) \mathbf{1} w^{1-\sigma},$$

where $\mathbf{Q} = (\mathbf{I} - \mathbf{G} \circ \mathbf{\Gamma} \circ \mathbf{A}^{\circ(\sigma-1)})^{-1}$ and we are using the fact that $\alpha = \mathbf{1} - (\mathbf{1}\mathbf{1}' \circ \mathbf{\Gamma})\mathbf{1}$ to obtain the second equality. Recall that, by assumption, matrix $\mathbf{H} = \mathbf{\Gamma} \circ \mathbf{A}^{\circ(\sigma-1)}$ has a subunit spectral radius. Therefore, $\mathbf{Q}$ is an inverse M-matrix and hence is element-wise nonnegative. The marginal cost of firm 0 thus satisfies

$$\mathrm{mc}_0^{1-\sigma} = \gamma_0' p^{\circ(1-\sigma)} = \gamma_0' \mathbf{Q}(\mathbf{I} - \mathbf{G} \circ \mathbf{\Gamma}) \mathbf{1} w^{1-\sigma},$$

where $\gamma_0 = (\gamma_{01}, \dots, \gamma_{0n})$. This, together with $\mathcal{A}(\mathbf{G}) = w / \mathrm{mc}_0$ leads to the following expression for aggregate productivity:

$$\mathcal{A}(\mathbf{G}) = \left[\gamma_0' \mathbf{Q}(\mathbf{I} - \mathbf{G} \circ \mathbf{\Gamma}) \mathbf{1} \right]^{1/(\sigma-1)}. \tag{B.12}$$

With the expression for aggregate productivity at hand, we next show that (B.12) satisfies the inequality in (9). Note that $\mathcal{A}(\mathbf{G})$ is a function defined over the set of binary matrices $\mathbf{G}$ with elements $g_{ij} \in \{0,1\}$. We consider the extension of the expression in (B.12), which we denote by $\bar{\mathcal{A}}(\mathbf{G})$, by assuming that g_{ij} can take any value in the unit interval $[0,1]$ and establish that

$$\frac{\partial^2 \bar{\mathcal{A}}}{\partial g_{ij} \partial g_{kr}} \geq 0 \tag{B.13}$$

for all $ij \neq kr$. If (B.13) is satisfied, then $\bar{\mathcal{A}}(\max\{\mathbf{G}_1, \mathbf{G}_2\}) + \bar{\mathcal{A}}(\min\{\mathbf{G}_1, \mathbf{G}_2\}) \geq \bar{\mathcal{A}}(\mathbf{G}_1) + \bar{\mathcal{A}}(\mathbf{G}_2)$ for any pair of matrices $\mathbf{G}_1$ and $\mathbf{G}_2$ with elements in the unit interval. The fact that $\bar{\mathcal{A}}(\mathbf{G}) = \mathcal{A}(\mathbf{G})$ for any binary matrix $\mathbf{G}$ then establishes that $\mathcal{A}(\mathbf{G})$ is a supermodular function of the production network $\mathbf{G}$, thus establishing (9) and Assumption 1(a).

To establish (B.13), observe that

$$\frac{\partial \bar{\mathcal{A}}}{\partial g_{ij}} = \frac{1}{\sigma - 1} \gamma_{ij} (\gamma_0' \mathbf{Q} e_i) \left(A_{ij}^{\sigma-1} (p_j / w)^{1-\sigma} - 1 \right) \bar{\mathcal{A}}^{2-\sigma}, \tag{B.14}$$

where e_i denotes the i-th unit vector. Since $\mathbf{Q}$ is element-wise nonnegative, it is immediate that $\gamma_0' \mathbf{Q} e_i \geq 0$. Furthermore, since $A_{ij} \geq 1$ and $p_j \leq w$, expression $A_{ij}^{\sigma-1} (p_j / w)^{1-\sigma} - 1$

has the same sign as $\sigma-1$. Consequently, $\partial\bar{A}/\partial g_{ij}\geq 0$. Next, observe that differentiating (B.14) implies that

$$\frac{\partial^2\bar{A}}{\partial g_{ij}\partial g_{kr}}=(2-\sigma)\frac{1}{\bar{A}}\frac{\partial\bar{A}}{\partial g_{ij}}\frac{\partial\bar{A}}{\partial g_{kr}}+\gamma_{ij}A_{ij}^{\sigma-1}\frac{\partial\bar{A}}{\partial g_{kr}}\frac{\gamma_0'\mathbf{Q}e_i}{\gamma_0'\mathbf{Q}e_k}e_j'\mathbf{Q}e_k+\gamma_{kr}A_{kr}^{\sigma-1}\frac{\partial\bar{A}}{\partial g_{ij}}\frac{\gamma_0'\mathbf{Q}e_k}{\gamma_0'\mathbf{Q}e_i}e_r'\mathbf{Q}e_i.$$

As we already established, $\partial\bar{A}/\partial g_{ij}\geq 0$. This, together with the fact that $\mathbf{Q}$ is element-wise nonnegative, guarantees that the second and third terms on the right-hand side of the above equation are nonnegative. Hence, (B.12) is trivially satisfied for all $\sigma\leq 2$. □

Proof of Proposition A.2

Proof of part (a). By (A.14), equilibrium gross profits satisfy

$$\theta_s(\pi_k(\mathbf{G})-\pi_k(\mathbf{G}\setminus\{ks\}))=\theta_k(\pi_s(\mathbf{G})-\pi_s(\mathbf{G}\setminus\{ks\}))\quad\text{for all } ks\in\mathbf{G}. \tag{B.15}$$

while at the same time (A.15) implies that

$$\sum_{k\in N}\pi_k(\mathbf{G})=(\mathcal{A}(\mathbf{G})-\mathcal{A}(\varnothing))L. \tag{B.16}$$

Next, consider the counterfactual economy in which after adding relationship ij to $\mathbf{G}$ the economy's aggregate productivity rises from $\mathcal{A}(\mathbf{G})$ to $\mathcal{A}(\mathbf{G}\cup\{ij\})$ but all disagreement points remain unchanged. Denoting firm k's gross profits in this economy by $\bar{\pi}_k(\mathbf{G}\cup\{ij\})$, the first-order conditions of pairwise bargaining problems imply that

$$\theta_s(\bar{\pi}_k(\mathbf{G}\cup\{ij\})-\pi_k(\mathbf{G}\setminus\{ks\}))=\theta_k(\bar{\pi}_s(\mathbf{G}\cup\{ij\})-\pi_s(\mathbf{G}\setminus\{ks\}))\quad\text{for } ks\in\mathbf{G} \tag{B.17}$$

$$\theta_j(\bar{\pi}_i(\mathbf{G}\cup\{ij\})-\pi_i(\mathbf{G}))=\theta_i(\bar{\pi}_j(\mathbf{G}\cup\{ij\})-\pi_j(\mathbf{G})), \tag{B.18}$$

where note that the disagreement points in (B.17) remain the same as in (B.15). At the same time, since aggregate productivity increases from $\mathcal{A}(\mathbf{G})$ to $\mathcal{A}(\mathbf{G}\cup\{ij\})$, an argument similar to the proof of (A.15) in Lemma A.1 implies that

$$\sum_{k\in N}\bar{\pi}_k(\mathbf{G}\cup\{ij\})=(\mathcal{A}(\mathbf{G}\cup\{ij\})-\mathcal{A}(\varnothing))L. \tag{B.19}$$

Recall that, by definition, the surplus expansion effect for any given firm k is given by $\Delta\pi_k^{\text{exp}}=\bar{\pi}_k(\mathbf{G}\cup\{ij\})-\pi_k(\mathbf{G})$. Therefore, combining (B.17) with (B.15) implies

$$\frac{1}{\theta_k}\Delta\pi_k^{\text{exp}}=\frac{1}{\theta_s}\Delta\pi_s^{\text{exp}}\quad\text{for all } ks\in\mathbf{G}. \tag{B.20}$$

Furthermore, (B.18) can be rewritten as

$$\frac{1}{\theta_i}\Delta\pi_i^{\text{exp}}=\frac{1}{\theta_j}\Delta\pi_j^{\text{exp}}. \tag{B.21}$$

Subtracting (B.16) from (B.19) leads to

$$\sum_{k\in N}\Delta\pi_k^{\text{exp}}=(\mathcal{A}(\mathbf{G}\cup\{ij\})-\mathcal{A}(\mathbf{G}))L. \tag{B.22}$$

Since production network $\mathbf{G}$ is connected by assumption, the system of equations (B.20)–(B.22) always has a unique solution, which is given by (A.4). □

Proof of part (b). Applying equation (A.15) to production networks $\mathbf{G}$ and $\mathbf{G}\cup\{ij\}$ implies that firms' equilibrium profits satisfy

$$\sum_{k\in N}\pi_k(\mathbf{G}\cup\{ij\})-\sum_{k\in N}\pi_k(\mathbf{G})=(\mathcal{A}(\mathbf{G}\cup\{ij\})-\mathcal{A}(\mathbf{G}))L,$$

This, together with (B.22) and the fact that $\Delta\pi_k^{\text{exp}}+\Delta\pi_k^{\text{red}}=\pi_k(\mathbf{G}\cup\{ij\})-\pi_k(\mathbf{G})$, then guarantees that $\sum_{k\in N}\Delta\pi_k^{\text{red}}=0$.

We next establish that $\Delta\pi_i^{\text{red}}\geq 0$. The argument for $\Delta\pi_j^{\text{red}}\geq 0$ is identical. By (6), the change in i's equilibrium profits as a result of adding relationship ij to $\mathbf{G}$ is given by

$$\pi_i(\mathbf{G}\cup\{ij\})-\pi_i(\mathbf{G})=\theta_i\sum_{T\subseteq N\setminus\{i\}}\psi_i(T)\left[\mathcal{A}(\mathbf{G}\cup\{ij\}|_{T\cup\{i\}})-\mathcal{A}(\mathbf{G}|_{T\cup\{i\}})\right]L, \tag{B.23}$$

where $\psi_i(T)$ is given by (7) and we are using the fact that $\mathcal{A}(\mathbf{G}\cup\{ij\}|_T)=\mathcal{A}(\mathbf{G}|_T)$ for all $T\subseteq N\setminus\{i\}$. When $T=N\setminus\{i\}$, the summand on the right-hand side of (B.23) reduces to

$$\theta_i\psi_i(N\setminus\{i\})\left[\mathcal{A}(\mathbf{G}\cup\{ij\})-\mathcal{A}(\mathbf{G})\right]L=\frac{\theta_i}{\theta_0+\cdots+\theta_n}\left[\mathcal{A}(\mathbf{G}\cup\{ij\})-\mathcal{A}(\mathbf{G})\right]L=\Delta\pi_i^{\text{exp}},$$

where the second equality follows from (A.4). Therefore, subtracting both sides of the above equation from (B.23) and using $\Delta\pi_i^{\text{red}}=\pi_i(\mathbf{G}\cup\{ij\})-\pi_i(\mathbf{G})-\Delta\pi_i^{\text{exp}}$, implies

$$\Delta\pi_i^{\text{red}}=\theta_i\sum_{T\subsetneq N\setminus\{i\}}\psi_i(T)\left[\mathcal{A}(\mathbf{G}\cup\{ij\}|_{T\cup\{i\}})-\mathcal{A}(\mathbf{G}|_{T\cup\{i\}})\right]L.$$

Recall from (7) that $\psi_i(T)\geq 0$ for all T. Furthermore, $\mathcal{A}(\mathbf{G}\cup\{ij\}|_{T\cup\{i\}})\geq\mathcal{A}(\mathbf{G}|_{T\cup\{i\}})$. Thus, the right-hand side of the above equation is nonnegative, implying that $\Delta\pi_i^{\text{red}}\geq 0$. □

Proof of part (c). Inequality (A.5) is simply a restatement of Theorem 3. To establish (A.6), note that equation (A.15) implies that

$$\sum_{k\in N}(\Delta\pi_k^{\text{exp}}+\Delta\pi_k^{\text{red}})=(\mathcal{A}(\mathbf{G}\cup\{ij\})-\mathcal{A}(\mathbf{G}))L.$$

This, together with in (A.5), then establishes (A.6). □

REFERENCES

Acemoglu, Daron and Pablo Azar (2020), "Endogenous production networks." *Econometrica*, 88(1), 33–82.