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e-companion

ONLY AVAILABLE IN ELECTRONIC FORM

Electronic Companion—"Risk Mitigation in Newsvendor Networks: Resource Diversification, Flexibility, Sharing, and Hedging" by Jan A. Van Mieghem, *Management Science*, DOI 10.1287/mnsc.1070.0700.

Online Appendix

Technical Assumptions

Clearly, $V(K, \omega)$ must have an expected value; that is, it is a measurable and integrable function of ω for every K. Interchanging differentiation and integration requires conditions that bound the derivatives such as requiring that all functions are Lipschitz, as in the general conditions in Appendix A of Broadie and Glasserman (1996). In our setting all functions are concave and thus absolutely continuous on any compact subset of $\mathbb{R}^n_+$, where its right- and left-hand partial derivatives exist (and thus are finite) and are monotone increasing. Let $\nabla f(K, \omega)$ denote this vector of right-hand partial derivatives of a concave function f with respect to K. An absolutely continuous function f satisfies a Lipschitz condition if and only if $\|\nabla f\|$ is bounded. Clearly, for a concave function, $\nabla f(K, \omega)$ is bounded, and thus Lipschitz, on any compact subset in $\mathbb{R}^n_+$. The only technical condition is to require finite derivatives also at 0 and ∞ and require a Lipschitz condition on the open set $\mathbb{R}^n_+$:

Assumption 1. *The value function $V(K, \omega)$ satisfies a Lipschitz condition on $\mathbb{R}^n_+$ almost surely: there is a $M_V(\omega)$ such that $|V(K_1, \omega) - V(K_2, \omega)| \le M_V(\omega)\|K_1 - K_2\|$ for all $K_1, K_2 \in \mathbb{R}^n_+$, where $\mathbb{E}M_V(\omega) < \infty$.*

Assumption 2. *The utility function $u(x)$ satisfies a Lipschitz condition on $\mathbb{R}$: there is a M_u such that $|u(x_1) - u(x_2)| \le M_u|x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}$.*

For newsvendor networks, π and thus also V are the solution of a linear program and have finite partial derivatives so that Assumption 1 is always satisfied. It is not unrealistic to assume that demands are bounded which then obviates Assumption 2.

Proof of Proposition 1. The function $V(\mathbf{K}, \omega)$ is concave in $\mathbf{K}$ for any ω as a sum of two concave functions. Because $u(\cdot)$ is concave increasing, the scalar composition $u(V(\mathbf{K}, \omega) + W)$ is also concave in $\mathbf{K}$ for any ω and W. (The latter is directly shown for twice differentiable functions, but also holds without assuming differentiability, see Boyd and Vandeberghe 2004, p. 84.) Finally, the expected utility function is concave as a linear combination of concave functions. □

Proof of Proposition 2. Let $f(\mathbf{K}, \omega) = u(V(\mathbf{K}, \omega) + W)$, which is concave in K for any ω and wealth W according to the proof of Proposition 1 so that $\nabla U(\mathbf{K}^*) = 0$ is necessary and sufficient for an interior maximum.

Because f is concave, its right-hand partial gradient $\nabla f(\mathbf{K}, \omega)$ exists for every ω. Thus, for all $\mathbf{K} \in \mathbb{R}^n_+$ and $m > 0$, $g_m = m(f(\mathbf{K} + m^{-1}\mathbf{e}_i, \omega) - f(x, \omega)) \to_m \nabla_i f(\mathbf{K}, \omega)$, where $\mathbf{e}_i$ is the ith unit vector. Given that the Lipschitz property is preserved by composition, the technical assumptions guarantee the existence of Lipschitz modulus $M(\omega)$ for f w.p. 1 that is integrable. Because $|g_m(\mathbf{K}, \omega)| < M(\omega)$ with $\mathbb{E}M(\omega) < \infty$, the dominated convergence theorem shows that $\lim_{m\to\infty} \mathbb{E}g_m = \mathbb{E}\lim_{m\to\infty} g_m$. Thus, differentiation and integration interchange so that $\nabla U(\mathbf{K}) = \mathbb{E}\nabla f = \mathbb{E}u'(V + W)\nabla V$. □

Proof of Proposition 3. Applying the definition of covariance and given that integration and differentiation can be interchanged, we have that:

$$\begin{aligned}\nabla\sigma^2(\mathbf{K}) &= \nabla_{\mathbf{K}}(\mathbb{E}\pi^2(\mathbf{K}, \mathbf{D}) - (\mathbb{E}\pi(\mathbf{K}, \mathbf{D}))^2) = \mathbb{E}\nabla_{\mathbf{K}}\pi^2(\mathbf{K}, \mathbf{D}) - \nabla_{\mathbf{K}}(\mathbb{E}\pi(\mathbf{K}, \mathbf{D}))^2 \\ &= 2\mathbb{E}[\pi(\mathbf{K}, \mathbf{D})\lambda(\mathbf{K}, \mathbf{D})] - 2(\mathbb{E}\pi(\mathbf{K}, \mathbf{D}))\mathbb{E}\lambda(\mathbf{K}, \mathbf{D}) = 2\,\mathrm{Cov}(\lambda, \pi).\end{aligned}$$

For the second part, again applying the definition of covariance, we have that

$$\begin{aligned}\mathbb{E}[(\boldsymbol{\lambda}(\mathbf{K}^n,\omega)-\mathbf{c})V(\mathbf{K}^n,\omega)] &= \operatorname{Cov}(\boldsymbol{\lambda}(\mathbf{K}^n,\omega)-\mathbf{c}, V(\mathbf{K}^n,\omega))+\mathbb{E}[\boldsymbol{\lambda}(\mathbf{K}^n,\omega)-\mathbf{c}]\mathbb{E}V(\mathbf{K}^n,\omega)\\ &= \operatorname{Cov}(\boldsymbol{\lambda}(\mathbf{K}^n,\omega)-\mathbf{c}, \pi(\mathbf{K}^n,\omega)-C(\mathbf{K}^n)) \quad (\text{second term}=0 \text{ by } (3))\\ &= \operatorname{Cov}(\boldsymbol{\lambda}(\mathbf{K}^n,\omega), \pi(\mathbf{K}^n,\omega)) \quad (\text{constants fall out}). \quad \square\end{aligned}$$

Proof of Proposition 4. According to the implicit function theorem, $\mathbf{K}(\gamma)$ is a continuous function of γ where $(d/d\gamma)\mathbf{K}(\gamma)$ is found by differentiating the first-order condition: $(d/d\gamma)\mathbb{E}[(\boldsymbol{\lambda}(\mathbf{K},\omega)-\mathbf{c})\cdot\exp(-\gamma V(\mathbf{K},\omega))]=0$ or

$$\begin{aligned}&\mathbb{E}[(\nabla_{\mathbf{K}}\boldsymbol{\lambda}(\mathbf{K},\omega))\exp(-\gamma V(\mathbf{K},\omega))+(\boldsymbol{\lambda}(\mathbf{K},\omega)-\mathbf{c})\exp(-\gamma V(\mathbf{K},\omega))(-\gamma)\nabla_{\mathbf{K}}V]\frac{d}{d\gamma}\mathbf{K}(\gamma)\\ &\quad+\mathbb{E}[(\boldsymbol{\lambda}(\mathbf{K},\omega)-\mathbf{c})\exp(-\gamma V(\mathbf{K},\omega))(-V(\mathbf{K},\omega))]=0\end{aligned}$$

Recall that the Hessian $H(\mathbf{K}^n)=\mathbb{E}[(\nabla_{\mathbf{K}}\boldsymbol{\lambda}(\mathbf{K}^n,\omega))]$ and evaluate at the risk-neutral case $\gamma=0$ to get:

$$H(\mathbf{K}^n)\frac{d}{d\gamma}\mathbf{K}(0)=\mathbb{E}[(\boldsymbol{\lambda}(\mathbf{K}^n,\omega)-\mathbf{c})V(\mathbf{K}^n,\omega)].$$

Given that Π is concave, its Hessian $H(\mathbf{K}^n)$ is negative-definite, and invertible. $\square$

Proof of (9). As illustrated in Figure 2, the MV-frontier function $\mathcal{F}$ and the maximal utility $U^{\mathrm{MV}}(\gamma)$, denoted as $\mathcal{U}(\gamma)$, are almost inverse functions in that they satisfy, except at possible inflection points: $\mathcal{U}'(\mathcal{F}'(x))=-x$ and thus $\mathcal{U}''(\mathcal{F}'(x))=-1/\mathcal{F}''(x)$. Evaluating at $x=\sigma^2(\mathbf{K}^n)$, where $z=\mathcal{F}'(x)=0$, directly yields $\mathcal{U}'(0)=-\sigma^2(\mathbf{K}^n)$ and $\mathcal{F}''(\sigma^2(\mathbf{K}^n))=-1/\mathcal{U}''(0)$. It only remains to find $\mathcal{U}''(0)$. Twice differentiate the defining condition of $\mathbf{K}(z)$:

$$\begin{aligned}\frac{d}{d\gamma}\mathcal{U}(\gamma)&=\nabla'\mu(\mathbf{K}(\gamma))\frac{d}{d\gamma}\mathbf{K}-\frac{\gamma}{2}\nabla'\sigma^2(\mathbf{K}(\gamma))\frac{d}{d\gamma}\mathbf{K}-\frac{1}{2}\sigma^2(\mathbf{K}(\gamma)),\\ \frac{d^2}{d\gamma^2}\mathcal{U}(\gamma)&=\left(H(\mathbf{K}(\gamma))\frac{d}{d\gamma}\mathbf{K}\right)'\frac{d}{d\gamma}\mathbf{K}+\nabla'\mu(\mathbf{K}(\gamma))\frac{d^2}{d\gamma^2}\mathbf{K}-\frac{1}{2}\nabla'\sigma^2(\mathbf{K}(\gamma))\frac{d}{d\gamma}\mathbf{K}\\ &\quad-\frac{\gamma}{2}\frac{d}{d\gamma}\left(\nabla'\sigma^2(\mathbf{K}(\gamma))\frac{d}{d\gamma}\mathbf{K}\right)-\frac{1}{2}\nabla'\sigma^2(\mathbf{K}(\gamma))\frac{d}{d\gamma}\mathbf{K}.\end{aligned}$$

Evaluate at $\gamma=0$ and recall that $\nabla\mu(\mathbf{K}(0))=0$ and $H(\mathbf{K}(0))(d/d\gamma)\mathbf{K}(0)=\nabla\sigma^2(\mathbf{K}(0))$:

$$\mathcal{U}''(0)=-\nabla'\sigma^2(\mathbf{K}(0))'\frac{d}{d\gamma}\mathbf{K}(0)=-\nabla'\sigma^2(\mathbf{K}^n)H^{-1}(\mathbf{K}^n)\nabla\sigma^2(\mathbf{K}^n). \quad \square$$

Proof of Property 1. Optimal activity $x=\min(K,D)$ so that $\lambda=v1_{\{D\geq K\}}$ with the familiar risk-neutral optimality condition $\mathbb{E}(\lambda-c)=v(1-F(K))-c=0$. Using the standard normal pdf ϕ and cdf Φ, $v(1-\Phi(z^n))=c$ where $z^n=(K^n-\mu_1)/\sigma_1$. Hence, $H(K^n)=(d/dK)\mathbb{E}\lambda=-(v/\sigma_1)\phi(z^n)<0$ and

$$\begin{aligned}\mathbb{E}[(\boldsymbol{\lambda}(\mathbf{K}^n,\omega)-\mathbf{c})V(\mathbf{K}^n,\omega)]&=\int_{-\infty}^{K^n}(-c)(vx-cK^n)\,dF+\int_{K^n}^{\infty}(v-c)(vK^n-cK^n)\,dF\\ &=-cv\underbrace{\int_{-\infty}^{K^n}x\,dF}_{A}+\underbrace{c^2K^nF(K^n)+(v-c)^2K^n(1-F(K^n))}_{B}\end{aligned}$$

For the normal distribution, integration by parts yields $A=\mu_1\Phi(z^n)-\sigma_1\phi(z^n)=\mu_1(1-c/v)-\sigma_1\phi(z^n)$. Moreover $B=[c^2(1-c/v)+(v-c)^2c/v]K^n=c(v-c)K^n$. Putting this together yields

$$\begin{aligned}\mathbb{E}[(\boldsymbol{\lambda}(\mathbf{K}^n,\omega)-\mathbf{c})V(\mathbf{K}^n,\omega)]&=-cv(\mu_1(1-c/v)-\sigma_1\phi)+c(v-c)(\mu_1+z\sigma_1)\\ &=\sigma_1cv(\phi(z^n)+(1-c/v)z^n),\end{aligned}$$

which is nonnegative because $f(z)=\phi(z)+z\Phi(z)$ has $f(-\infty)=0$ and is nondecreasing ($f'(z)=\Phi(z)\geq 0$). $\square$

Proof of Property 3. Using the notation of the proof of Property 2:

$$\begin{aligned}\tfrac{1}{2}\nabla_2\sigma^2(\mathbf{K}^n) &= \mathbb{E}(\lambda_2(\mathbf{K}^n,\omega)-c_2)V(\mathbf{K}^n,\omega)\\ &= (v_2-c_2)\mathbb{E}_{13}V(\mathbf{K}^n,\mathbf{D})-c_2\mathbb{E}_{04}V(\mathbf{K}^n,\mathbf{D})\\ &\le V_1(\mathbf{K}^n)[(v_2-c_2)P_1(\mathbf{K}^n)-c_2P_{04}(\mathbf{K}^n)]=0.\end{aligned}$$

Also $H_{12}=(\partial/\partial K_2)\mathbb{E}\lambda_1(\mathbf{K}^n,\omega)=(\partial/\partial K_2)v_1P(D_1>K_1^n)=0$, so that

$$H^{-1}(\mathbf{K}^n)=\begin{bmatrix}\dfrac{-1}{v_1f_1(K_1^n)} & 0\\ 0 & \dfrac{-1}{v_2f_2(K_2^n)}\end{bmatrix},$$

where $f_i(\cdot)$ is the p.d.f. of D_i. □

Proof of Property 4 is similar to that of Property 2: The activity vector $\mathbf{x}(\mathbf{K},\mathbf{D})$ again is a simple greedy solution: $x_1=\min(D_1,K_1,K_3)$ and $x_2=\min(D_2,K_2,K_3-x_1)$. The optimal risk-neutral $\mathbf{K}^n$ satisfies the optimality conditions $\mathbb{E}\boldsymbol{\lambda}(\mathbf{K}^u,\omega)=\mathbf{c}$:

$$(v_1-v_2)P_3(\mathbf{K}^n)+v_1P_4(\mathbf{K}^n)=c_1,\qquad v_2P_1(\mathbf{K}^n)=c_2,\qquad v_2P_{2+3}(\mathbf{K}^n)=c_3. \tag{EC1}$$

According to Proposition 2, the (second) optimality condition for the risk-averse resource vector $\mathbf{K}^u$ is:

$$0=(v_2-c_2)\mathbb{E}_1u'(V(\mathbf{K}^u,\mathbf{D})+W)-c_2\mathbb{E}_{0234}u'(V(\mathbf{K}^u,\mathbf{D})+W).$$

Case 1. $\sigma_1\ge\sigma_2$ as shown in Figure EC.1. Notice that it is suboptimal for the point $(K_1^u,K_3^u-K_1^u)$ to be above the demand line because reducing K_3^u by ϵ would not change operating profits but reduce investment costs. Thus there are two possible cases: moderate c_3 so that $(K_3^n-K_2^n,K_2^n)$ falls above/at the demand line; with high c_3 it falls below. The property applies in the former case. The proof is similar to that of Property 2: define the vector $\mathbf{x}_1(\mathbf{K})$ and scalar $k_1(\mathbf{K})$ and establish that, if $k_1(\mathbf{K}^u)<K_1^u$, then $P_1(\mathbf{K}^u)\le c_2/v_2=P_1(\mathbf{K}^n)$. Note that $P_1(\mathbf{K})=P(D_2>K_2)$ so that $K_2^u\ge K_2^n$. The required conditions are (1) and (2) of the proof of Property 2 and (3) the point $(K_3^n-K_2^n,K_2^n)$ falls above the demand line or $z_1(K_3^n-K_2^n)+z_2(K_2^n)>0$. The increasing-in-risk aversion is proved similarly to the proof of Property 2.

Case 2. $\sigma_1<\sigma_2$ proceeds similarly but uses the point $\mathbf{x}_1(\mathbf{K})=(K_3-K_2,K_2)$. The required conditions are (1) of the proof of Property 2; (2) $k_1(\mathbf{K}^n)<K_1^n$ or $K_3^n<K_1^n+(1-v_2/v_1)K_2^n$; and (3) the point $(K_1^n,K_3^n-K_1^n)$ falls above the demand line or $z_1(K_1^n)+z_2(K_3^n-K_1^n)>0$. □

Proof of Property 5. Use the notation of the proof of Property 4.

$$\begin{aligned}\tfrac{1}{2}\nabla_2\sigma^2(\mathbf{K}^n) &= \mathbb{E}(\lambda_2(\mathbf{K}^n,\omega)-c_2)V(\mathbf{K}^n,\omega)\\ &= (v_2-c_2)\mathbb{E}_1V(\mathbf{K}^n,\mathbf{D})-c_2\mathbb{E}_{0234}V(\mathbf{K}^n,\mathbf{D})\\ &\le V_1(\mathbf{K}^n)[(v_2-c_2)P_1(\mathbf{K}^n)-c_2P_{0234}(\mathbf{K}^n)]=0.\end{aligned}$$

Figure EC.1 The Activity Vector x for the Serial Network When $\sigma_1\ge\sigma_2$ and $\rho=-1$

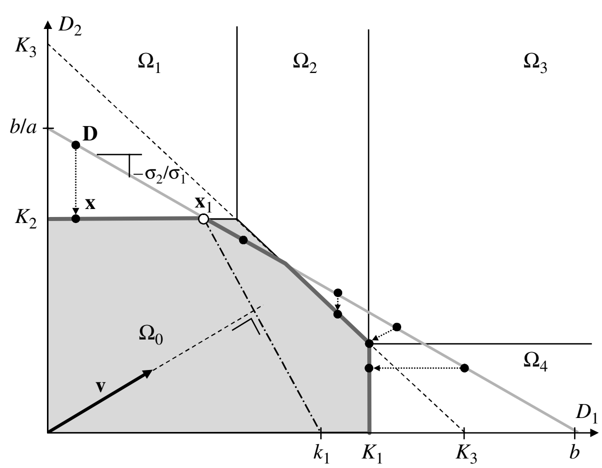
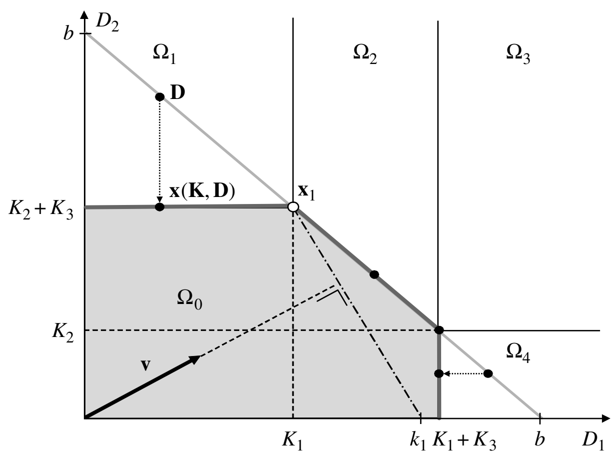


Figure EC.2 The Activity Vector x for the Parallel Network When $\sigma_1 = \sigma_2$ and $\rho = -1$



It is also easily verified that

$$H^{-1}(\mathbf{K}^n) = \frac{1}{|H|}\begin{bmatrix} \cdots & 0 & \cdots \\ 0 & \cdots & 0 \\ \cdots & 0 & \cdots \end{bmatrix},$$

where all nonzero elements (denoted by $\cdots$) are positive and $|H| < 0$. Similarly, if $v_1 = v_2$:

$$\begin{aligned} \tfrac{1}{2}\nabla_3\sigma^2(\mathbf{K}^n) &= \mathbb{E}(\lambda_3(\mathbf{K}^n, \omega) - c)V(\mathbf{K}^n, \omega) \\ &= (v_2 - c_3)\mathbb{E}_{23}V(\mathbf{K}^n, \mathbf{D}) - c_3\mathbb{E}_{014}V(\mathbf{K}^n, \mathbf{D}) \\ &\geq V_3(\mathbf{K}^n)[(v_2 - c_3)P_{23}(\mathbf{K}^n) - c_3P_{014}(\mathbf{K}^n)] = 0, \end{aligned}$$

where $V_3(\mathbf{K}^n) = V(\mathbf{K}^n, \mathbf{x}_3(\mathbf{K}^n))$ where $\mathbf{x}_3(\mathbf{K}) = (K_1, K_3 - K_1)$. □

Proof of Property 6 is similar to that of Property 2: Notice that $K_1^u + K_2^u + K_3^u > b$ is suboptimal because reducing K_3^u by ϵ and increasing K_1^u by ϵ would not change operating profits but would decrease investment cost by $(c_3 - c_1)\epsilon > 0$. Thus there are two possible cases: $K_1^u + K_2^u + K_3^u < b$ if c_3 is high and $K_1^u + K_2^u + K_3^u = b$ otherwise. The property applies in the latter boundary case which is shown in Figure EC.2.

Let $\mathbf{K}_{1:2} = (K_1, K_2)$ be the independent variable for this boundary case where $K_3 = b - K_1 - K_2$. The associated two-dimensional shadow vector on this boundary has components $\lambda_1^b(\mathbf{K}_{1:2}^u, \mathbf{D}) = -v_2 1_{\{\mathbf{D}\in\Omega_1(\mathbf{K}_{1:2}^u)\}}$ and $\lambda_2^b(\mathbf{K}_{1:2}^u, \mathbf{D}) = -v_1 1_{\{\mathbf{D}\in\Omega_4(\mathbf{K}_{1:2}^u)\}}$ with effective marginal cost $c^b = (c_1 - c_3, c_2 - c_3) < 0$. The risk-neutral boundary solution satisfies $\mathbb{E}\boldsymbol{\lambda}^b(\mathbf{K}^n, \mathbf{D}) = \mathbf{c}^b$ so that $P_1(\mathbf{K}^n) = (c_3 - c_1)/v_2$ and $P_4(\mathbf{K}^n) = (c_3 - c_1)/v_2$. Define the vector $\mathbf{x}_1(\mathbf{K}) = (K_1, K_2 + K_3)$ to partition and bound marginal utilities similar to the proof of Property 2: The optimality conditions for $\mathbf{K}^u$ include

$$\begin{aligned} 0 &= \mathbb{E}(\lambda_1^b(\mathbf{K}^u, \mathbf{D}) - c_1^b)u'(V(\mathbf{K}^u, \mathbf{D}) + W) \\ &= (-v_2 - c_1 + c_3)\mathbb{E}_1 u'(V(\mathbf{K}^u, \mathbf{D}) + W) + (-c_1 + c_3)\mathbb{E}_{24}u'(V(\mathbf{K}^u, \mathbf{D}) + W) \\ &\leq u'(V_1(\mathbf{K}^u))[-v_2P_1(\mathbf{K}^u) - c_1 + c_3] \Rightarrow P_1(\mathbf{K}^u) \leq (c_3 - c_1)/v_2. \end{aligned}$$

Thus, $P_1(\mathbf{K}^u) \leq P_1(\mathbf{K}^n)$ so that $K_1^u \leq K_1^n$ and $K_2^u + K_3^u \geq K_2^n + K_3^n$. The increase in risk aversion is proved similarly to the proof of Property 2. The required conditions are (1) $v_1 > v_2$ and $\sigma_1 = \sigma_2$; (2) $k_1(\mathbf{K}^n) < K_1^n + K_3^n$ or $v_2K_2^n < (v_1 - v_2)K_3^n$; (3) $K_1^n + K_2^n + K_3^n = b$ or conditions (c) of Proposition 7 of Van Mieghem (1998, Proposition 7). □

References

See references list in the main paper.

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