Rationality and equilibrium in perfect-information games

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Abstract

In generic perfect-information games the unique Subgame-Perfect Equilibrium (SPE) outcome is identical to the one predicted by several rationalizability notions, like Extensive-Form Rationalizability (EFR), the Backward Dominance Procedure (BDP), and Extensive-Form Rationalizability of the Agent form (AEFR). We show that, in contrast, within the *general* class of perfect information games all these solution concepts are genuinely distinct in terms of the outcomes that they predict: SPE is more restrictive than EFR, which is in turn more restrictive than BDP, which is, finally, more restrictive than AEFR.

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1 Introduction

In a large, robust class of strategic-form games Nash equilibrium is a more restrictive solution concept than rationalizability, as the set of Nash equilibria is strictly contained in the set of rationalizable strategy profiles. In contrast, the relation between equilibrium and rationalizability seems to be different in extensive-form games with perfect information. The basic equilibrium notion for extensive-form games with perfect information is *Subgame-Perfect Equilibrium (SPE)*, and in generic perfect-information games the unique *SPE* outcome *conicides* with the predictions of a variety of rationalizability notions with different corresponding epistemic characterizations:

1. *Extensive-Form Rationalizability (EFR)*, introduced by Pearce (1984)[1], is a solution concept that embodies *forward induction reasoning*: In each decision node the active player asks herself how she can best rationalize her opponents' past behavior, and what could be deduced from this best rationalization about these opponents' future behavior. When the player knows that her opponents did not behave irrationally in the past, she assumes they will continue to be rational in the future; if, on top of that, she observes that her opponents' past behavior can be rationalized under the assumption that each of them believed *their* opponents to be rational, she assumes that they will continue to optimize under this belief in the future (as long as such a belief is sustainable given the history of play); and so forth. *EFR* outcomes are characterized epistemically as those attained under *rationality and common strong belief of rationality* (Battigalli and Siniscalchi 2002). In generic perfect-information games there is a unique EFR outcome, which coincides with the unique SPE outcome, despite of the fact that the EFR strategies of some players may be *distinct* from their SPE strategies[2]. This result was proved by Reny (1993) and Battigalli (1997) using Kohlberg-Mertens stability, and different, more elementary proofs were recently given by Robles (2006), Perea (forthcoming), and Chen and Micali (2011).

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[1] See also Battigalli (1997) for an equivalent definition. In the case of more than two players, we refer here to the notion of *correlated* extensive-form rationalizability.

[2] Reny (1992) gave an example of a game possessing this property (see also Perea, forthcoming).

2. *Backward Dominance Procedure (BDP)*, whose outcomes are characterized epistemically as those attained under *rationality and common belief in future rationality,* was recently introduced by Perea (2010). With this solution concept the players are assumed to be relentlessly optimistic: no matter how often they have already witnessed irrational behavior (or behavior of a player which could only be rationalized by assuming the player believed that her opponents will behave foolishly in the future, or that she believed that these opponents will believe that others will behave irrationally, etc.), in the remaining subgame they optimize under the (recursively-defined) assumption of common future belief in rationality in the current and future decision nodes.

3. *Agent Extensive-Form Rationalizability (AEFR)* is the *EFR* notion applied to the agent-form of the game, in which an independent agent acts at each decision node, and different agents of the same player only happen to have the same payoff function. Hence, again by Battigalli and Siniscalchi (2002), *AEFR* is characterized by rationality and common strong belief of rationality among the agents. *AEFR* is directly akin to Backward Induction (BI) – indeed, since no agent plays more than once, no forward-induction reasoning can take place. Each agent maximizes its expected payoff contigent on its node being reached (whether or not this is consistent with rationality or mutual belief in rationality of preceding agents). In generic perfect-information games *AEFR* thus gives rise to the unique *BI* strategy of each player.

This state of affairs is puzzling, because it suggests that in spite of the conceptual differences among these three rationalizability notions, and in spite of the conceptual wedge between them and the idea of subgame perfection, it is conceivable that all these distinct solution concepts are nevertheless equivalent in perfect-information games.

Alternatively, it is conceivable that there is no inclusion relation, strict or otherwise, between these rationalizability notions and subgame perfection in general perfect-information games. That is, it is possible that in some games there are subgame perfect equilibrium outcomes that are not rationalizable and some rationalizable outcomes that are not the outcomes of subgame perfect equilibria.

In this paper we show that neither of these conceivable scenarios occurs. More specifically, we show that in general, within the entire class of perfect-information games, (i) SPE outcomes are strictly contained in the set of EFR outcomes; (ii) EFR outcomes are strictly contained in the set of BDP outcomes; and (iii) BDP outcomes are strictly contained in the set of of AEFR outcomes. Result (i) is proved in Theorem 1 and Figure 1; as for (ii), weak inclusion was proved by Perea (forthcoming) and Chen and Micali (2011), and strict inclusion is demonstrated in Figure 2; finally, (iii) is proved in Theorem 2 and Figure 3.

Thus, while the absence of simultaneous moves masks the difference between equilibrium and rationalizability in generic perfect-information games, the difference gets *unveiled* once perfect-information games with payoff ties are considered. Payoff ties are natural in various entry/exit games and in bargaining games, as demonstrated for example in Figure 1.

The paper is organized as follows. Section 2 is devoted to general definitions and notation for perfect-information games. Section 3 presents the three rationalizability solution concepts that we study, along with algorithms that facilitate their computation. Sections 4, 5, and 6 prove that these solution concepts are strictly ordered in terms of the sets of outcomes they induce, and that the equilibrium notion of subgame perfection is strictly more demanding than all of them.

2 Perfect-information games

An extensive-form game G with perfect-information is defined by a finite set of players I, a finite set of decision nodes N, an active player $i \in N$ at each node $n \in N$ along with her action set A_n^i there, and terminal nodes Z, each $z \in Z$ with a payoff vector $(p_i^z)_{i \in I} \in \mathbb{R}^I$ for the players. The nodes $\bar{N} = N \cup Z$ constitute a tree, i.e. they are partially ordered by a precedence relation $\rightsquigarrow$ with which the $\left(\bar{N}, \rightsquigarrow\right)$ is an arborescence (that is, the predecessors of each node in $\bar{N}$ are totally ordered by $\rightsquigarrow$). Furthermore, there is a unique node $r \in N$ with no predecessors – the root of the tree. All the terminal nodes $z \in Z$ have no successors, and hence they are also called the leaves of the tree. $N_i \subseteq N$ denotes the set of nodes at which player i is active. Each action $a_n^i \in A_n^i$ leads to another node $n' \in \bar{N}$ which immediately succeeds n according to $\rightsquigarrow$.

A strategy s_i of player i specifies for every decision node $n \in N_i$ the action $s_i(n) \in A_n^i$ that player i takes at n. The set of strategies of player

i is denoted by S_i, and $S = \prod_{i \in I} S_i$ denotes the players' strategy profiles. $S_{-i} = \prod_{j \neq i} S_j$ is the set of strategy profiles of all players but i.

Each strategy profile $s = (s_i)_{i \in I} \in S$ defines a path in the tree from the root to one of the leaves. We say that s reaches the node $n \in \bar{N}$ if n is on this path. We further say that $s_i \in S_i$ reaches n if there exists a profile $s_{-i} \in S_{-i}$ such that (s_i, s_{-i}) reaches n. Similarly, we say that $s_{-i} \in S_{-i}$ reaches n if there exists a profile $s_i \in S_i$ such that (s_i, s_{-i}) reaches n. Denote by $S_i(n)$ the set of player i's strategies that reach n and by $S_{-i}(n)$ is the set of the opponents' strategy profiles that reach n.

For a subset of strategies $D_i \subseteq S_i$ we say that D_i reaches the node n if there exists a strategy $s_i \in D_i$ that reaches n. Similarly, we say that a subset $D_{-i} \subseteq S_{-i}$ reaches n if there exists $s_{-i} \in D_{-i}$ that reaches n.

For each $n \in N$ denote by G^n the subgame defined by the subtree of nodes emanating from n as the root. Each strategy $s_i \in S_i$ induces a strategy s_i^n in G^n; the set of these induced strategies is denoted by S_i^n. When there is no risk of confusion, we sometimes abuse notation slightly and identify G^n with the set of nodes in G^n.

Finally, for each node $n \in N$, denote by $d(n)$ the maximal distance from n to a terminal node.

3 Rationalizability Solution Concepts

We now turn to describe the rationalizability solution concepts that we will compare in the next section.

3.1 Extensive-form rationalizability (EFR)

Pearce (1984) defined extensive-form (correlated) rationalizable strategies by a procedure of an iterative elimination of strategies. The idea behind the definition involves a notion of forward induction. In generic perfect-information games, rationalizable strategy profiles yield the backward induction outcome, though they need not be subgame-perfect equilibrium strategies (Reny 1992, Battigalli 1997).

In a perfect-information game, a *belief system* of player i

$$b_i = (b_i(n_i))_{n_i \in N_i} \in \prod_{n_i \in N_i} \Delta(S_{-i})$$

is a profile of beliefs - a belief $b_i(h_i) \in \Delta(S_{-i})$ about the other players' strategies for each decision node $n_i \in N_i$, with the following properties

- $b_i(n_i)$ reaches n_i, i.e. $b_i(n_i)$ assigns probability 1 to the set of strategy profiles of the other players that reach n_i.
- If n_i precedes n'_i $(n_i \rightsquigarrow n'_i)$ then $b_i(n'_i)$ is derived from $b_i(n_i)$ by Bayes rule whenever possible.

We say that the belief system b_i 'starts anew' at a node n'_i in which Bayes rule cannot be applied (because at the preceding node n_i, $b_i(n_i)$ assigned probability 0 to the other players' strategy profiles that reach n'_i).

Denote by B_i the set of player i's belief systems.

For a belief system $b_i \in B_i$, a strategy $s_i \in S_i$ and a node $n_i \in N_i$, define player i's expected payoff *at* n_i to be the payoff for player i given $b_i(n_i)$, the actions prescribed by s_i at n_i and its successors, and conditional on the fact that n_i has been reached.[3]

We say that with the belief system b_i and the strategy s_i player i is *rational* at the node $n_i \in N_i$ if there exists no action $a'_{n_i} \in A_{n_i}$ such that only replacing the action $s_i(n_i)$ by a'_{n_i} results in a new strategy, denoted s_i/a'_{n_i}, which yields player i a higher expected payoff at n_i.

We say that the strategy s_i is *supported* by the belief system b_i if s_i is rational for i vis-a-vis b_i at *all* of her decision nodes $n_i \in N_i$.

We now turn to define extensive-form rationalizability in extensive-form games.[4]

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[3] Even if this condition is counterfactual due to the fact that the strategy s_i does not reach n_i. The conditioning is thus on the event that player i's past actions (in the nodes preceding n_i) have led to n_i even if these actions are distinct than those prescribed by s_i.

[4] The following definition is a slight modification of Battigalli's (1997) definition of (correlated) extensive-form strategies (which he proved to be equivalent to that of Pearce 1984): our definition requires an extensive-form rationalizable strategy s_i to be optimal w.r.t. some belief also at nodes which are excluded by the actions of s_i at some preceding node.

This means that in perfect-information games our definition refines the Pearce-Battigalli definition, but gives rise to the same plans of action. (A plan of action of a player is an equivalence class of her strategies which are identical in all the player's nodes that are not excluded by any of these strategies.)

Another slight difference between the definition here and that of Battigalli (1997) is

Definition 1 (EFR) *Define, inductively, the following sequence of belief systems and strategies of player i.*

$B_i^1 = B_i$

$R_i^1 = \{s_i \in S_i$: there exists a belief system $b_i \in B_i^1$ supporting s_i $\}$

$\vdots$

$B_i^k = \{b_i \in B_i^{k-1}$: for every node n_i, if $R_{-i}^{k-1} = \prod_{j \neq i} R_j^{k-1}$ reaches n_i then $b_i(n_i)$ assigns probability 1 to $R_{-i}^{k-1}\}$

$R_i^k = \{s_i \in S_i$: there exists a belief system $b_i \in B_i^k$ supporting s_i $\}$

In the limit, define

$$B_i^{\infty} = \bigcap_{k=1}^{\infty} B_i^k.$$

Define also the set of player i's **extensive-form rationalizable strategies** to be

$$R_i^{\infty} = \bigcap_{k=1}^{\infty} R_i^k.$$

The definition captures rationality and common strong belief in rationality (Battigalli and Siniscalchi 1992): At each decision node, a rationalizable strategy should be optimal vis-a-vis some belief over the opponents strategy; if the decision node is reached by some profile of *optimal* opponents' strategies (vis-a-vis some beliefs of theirs), then the player's belief is further required to be concentrated on such profiles; if, furthermore, the decision node is reached by some profile of the opponents' strategies which are optimal vis-a-vis a belief system of theirs concentrated on optimal strategies of

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that for a given belief system and a strategy s_i, at a node n_i not excluded earlier by s_i Battigalli compares s_i to all its n_i-replacements (strategies identical to s_i except, possibly, at n_i and its successors), while we restrict attention only to 'local' n_i-replacements which alter s_i solely at n_i. By the one-deviation principle (see e.g. Perea 2002), for a given belief system b_i a strategy s_i is dominated by no n_i-replacement at no node n_i unexcluded by s_i if and only if s_i is dominated by no 'local' n_i-replacement at no node n_i unexcluded by s_i. Hence, at each iteration of the inductive definition below the plans of actions of the surviving strategies are identical to those surviving Battigalli's definition.

their opponents, the player's belief should be concentrated on *those* profiles; and so forth.

In other words, along each feasible path of play, in the first decision node an active player believes that all her opponents will behave rationally, will believe that *their* opponents will behave rationally, etc. If at some decision node in the game all the opponents' strategy profiles which could lead to that decision node fail this ideal criterion, the player seeks a *best rationalization* (Battigalli 1996) which could have led to that decision node.

For example, if player i has a unique opponent j, who has only two strategies that lead to a decision node of i – s'_j which is strictly dominated for j, and s_j which is optimal for j but only under a belief of j that i is (or was, or will be) irrational, then at that decision node i is required to believe that in the sequel j will continue to employ s_j (because s_j embodies a better rationalization of j's past behavior than does s'_j). Forward induction reasoning then implies that from that decision node onwards, i's rationalizable strategy should be optimal vis-a-vis s_j,unless a further decision node n'_i is reached which is compatible only with s'_j; at n'_i player i has no choice but to revert to the belief that j is irrational, and react accordingly.

We denote by

$$k_{n_i} = \sup\left\{k : \exists s_{-i} \in R^k_{-i} \text{ that reaches } n_i\right\}$$

(with the caveat that $\sup\{\emptyset\} = 0$, and $\sup\{1, 2, 3, \ldots\} = \infty$). Intuitively, k_{h_i} is the maximal level of rationality and mutual strong belief of rationality that player i can attribute to the other players at n_i.

The *EFR* outcomes are those induced by the *EFR* strategy profiles.

3.1.1 An algorithm for computing EFR strategies

We begin with some definitions from Perea (2010, forthcoming).

Definition 2 (full decision problem) *The* full decision problem *at a decision node n is the I-tuple $S(n) = (S_i(n))_{i \in I}$, where $S_i(n)$ is the set of player i's strategies that reach n.*

Definition 3 (reduced decision problem) *A* reduced decision problem *at a decision node n is an I-tuple $D(n) = (D_i(n))_{i \in I}$, where $D_i(n) \subseteq S_i(n)$.*

Definition 4 (strict domination) *For a decision problem $D(n)$ at a decision node $n \in N_i$, a strategy $s_i \in D_i(n)$ is strictly dominated at $D(n)$ if for every belief $\mu_i \in \Delta(D_{-i}(n))$ there exists a strategy $s_i' \in D_i(n)$ such that w.r.t. μ_i the strategy s_i' yields player i a higher expected utility than does s_i.*[5]

The following algorithm of Shimoji and Watson (1998) is useful for computing EFR strategies.

Algorithm (Iterated Conditional Dominance Procedure (ICDP))

- **Initial Step:** For every decision node n let $\Phi^0(n) = S(n)$ be the full decision problem at n.
- **Inductive Step:** Let $k \geq 1$, and suppose that the decision problems $\Phi^{k-1}(n)$ have already been defined for every node n. Then for every player $i \in I$ and each decision node $n \in N_i$ delete from $\Phi_i^{k-1}(n)$ all the strategies of player i that are strictly dominated at some $\Phi^{k-1}(n')$, $n' \in N_i$, *unless* this would remove all the strategies in $\Phi_i^{k-1}(n)$. In the latter case, do not remove any strategies from $\Phi_i^{k-1}(n)$. The resulting reduced decision problem is denoted by $\Phi^k(n)$.

At some point no more strategies are eliminated at any node n. Denote the resulting reduced decision problem at n by $\Phi(n)$.

Shimoji and Watson (1998) showed that a strategy s_i in a game G is extensive-form rationalizable if and only if $s_i \in \Phi_i(r)$, where r is the root of the game tree.

3.2 The Backward Dominance Procedure (BDP)

Perea (2010) defined the *Backward Dominance Procedure (BDP)*, and proved that its outcomes are characterized epistemically as those attained under *rationality and common belief in future rationality.*

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[5] As Pearce (1984, lemma 3) has shown, this is equivalent to i having a mixed strategy which strictly dominates s_i in the strategic-form game with strategy sets $(D_j(n))_{j \in I}$ – hence the term "strict domination."

Algorithm (Backward Dominance Procedure (BDP))

- **Initial Step:** For every decision node n let $\Gamma^0(n) = S(n)$ be the full decision problem at n.

- **Inductive Step:** Let $k \geq 1$, and suppose that the decision problems $\Gamma^{k-1}(n)$ have already been defined for every node n. Then for every player $i \in I$ and each decision node $n \in N_i$ delete from $\Gamma_i^{k-1}(n)$ all the strategies of player i that are strictly dominated at $\Gamma^{k-1}(n')$ for some node $n' \in N_i$ that weakly follows n (i.e. $n' = n$ or $n \rightsquigarrow n'$).

At some point no more strategies are eliminated at any decision node n. Denote the resulting reduced decision problem at n by $\Gamma(n)$.

The BDP strategies of a player i are those strategies that survive the Backward Dominance Procedure at the root r of the game tree, namely $\Gamma_i(r)$.

This is a solution concept with which the players always disregard the past behavior in the game (and any deviations from rationality, or from belief in rationality etc. that have already been manifested), and are "optimistic" regarding the future behavior of their opponents in terms of its rationality, its reliance on belief in future rationality, and so forth.

Perea (2010) also showed that the BDP procedure is *order independent*: the order in which strategies are eliminated does not matter. Even if at each iteration of the inductive step only *some* of the strictly dominated strategies are deleted (as opposed to all of them), the eventual strategies that remain in $\Gamma(r)$ are the same as those that remain in the unaltered BDP procedure described above.

Of course, the *BDP* outcomes are those induced by the *BDP* strategy profiles.

3.3 Agent Extensive-Form Rationalizability (AEFR)

Let G_A be the agent form of the perfect-information game G: The player set of G_A is $\{i_n\}_{n \in N_i,\ i \in I}$, and i_n plays only at the node n. The utility of a player i_n in G_A is the same as that of player i in G, and we call player i_n the n-agent of player i.

We denote a strategy of the n-agent i_n of player i by s_{i_n}, and observe that this strategy is just a "move" of player i at node n in the original game G.

We sometimes write $s_{i_n} \equiv s_i(n)$ to mean that the strategy s_{i_n} of the n-agent i_n of player i is the one in which i_n makes the move $s_i(n)$.

Consider the EFR strategy profiles of the agent-form G_A. The set of paths induced by these profiles is called the set of *Agent Extesnive-Form Rationalizable (AEFR) outcomes.*

4 *SPE* is strictly more restrictive than *EFR*

Reny (1992) gave an example of a perfect-information game in which the strategy of one of the players in the unique SPE of the game is distinct from that player's unique EFR strategy. However, in that game the unique SPO outcome nevertheless coincides with the unique EFR outcome, as is the case in generic perfect-information games: This was proved by Reny (1992) and Battigalli (1997) using Kohlberg-Mertens stability, and independent, more elementary proofs were given by Robles (2006) and Perea (forthcoming).

In this section we show that in the general class of perfect-information games, the set of Subgame-Perfect Equilibrium (SPE) outcomes is a subset of the set of Extensive-Form Rationalizable (EFR) outcomes (theorem 1), but that in some games the inclusion is strict (example 1).

4.1 *SPE* is more restrictive than *EFR*

Theorem 1 (SPE $\subseteq$ EFR). Let $s = (s_i)_{i\in I}$ be a subgame-perfect equilibrium profile of the perfect-information game G. Then there exists a strategy profile $\tilde{s} = (\tilde{s}_i)_{i\in I}$ of EFR strategies which induces the same path as the one induced by s.

Proof. The proof is by induction on the structure of game trees.

The theorem is trivial for a game in which all the actions of the active player at the root lead to leaves: this is a single-person decision problem, and each subgame-perfect equilibrium strategy of that player is simply an optimal strategy, which is also an extensive-form rationalizable strategy for that single player.

Consider a more complicated game G, and assume the theorem has already been proved for all its proper subgames.

Let $s = (s_i)_{i \in I}$ be a subgame-perfect equilibrium profile in G. Define the strategies $\tilde{s} = (\tilde{s}_i)_{i \in I}$ in G along with belief systems $\tilde{b} = \left(\tilde{b}_i\right)_{i \in I}$ as follows.

For every player $i \in I$ and every node $n \in N_i$ on the equilibrium path of s,

1. define $\tilde{s}_i(n) = s_i(n)$.

2. For every $n' \in N$ reached immediately by a deviation $a_i^n \neq s_i(n)$ of player i at n,

2a If it is the case that $k_{n'} = \infty$, use the induction hypothesis to find an EFR strategy profile $\tilde{s}^{n'}$ in the subgame $G^{n'}$ which induces the same path as $s^{n'}$, supported by a corresponding profile of belief systems $\tilde{b}^{n'}$ in B^{∞} of the game $G^{n'}$.

2b If it is the case that $k_{n'} < \infty$, then $\forall j \neq i$ choose some belief system $\tilde{b}_j^{\infty} \in B_j^{\infty}$ and some strategy $\tilde{s}_j^{n'}$ in $G^{n'}$ which is supported by the restriction $\tilde{b}_j^{n'}$ of $\tilde{b}_j^{\infty}$ to $N_j \cap G^{n'}$; and finally, choose some $\tilde{s}_i^{n'} \in S_i^{n'}$ supported by the belief system $\tilde{b}_i^{n'}$ in $N_i \cap G^{n'}$ according to which at each $n'' \in N_i \cap G^{n'}$ player i is certain that in the past the other players chose the unique behavior leading to n'' and that in the sequel they will employ $\tilde{s}_{-i}^{n'} = \left(\tilde{s}_j^{n'}\right)_{j \neq i}$.

In each of the cases 2a and 2b, define $\tilde{s}_j(n'') = s_j^{n'}(n'')$ for every n'' in $G^{n'}$ in which j is the active player, for every player $j \in I$ (including the case $j = i$). This completes the definition of the strategy profile $\tilde{s}$. In view of 1., the equilibrium path of $\tilde{s}$ is the same as the equilibrium path of s.

For nodes $n \in N_i$ on the equilibrium path of s define $\tilde{b}_i(n)$ to be the belief on S_{-i} concentrated on the unique strategy profile $\tilde{s}_{-i}$. In subgames $G^{n'}$ in each of the cases 2a and 2b, define $\tilde{b}_j(n'') = \tilde{b}_j^{n'}(n'')$ for every $n'' \in N_j \cap G^{n'}$, for every player $j \in I$ (again, including the case $j = i$). This completes the definition of a belief $\tilde{b}_j(\hat{n}) \in \Delta(S_{-j})$ in all the decsion nodes N_j of player j in G, for every player $j \in I$. $\tilde{b}_j$ is indeed a belief system – by construction $\tilde{b}_j$ obeys Bayes rule whenever possible, and at each node $\hat{n} \in N_j$, $\tilde{b}_j(\hat{n})$ assigns probability 1 to the strategy profiles in S_{-j} that reach $\hat{n}$.

Our task is now to prove that $\tilde{s} = (\tilde{s}_i)_{i \in I}$ is a profile of EFR strategies. To this effect, we will show that $\forall i \in I$ it is the case that (i) $\tilde{s}_i$ is supported by $\tilde{b}_i$, and that (ii) $\tilde{b}_i \in B_i^{\infty}$.

By construction, in all the subgames starting off the equilibrium path of s (cases 2a and 2b), $\tilde{s}_i$ is supported by (the restriction of) $\tilde{b}_i$ within each of these subgames. To prove (i) it therefore remains to show that $\tilde{s}_i(n)$ is rational w.r.t. $\tilde{b}_i$ at decision nodes $n \in N_i$ on the equilibrium path of s.

Indeed, assume by contradiction that there exists a node $n \in N_i$ on the equilibrium path of s for which, under the belief system $\tilde{b}_i$, player i has an alternative action $a_i^n \neq \tilde{s}_i(n)$ the switch to which yields her a higher payoff. Let n be the first node on the equilibrium path of s at which this occurs for some player.

Observe that a_i^n cannot immediately lead to a node n' of type 2a (for which $k_{n'} = \infty$): in that case the payoff from the switch to a_i^n will be identical to the payoff player i would obtain from deviating to a_i^n in the original subgame-perfect equilibrium s, and by the definition of subgame-perfect equilibrium such a deviation does not improve i's payoff.

The remaining possibility is that a_i^n immediately leads to a node n' of type 2b (for which $k_{n'} < \infty$). The fact that n is the first node on $\tilde{s}$'s equilibrium path at which a profitable deviation exists for some player implies that for each of the other players $j \neq i$, the node n' is reached by EFR strategies $s_j \in R_j^\infty$. Hence, by the construction in 2a, $\tilde{s}_j \in R_j^{k_{n'}+1}$ for every $j \neq i$. However, the fact that within $G^{n'}$ the strategy $\tilde{s}_i$ is by construction a best reply to $\tilde{s}_{-i}$, together with the assumption that the deviation from $\tilde{s}_i(n)$ to a_i^n is profitable (given $\tilde{s}_{-i}$), imply that the amended strategy $\tilde{s}_i/a_i^n$ is in $R_i^{k_{n'}+2}$. However, since $\tilde{s}_i/a_i^n$ reaches n' this means that

$$\sup\left\{k : \exists s_{-j} \in R_{-j}^k \text{ that reaches } n'\right\} \geq k_{n'} + 2$$

which is a contradiction to the definition of $k_{n'}$.

This concludes the proof of (i) that $\tilde{s}_i$ is supported by $\tilde{b}_i$ for every $i \in I$.

It remains to show (ii), namely that $\tilde{b}_i \in B_i^\infty$ for every player $i \in I$, i.e. that $\tilde{b}_i \in B_i^\ell$ for every $\ell \geq 1$.

We have already shown that $\tilde{b}_i$ is a belief system, i.e. that $\tilde{b}_i \in B_i^1$. Suppose, inductively, that we have already proved that $\tilde{b}_i \in B_i^{k-1}$ for every $i \in I$. By (i) this implies that also $\tilde{s}_i \in R_i^{k-1}$ and hence $\tilde{s}_{-i} \in R_{-i}^{k-1}$.

We have to prove that $\tilde{b}_i \in B_i^k$. That is, for every player $i \in I$ and every decision node $\hat{n} \in N_i$ we have to prove that if $k_{\hat{n}} \geq k-1$ then $\tilde{b}_i(n)\left(R_{-i}^{k-1}\right) = 1$.

Suppose, by way of contradiction, that this is not the case. Then there exists a player i and a node $\hat{n} \in N_i$ such that $k_{\hat{n}} \geq k-1$ but $\tilde{b}_i(\hat{n})\left(R_{-i}^{k-1}\right) < 1$. Let n be the first node on the equilibrium path of s with the property that either $\hat{n} = n$ or $\hat{n}$ belongs to a subtree which follows a deviation from s at n.

Notice that it cannot be the case that $\hat{n} = n$: by the induction hypothesis $\tilde{s}_{-i} \in R_{-i}^{k-1}$, and since $\tilde{s}_{-i}$ reaches n and $\tilde{b}_i(n)(\{\tilde{s}_{-i}\}) = 1$ it follows that $\tilde{b}_i(n)\left(R_{-i}^{k-1}\right) = 1$.

Notice next that $\hat{n}$ cannot belong to a subtree $G^{n'}$ of type (2b), because for every player $j \neq i$, the beliefs of player j in $N_j \cap G^{n'}$ are by definition part of a belief system in B_j^∞, which furthermore supports $\tilde{s}_j$; and as for player i herself – since she was the only player who deviated at n so as to reach $G^{n'}$, if for $n'' \in N_i \cap G^{n'}$ it is the case that $k_{n''} \geq k-1$ then by the definition in case (2b), $\tilde{b}_i(n'')$ ascribes probability 1 to a particular profile of strategies in R_{-i}^{k-1} – either to $\tilde{s}_{-i}$ itself (which is in R_{-i}^{k-1} by the induction hypothesis) or to the strategy profile composed of the past actions of the players $j \neq i$ that led to n'', continued by $\tilde{s}_{-i}$ in $G^{n''}$.

Finally, notice also that $\hat{n}$ cannot belong to a subtree described in case (2a) : since R_i^∞ reaches n' for all players $i \in I$ and by definition $\tilde{b}_i$ coincides in $G^{n'}$ with a belief system $\tilde{b}_i^{n'}$ in B_i^∞ of the game $G^{n'}$, it follows that $\tilde{b}_i(n'')\left(R_{-i}^{k_{n''}}\right) = 1$ for every $n'' \in G^{n'} \cap N_i$, and in particular for $n'' \in G^{n'} \cap N_i$ for which $k_{n''} \geq k-1$.

This completes the proof that for every player $i \in I$ and every decision node $\hat{n} \in N_i$ if $k_{\hat{n}} \geq k-1$ then $\tilde{b}_i(\hat{n})\left(R_{-i}^{k-1}\right) = 1$, i.e. that $\tilde{b}_i \in B_i^k$. This concludes the inductive step, yielding the conclusion that $\tilde{b}_i \in B_i^\infty$. ■

4.2 EFR outcomes are not a subset of SPE outcomes

Figure 1 is an example of a game in which an EFR outcome is not a subgame perfect equilibrium outcome. This game is a simple ultimatum game: Player

1 makes an offer to give player 2 a value $v \in \{0, 1, 2\}$. If player 2 accepts, he obtains a payoff of v, while player 1 gets $2 - v$. If he rejects the offer, then both players get 0.

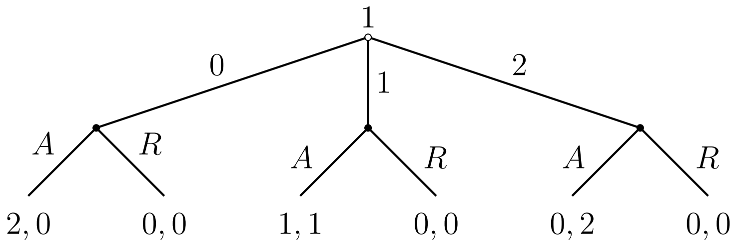

Figure 1: EFR $\not\subseteq$ SPE

This game has two SPE profiles: (i) Player 1 offers 0 and player 2 accepts after any offer, and (ii) player 1 offers 1 and player 2 rejects after an offer of 0 but accepts offers 1 and 2.

Observe that the outcome in which player 1 offers 0 and player 2 rejects is rationalizable, but cannot be obtained as a SPE.

5 *EFR* is strictly more restrictive than *BDP*

Perea (forthcoming) and Chen and Micali (2011) proved that the set of *EFR* outcomes is contained in the set of *BDP* outcomes. The game in Figure 2 shows that the inclusion is strict:

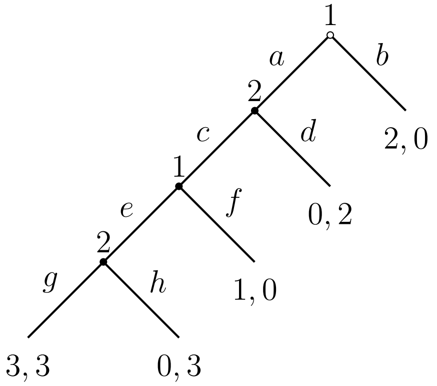

Figure 2: BDP $\not\subseteq$ EFR

The only rationalizable strategies for player 1 are (b,e), (b,f), and (a,e) (because (a,f) is dominated by (b,f)). Hence the EFR strategies of player 2 are (c,g) and (c,h). The EFR paths are therefore those ending by b, g, and h.

The BDP procedure maintains also the strategies (d,g) and (d,h) for player 2. If player 2 at his first decision node reasons only backwards, d is a best reply against f, and player 2 cannot rule out that player 1 will choose f. But (c,g) and (c,h) are also BDP strategies for player 2, because they are best replies against the BDP strategy (a,e) of player 1 (which is optimal against (c,g)). Thus, the BDP outcomes are those ending with b, d, g, and h.

6 *BDP* is strictly more restrictive than *AEFR*

Finally, we show that the set of BDP outcomes is contained in the set of AEFR outcomes, and that the inclusion is strict.

6.1 *BDP* is more restrictive than *AEFR*

Lemma 1 For an extensive-form game G of perfect information, let $(\Gamma(n'))_{n'\in N}$ be the outcome of the BDP algorithm, and let $(\Phi(n'))_{n'\in N}$ be the outcome of the ICDP algorithm in G_A. Then for every $i \in I$ and $n' \in N_i$, if $s_i \in \Gamma_i(n')$ then for every $n \in N_i \cap G^{n'}$ reachable by s_i, the strategy $s_{i_n} \equiv s_i(n)$ of agent i_n is in $\Phi_{i_n}(n')$.

Proof. The proof is by induction on the depth $d(n')$. For the base case $d(n') = 1$: If $s_i \in \Gamma_i(n')$ then, conditional on reaching n', $s_i(n')$ is an optimal move for player i. Thus, conditional on reaching n', $s_{i_{n'}} \equiv s_i(n')$ is an optimal strategy for the n'-agent $i_{n'}$ of player i. Therefore, $s_{i_{n'}} \in \Phi_{i_{n'}}(n')$.

Suppose now that the lemma has been proved for all n' that satisfy $d(n') < \ell$. We need to show that it holds also for nodes n' with $d(n') = \ell$. To this end, fix a node $n' \in N_i$ with $d(n') = \ell$ and a strategy $s_i \in \Gamma_i(n')$, and consider a node $n \in G^{n'} \cap N_i$. There are two cases:

$\mathbf{n} \neq \mathbf{n}'$: It must be the case that $s_i \in \Gamma_i(n)$ – otherwise, s_i would have been strictly dominated at n or at a node that weakly follows it. But this would have meant that $s_i \notin \Gamma_i(n')$, a contradiction. By the induction hypothesis, the fact that $s_i \in \Gamma_i(n)$ implies that $s_{i_n} \equiv s_i(n)$ is in $\Phi_{i_n}(n)$. This implies that $s_{i_n} \in \Phi_{i_n}(n')$ as well, since s_{i_n} can only be strictly dominated for agent i_n at the decision problem $\Phi(n)$, and it is not.

$\mathbf{n} = \mathbf{n}'$: Let $\{n_1, \ldots, n_t\}$ be the nodes that immediately follow n', and suppose that $s_i(n') = n_1$. For each $n_k \in \{n_2, \ldots, n_t\}$, let s_i^k be some player i strategy such that $s_i^k \in \Gamma_i(n_k)$. Also set $s_i^1 \equiv s_i$ for consistency of notation. By the induction hypothesis we know that for all $k \in \{1, \ldots, t\}$ and all nodes $n'' \in G^{n_k} \cap N_i$, if n'' is reachable by s_i^k then $s_{i_{n''}} \equiv s_i^k(n'') \in \Phi_{i_{n''}}(n'')$, and so $s_{i_{n''}} \in \Phi_{i_{n''}}(n')$ as well.

For every node $n'' \in N_i \cap (\cup_{k=1}^t G_k)$ reachable by some strategy s_i^k define $\tilde{s}_{i_{n''}} \equiv s_{i_{n''}}$, and for every other node n'' in $N_i \cap (\cup_{k=1}^t G_k)$ let $\tilde{s}_{i_{n''}}$ be some *AEFR* strategy of agent $i_{n''}$. Then $\left(\tilde{s}_{i_{n''}}\right)_{n'' \in N_i}$ is a strategy profile of i's agents in $N_i \cap (\cup_{k=1}^t G_k)$.

Now, observe that $\Gamma(n') \subset \bigcup_{k \in \{1,\ldots,t\}} \Gamma(n_k)$ – this follows from the fact that if the BDP procedure deletes a strategy from some $\Gamma(n_k)$ then that strategy must also be deleted from $\Gamma(n')$. Consider now a partial run of the BDP procedure for which the current reduced decision problem at n' is $\Gamma'(n') = \bigcup_{k \in \{1,\ldots,t\}} \Gamma(n_k)$. By order independence of BDP, continuing the deletion of strategies at n' from this point onwards would eventually yield the BDP strategies. Observe furthermore that $s_i^k \in \Gamma_i'(n')$ for all $k \in \{1, \ldots, t\}$. Now, suppose that at this stage the algorithm checks whether to delete s_i from $\Gamma_i'(n')$. The reason it does not is that there exists a belief $\mu_i \in \Delta\left(\Gamma_{-i}'(n')\right)$ with which no other strategy $s_i' \in \Gamma_i'(n')$ yields player i a higher expected utility than does s_i. In particular, s_i is a better response to μ_i than any of $s_i^2, \ldots, s_i^t$.

Note that each strategy $s_j \in \mathrm{supp}(\mu_i)$ satisfies $s_j \in \Gamma_j'(n')$, which implies that $s_j \in \Gamma_j(n_k)$ for some $k \in \{1, \ldots, t\}$. Thus, by the induction hypothesis, $s_{j_{n''}} \equiv s_j(n'') \in \Phi_{j_{n''}}(n')$ for all $n'' \in G^{n_k} \cap N_j$ that are reachable by s_j. The belief μ_i then induces a belief $\tilde{\mu}_i \in \Delta\left(\prod_{j \neq i, n'' \in N_j} \Phi_{j_{n''}}(n')\right)$ over the agent strategies of players $j \neq i$.

Consider now the product $\tilde{\nu}_i$ of $\tilde{\mu}_i$ with a point belief on $\left(\tilde{s}_{i_{n''}}\right)_{n'' \in N_i}$. This is a belief over agent strategies at nodes that strictly follow $n' = n$. We claim that for this belief, agent i_n's best response is the strategy $s_{i_n} \equiv s_i(n)$: If agent i_n plays the strategy s_{i_n}, his expected payoff is the same as that obtained when s_i is played with the belief μ_i. If agent i_n plays a different strategy s_{i_n}' that moves toward a node n_k, then his expected payoff is the same as that obtained when s_i^k is played with belief μ_i. Thus, the optimality of s_i with belief μ_i guarantees that s_{i_n} is optimal with respect to $\tilde{\nu}_i$. Finally, this implies that the agent strategy $s_{i_n} \in \Phi_{i_n}(n)$ as required. ■

Theorem 2 (*BDP* $\subseteq$ *AEFR*). *For any extensive form game G of perfect information and any player i in G, if an outcome is reachable by a profile of BDP strategies, then it is reachable by a profile of AEFR strategies.*

Proof. Let $(\Gamma(n))_{n \in N}$ be the outcome of the *BDP* procedure in G and $(\Phi(n))_{n \in N}$ the outcome of the *ICDP* procedure in G_A. Since s_i survives the BDP procedure, $s_i \in \Gamma(r)$. Lemma 6.1 implies that for all $n \in N_i$ reachable by s_i the strategy $s_{i_n} \equiv s_i(n)$ of the n-agent i_n of player i satisfies $s_{i_n} \in \Phi(r)$, and so s_i is agent extensive-form rationalizable. Thus, since every profile s of BDP strategies induces also a profile of AEFR strategies (where the strategy of agent $j_{n'}$ for n' not reached by s_j is any AEFR strategy), any outcome reachable by the former is also reachable by the latter. ■

6.2 AEFR outcomes are not a subset of BDP outcomes

In the game of Figure 3, the outcome $(1, 1)$ is reachable by strategies that are AEFR but not by strategies that are BDP.

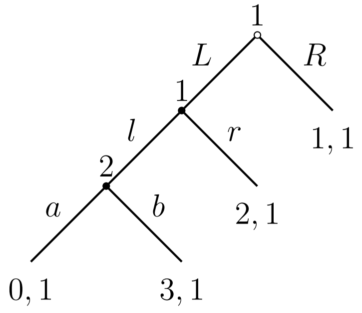

Figure 3: AEFR $\not\subseteq$ BDP

In this game, the strategies a and b are both BDP and EFR for player 2. Observe, however, that the strategies (R,l) and (R,r) are dominated by (L,r) at the root of the tree, and so no BDP strategy of player 1 has R as a first move. But when two independent agents play at player 1's decision nodes, l is EFR for the second agent (it is supported by the belief that 2 will choose b), and R is EFR for the first agent (it is supported by the belief that the second agent will choose the EFR move l and that 2 will choose a). Thus, the profile ((R,l), b) is an AEFR profile leading to the outcome (1,1).

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