The Price of Fixed Income Market Volatility

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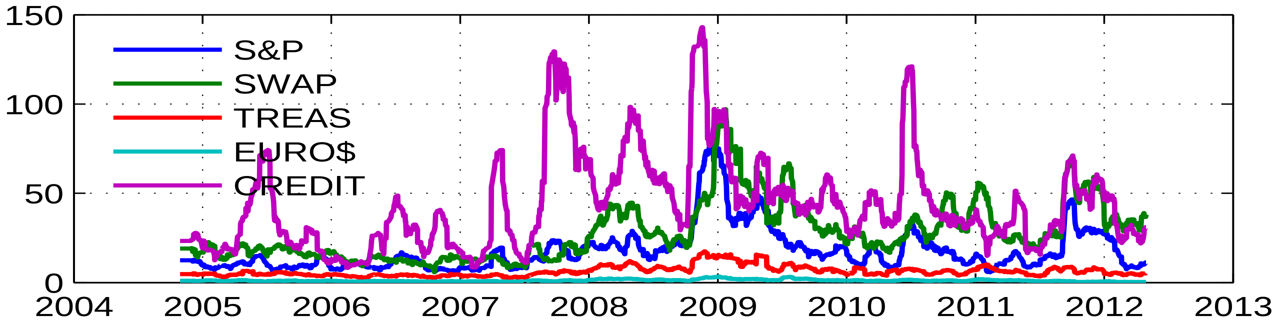
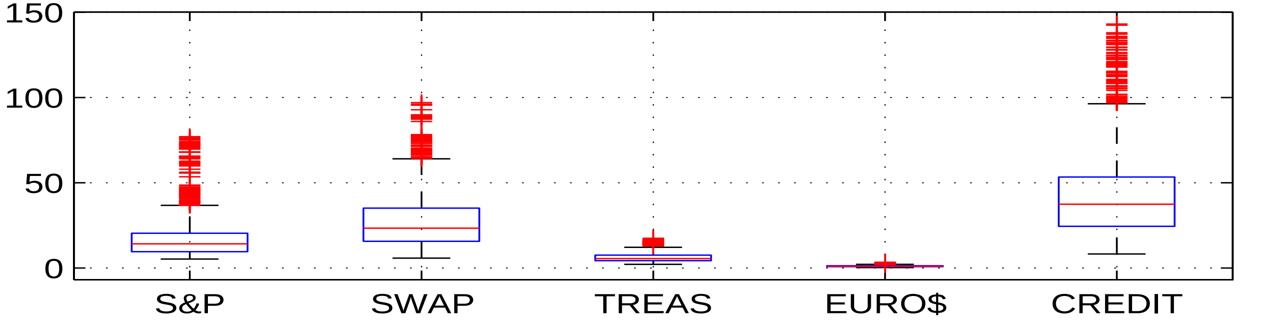
Why fixed income market volatility?

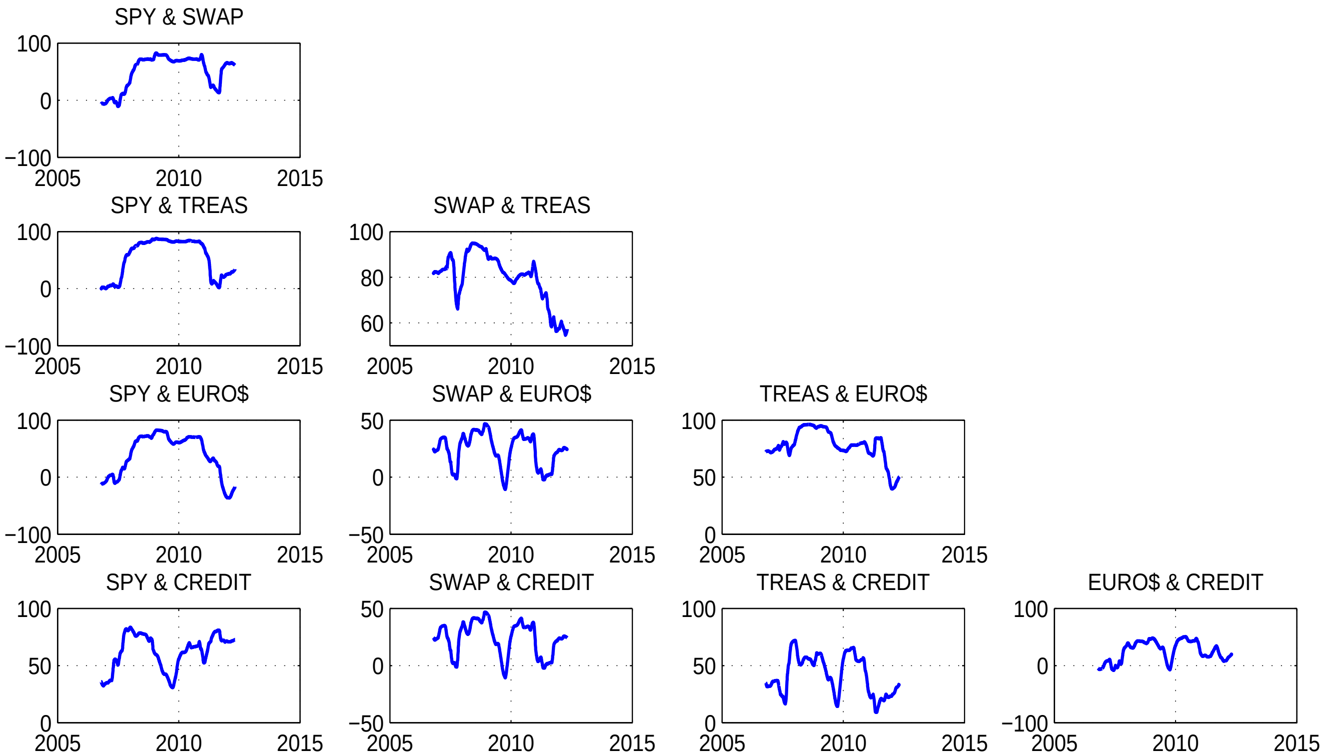
- \$14.5 trillions notional US dollar interest rate option markets
- Hedging interest rate risk could not be more timely over ongoing uncertainties
- Fixed income volatility and equity volatility evolve heterogeneously over time, co-moving disproportionately during periods of global imbalances and each reacting to events of different nature
  - VIX derivatives are not necessarily enough to hedge risks arising in fixed income markets

Realized volatility across equity and fixed income markets





Correlations



Purpose

- Methodology for options-based “model-free” pricing of equity volatility has been known for some time
  - Little is known about analogous methodologies for pricing various fixed income volatilities

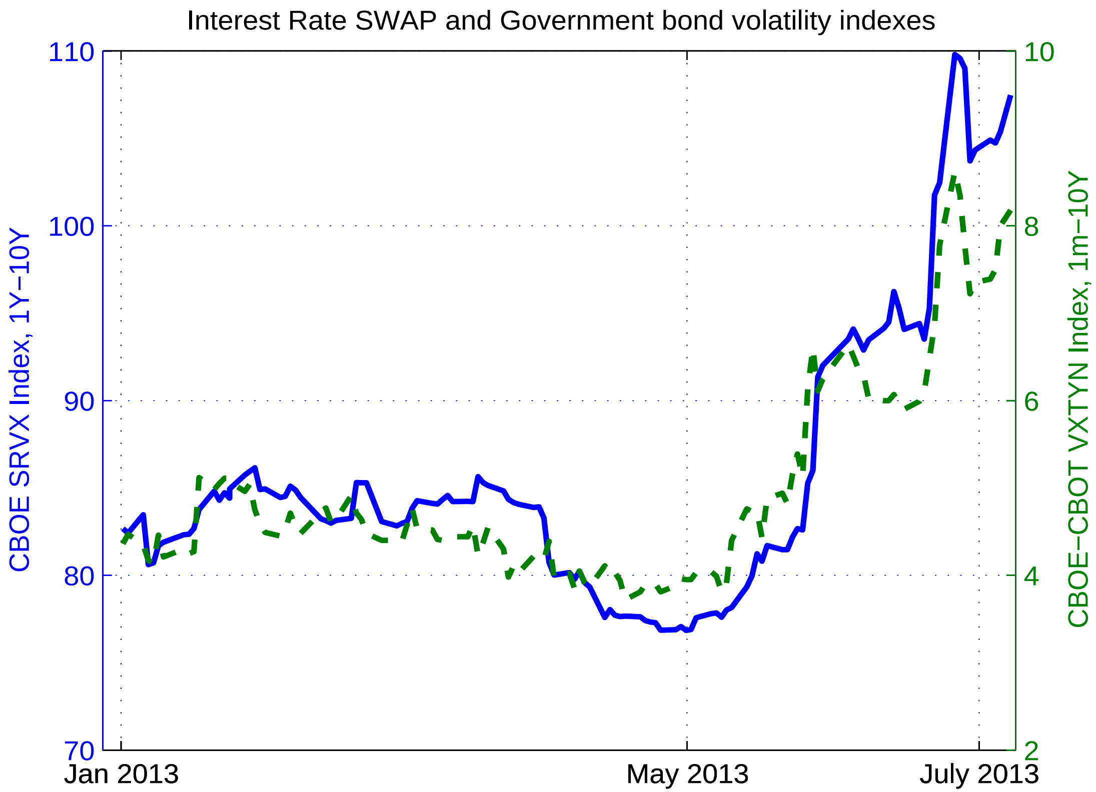
- Aim to provide a unified evaluation framework of fixed income volatility
  - Deal with disparate markets such as interest rate swaps, government bonds, time-deposits and credit
  - Develop model-free, forward looking indexes of fixed income volatility that match different quoting conventions across various markets
  - Uncover subtle yet important pitfalls arising from naïve superimpositions of the standard equity volatility methodology when pricing various fixed income volatilities

CBOE interest rate volatility indexes

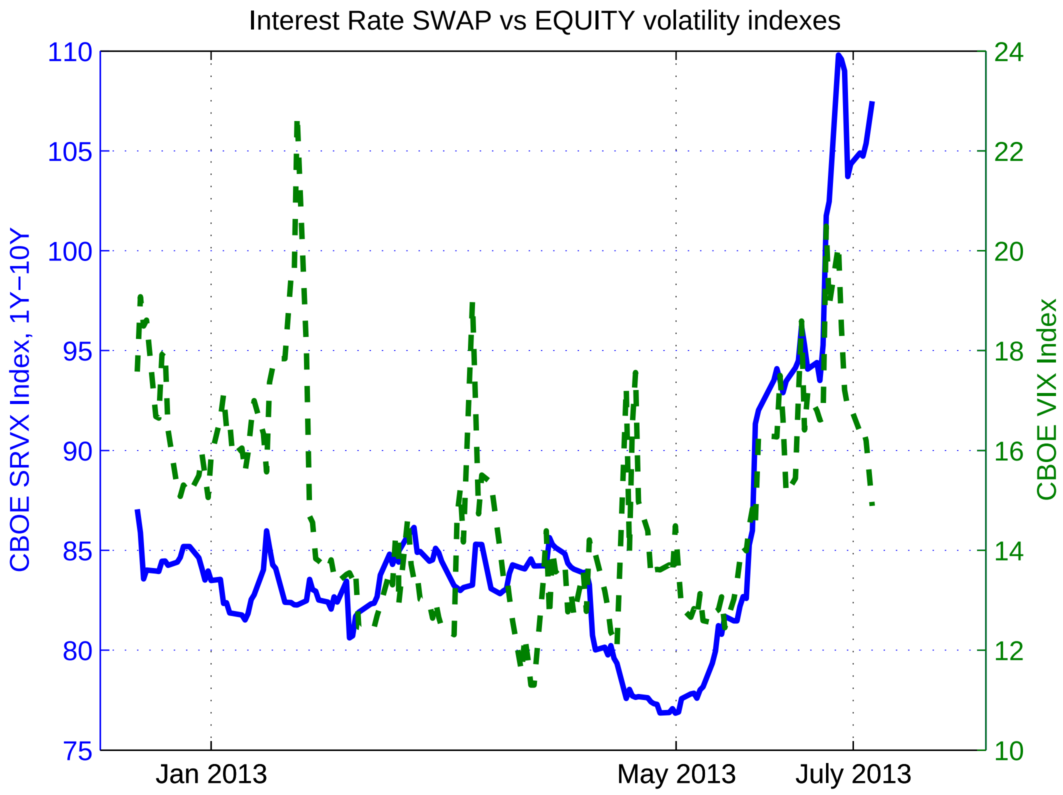
- Some of the interest rate volatility indexes in this book are currently being implemented by the Chicago Board Options Exchange (CBOE)

- Behavior of two recently launched indexes of fixed income volatility that parallel the equity VIX

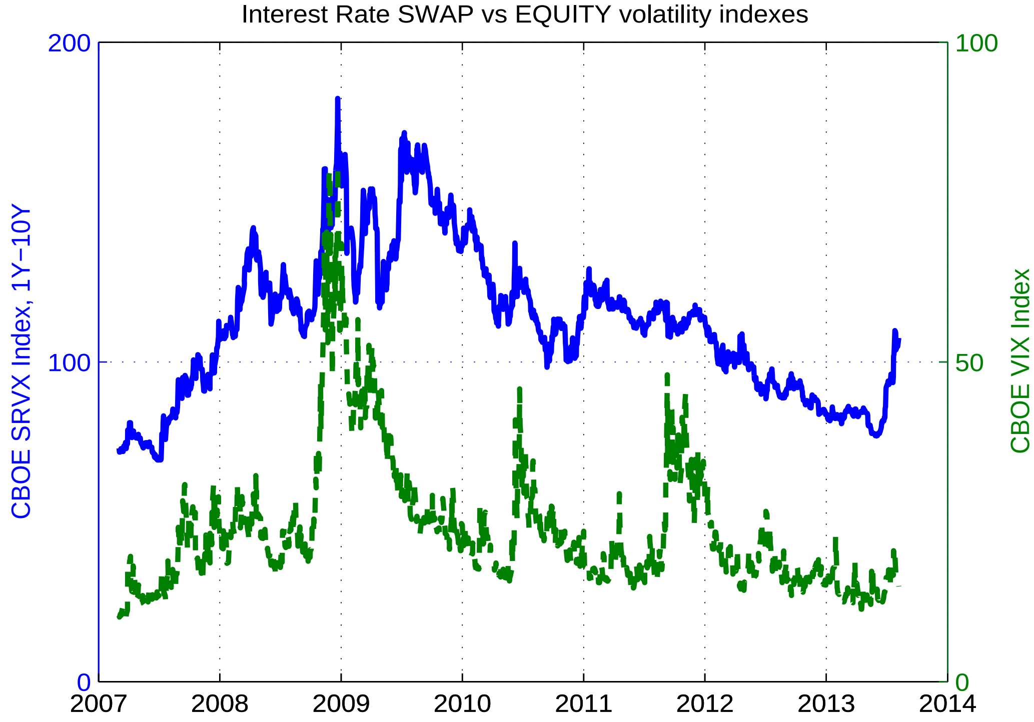
A snapshot


Interest Rate SWAP and Government bond volatility indexes
110
100
90
80
70
10
8
6
4
2
CBOE SRVX Index, 1Y−10Y
CBOE−CBOT VXTYN Index, 1m−10Y
Jan 2013
May 2013
July 2013

Tete-a-tete, CBOE interest rate volatility & VIX indexes



Tete-a-tete, II



Rates versus Equity: Key issues of methodology

- Extends the work of Carr & Madan (2001) to interest rate and credit sensitive securities

- Rates & Fixed income market numéraires
  - The nature of FI security evaluation involves numéraires beyond the MMA numéraire

- Model-free pricing requires dedicated contract designs
  - Match market practice—Basis point vs logN, price vs rate vols
  - Key insight: link BP variance to Quadratic contracts, and then price Q-contracts à la Carr-Madan

Plan

Part 1: Variance contracts: fixed income security design

Part 2: Interest rate swap markets

Part 3: Government bonds and time deposits

Part 4: Credit

Sources

- Mele, Antonio, and Yoshiki Obayashi, 2013. "The Price of Fixed Income Market Volatility." Book manuscript (≈ 200 pages)

- Mele, Antonio, and Yoshiki Obayashi, 2012. "An Interest Rate Swap Volatility Index and Contract." Technical white paper underlying the CBOE Interest Rate Swap Volatility Index. Available from `http://www.cboe.com/micro/srvx/default.aspx`.

- Mele, Antonio, and Yoshiki Obayashi, 2013. "Volatility Indexes and Contracts for Eurodollar and Related Deposits." Swiss Finance Institute Working paper.

- Mele, Antonio, and Yoshiki Obayashi, 2013. "The Price of Government Bond Volatility." Swiss Finance Institute Working paper.

- Mele, Antonio, and Yoshiki Obayashi, 2013. "Credit Variance Swaps and Volatility Indexes." Swiss Finance Institute Working paper.

And

- Mele, Antonio, and Yoshiki Obayashi, US Patent App. 13/528,150: "Methods and Systems for Creating an Interest Rate Swap Volatility Index and Trading Derivative Products Based Thereon." Patent issued.

- Mele, Antonio, and Yoshiki Obayashi, US Patent App. 13/842,197: "Methods and Systems for Creating a Government Bond Volatility Index and Trading Derivative Products Based Thereon." Patent pending.

- Mele, Antonio, and Yoshiki Obayashi, US Patent App. 13/842,050: "Methods and Systems for Creating a Time Deposit Volatility Index and Trading

Derivative Products Based Thereon." Patent pending.

- Mele, Antonio, and Yoshiki Obayashi, US Patent Appl. 13/841,653: "Methods and Systems for Creating a Credit Volatility Index and Trading Derivative Products Based Thereon." Patent pending.

1/4 Variance contracts: fixed income security design

Variance swaps in fixed income markets: overview of methodology

- Search for:

(i) a numéraire in each FI security market, such that all security prices in terms of this numéraire are martingales (*absence of arbitrage*) under Q^N; and
(ii) a numéraire in the variance swap market for the given security such that variance swaps are priced in a model-free fashion
Numéraires (i) and (ii) are not necessarily the same, although they could be

- Basis point variance contracts are quite relevant in FI markets, and deserve a special treatment
  - Rely on "Quadratic contracts" as mentioned which we span through Carr-Madan expansions

Additional issues

- How *BP* volatility indexes work compared to *percentage*?
- How truncating calculations to include a finite number of options affect the behavior of *BP* and *percentage* volatility indexes?
- Pitfalls from applying equity methodology to fixed income are explained in Part III

The right numéraire

- Aim to price volatility in a "model-free" fashion
  - "Model-free"—only relies on the price of traded assets (e.g., European-style interest rate derivatives, zero coupon bonds or defaultable bonds)
- Design variance swap contracts with fair value leading to indexes that reflect market expectations of fixed income market volatility, adjusted for the "relevant notion" of market risk

- What is the "relevant notion" of market risk-adjustment in our context?
  - Absence of arbitrage implies there is a unit of account, aka *numéraire*, such that the prices of all securities specified in terms of this unit are martingales under a certain probability—facilatate calculations via Black's pricers
  - This *numéraire* is the relevant notion of risk-adjustment in our context

- In the equity case, the numéraire is the money market account *assuming interest rates are constant*

- In the FI market, there is a notion of numéraire specific to each market
  - We provide a unifying methodology

Market numéraires and volatilities: definitions

- *Forward starting agreement* originated at t, with delivery of

$$\Pi_T \equiv N_T \times (X_T - K), \quad \text{at } T,$$

  where X_T and N_T are measurable wrt information at T, $\mathbb{F}_T$, and K is chosen at t, such that the value of the contract is zero at inception

- Q is the RN prob, $\mathbb{E}_t(\cdot)$ the expectation under Q conditional on $\mathbb{F}_t$, r_τ the short-term rate process

- N_τ is the price of a tradeable asset

- Under regularity conditions, there exist (i) a probability Q^N, and (ii) a martingale process X_τ under Q^N that clears the agreement, $X_t = K$

- X_T is the *forward risk process*, and N_T is the *market numéraire* at T, such that any asset price process S_t normalized by N_t is a martingale under Q^N,

$$\frac{S_t}{N_t} = \mathbb{E}_t^{Q^N}\left(\frac{S_T}{N_T}\right),$$

where $\mathbb{E}_t^{Q^N}$ denotes the conditional expectation under Q^N

- Q^N *market numéraire probability*
  - the annuity probability in interest rate swap markets
  - the forward probability in government bond and time deposit markets
  - the defaultable annuity probability in credit markets
  - . . .

- It is the volatility of X_τ that we are interested in pricing

- Take X_t to be a strictly positive diffusion process with stochastic volatility (we consider jumps later),

$$\frac{dX_\tau}{X_\tau} = \sigma_\tau \cdot dW_\tau, \quad \tau \in [t, T]$$

where W_t is a multidimensional Wiener process under Q^N

- Integrated variance over a time interval $[t, T]$. Based on
  - *Arithmetic* or *basis point* (BP henceforth) changes of X_t,

$$V^{\mathrm{bp}}(t,T) \equiv \int_t^T X_\tau^2 \left\|\sigma_\tau\right\|^2 d\tau$$

  - *Logarithmic* or *percentage* changes of X_t,

$$V(t,T) \equiv \int_t^T \left\|\sigma_\tau\right\|^2 d\tau$$

Example

- Forward starting swap payer with fixed interest K has payoff,

$$\mathrm{Swap}_T(K; T_1, \cdots, T_n) = \mathrm{PVBP}_T(T_1, \cdots, T_n)\left[R_T(T_1, \cdots, T_n) - K\right],$$

where $R_t(T_1, \cdots, T_n)$ is the fwd swap rate at t for a contract beginning at T_0 and reset dates $T_1, \cdots, T_n$, and $\mathrm{PVBP}_T(T_1, \cdots, T_n)$ is the annuity factor

- The annuity probability Q_{sw} has Radon-Nikodym against Q,

$$\left.\frac{dQ_{\mathrm{sw}}}{dQ}\right|_{\mathbb{F}_T} = e^{-\int_t^T r_s ds}\frac{\mathrm{PVBP}_T(T_1, \cdots, T_n)}{\mathrm{PVBP}_t(T_1, \cdots, T_n)}$$

- In a diffusion environment,

$$\frac{dR_\tau(T_1,\cdots,T_n)}{R_\tau(T_1,\cdots,T_n)}=\sigma_\tau\cdot dW_\tau^{\mathrm{sw}},\quad \tau\in[t,T]$$

where W_τ^{sw} is a Brownian motion under Q_{sw}

Back to general $\longrightarrow$

Forward contract with stochastic multiplier

- It's the contract with payoff,

$$Y_T \times \left(\Psi \left(\{X_\tau\}_{\tau \in [t,T]} \right) - K_Y \right),$$

where Y_T is $\mathbb{F}_T$-measurable, Ψ is a functional of the entire path of X, and K_Y is the fair value of the contract,

$$K_Y = E_t^{Q^Y} \left(\Psi \left(\{R_\tau\}_{\tau \in [t,T]} \right) \right), \quad \text{where } \left. \frac{dQ^Y}{dQ} \right|_{\mathbb{F}_T} = \frac{e^{-\int_t^T r_u du} Y_T}{\mathbb{E}_t \left[e^{-\int_t^T r_u du} Y_T \right]}$$

Refer to Q^Y as the forward multiplier probability

- **Example** Basis point variance contracts: $\Psi \left(\{R_\tau\}_{\tau \in [t,T]} \right) = V^{\text{bp}} (t, T)$

Model-free pricing

- **Definition**: K_Y is model-free if we can find a numéraire N_τ and a contract multiplier Y_T such that K_Y equals the value of a portfolio of European call and option prices with strike K, say $\mathrm{Call}_t(K)$ and $\mathrm{Put}_t(K)$, where:

$$\frac{\mathrm{Call}_t(K)}{N_t} = \mathbb{E}_t^{Q^N}\left(\frac{\max\{\Pi_T, 0\}}{N_T}\right), \quad \frac{\mathrm{Put}_t(K)}{N_t} = \mathbb{E}_t^{Q^N}\left(\frac{\max\{-\Pi_T, 0\}}{N_T}\right)$$

Model-free contracts

- **Proposition** K_Y is model-free if and only if the Radon-Nikodym derivative of the fwd multiplier prob Q^Y against the mkt numéraire prob Q^N, is uncorrelated with $V^{\mathrm{bp}}(t,T)$ and $V(t,T)$

For $\Psi(\cdot) = V^{\mathrm{bp}}(t,T)$ (*Basis Point variance pricing*) it is given by:

$$K_Y = \frac{2}{N_t}\left(\int_0^{X_t} \mathrm{Put}_t(K)\,dK + \int_{X_t}^{\infty} \mathrm{Call}_t(K)\,dK\right)$$

For $\Psi(\cdot) = V(t,T)$ (*Percentage variance pricing*) it is given by:

$$K_Y = \frac{2}{N_t}\left(\int_0^{X_t} \frac{\mathrm{Put}_t(K)}{K^2}dK + \int_{X_t}^{\infty} \frac{\mathrm{Call}_t(K)}{K^2}dK\right)$$

Remarks

(i) Suggest choices for the random multiplier Y_T in each context of interest

- Equity & constant interest rates: $N_t = e^{-\bar{r}(T-t)}$, $Y_T = 1$
- Fixed income: necessary and sufficient conditions under which variance swaps are model-free. Most intuitive case

$$\begin{aligned} Y_T &= N_T \\ \text{(stochastic contract multiplier} &= \text{market numéraire)} \end{aligned}$$

(ii) We are not merely re-stating that in the absence of arbitrage, security prices rescaled by some N_t are martingales under Q^N

- Saying something stronger: variance swaps are *model-free* if their payoffs are re-scaled by the market numéraire

(iii) The situation is actually intricate—Proposition identifies a host of possible variance swap "tilters"

– Consider the following example of stochastic multiplier

$$Y_T = N_T \epsilon_T,$$

where ϵ_T is any $\mathbb{F}_T$-measurable random variable satisfying $\mathrm{cov}^{Q^N}\left[V^{\mathrm{bp}}(t,T), \epsilon_T\right] = 0$

– *However, the market numéraire is the benchmark of this book* because of its economic appeal and the familiarity with it by academics and practitioners

(iv) Finally, we check the internal consistency of our contract design

Consider highly idealized case of a Gaussian model,

$$dX_\tau = \sigma_{\mathrm{n}} \cdot dW_\tau$$

Obvious counterpart to the standard Black-Scholes. We have,

$$K_Y = \frac{2}{N_t}\left(\int_0^{X_t} \mathrm{Put}_t\left(K\right)dK + \int_{X_t}^{\infty} \mathrm{Call}_t\left(K\right)dK\right) = \left\|\sigma_{\mathrm{n}}\right\|^2 \left(T - t\right).$$

Log versus quadratic contracts

- We hinge on a key and novel insight—the price of a BP variance swap links to that of a "quadratic contract,"

$$X_T^2 - X_t^2 = V^{\mathrm{bp}}(t,T) + 2\int_t^T X_\tau dX_\tau,$$

such that,

$$\mathbb{E}_t^{Q^N}\left(V^{\mathrm{bp}}(t,T)\right) = \mathbb{E}_t^{Q^N}\left(X_T^2 - X_t^2\right)$$

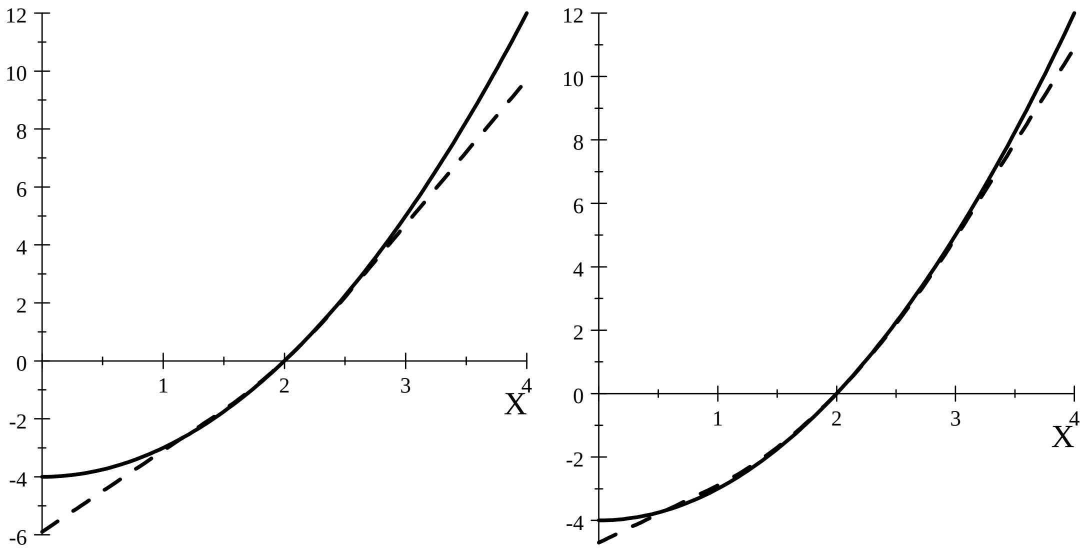
- Radically new hedging implications

- Payoff of a quadratic contract is roughly the sum of (i) two forwards, and (ii) two portfolios comprising OTM and one ATM option,

$$X_T^2 - X_o^2 \approx 2X_o\left(X_T - X_o\right) + 2\left(\sum_{j:K_j<X_0}\left(K_j - X_T\right)^+ + \sum_{j:K_j\geq X_o}\left(X_T - K_j\right)^+\right)\Delta K$$

- In contrast, the payoff of a logarithmic contract can be approximated as,

$$\ln\frac{X_T}{X_o} \approx \frac{1}{X_o}\left(X_T - X_o\right) - 2\left(\sum_{j:K_j<X_o}\frac{1}{K_j^2}\left(K_j - X_T\right)^+ + \sum_{j:K_j\geq X_o}\frac{1}{K_j^2}\left(X_T - K_j\right)^+\right)\Delta K$$



Hedging quadratic contracts with options. Solid lines Quadratic contract, $X^2 - X_o^2$, with $X_o = 2$. Dashed lines Replicating portfolios comprising: (i) two forwards struck at $X_o = 2$; and (ii) two additional equally weighted portfolios, with $\Delta K = \frac{1}{10}$, each including one ATM option, *and* 10 OTM options (left panel) or 20 OTM options (right panel)

Volatility indexes

- *Model-free indexes* of expected volatility,

$$\mathrm{VX}_t^j(T) \equiv \sqrt{(T-t)^{-1}\,\mathrm{V}_t^j}, \quad j \in \{\mathrm{bp}, \mathrm{p}\},$$

  where $\mathrm{V}_t^{\mathrm{bp}}$ is the strike K_Y for the basis point variance contract, and $\mathrm{V}_t^{\mathrm{p}}$ is the strike K_Y for the percentage contract

- *Index decompositions* Assume "sticky smile," i.e. $\sigma(X,K) = \sigma(\lambda X, \lambda K)$, $\lambda > 0$. Then,

  (i) there exists a function $\xi(t,T)$ independent of X, such that $\mathrm{V}_t^{\mathrm{bp}} = X_t^2 \times \xi(t,T)$
  (ii) $\mathrm{V}_t^{\mathrm{p}}$ is independent of X

Skew shifts and the dynamics of volatility indexes

- *Truncations* In practice, rely on a finite number (/bounded set) of options. Suppose

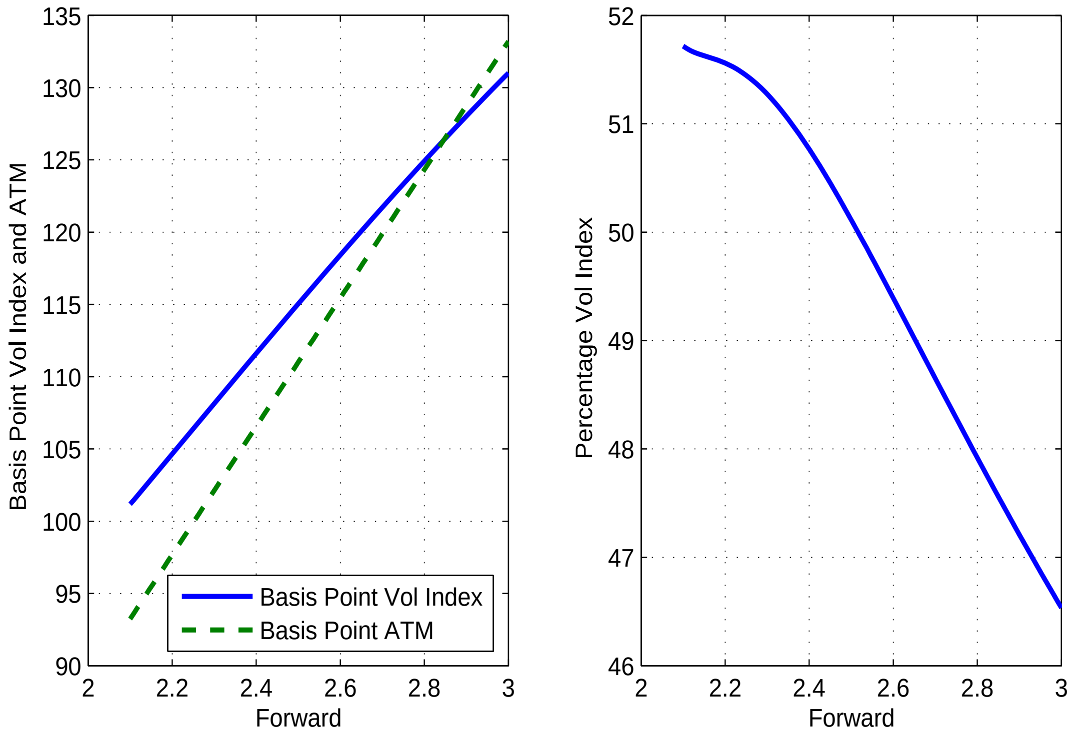
$$\mathrm{V}_{\ell}^{\mathrm{bp}} \equiv \int_{X-\ell}^{X} \mathrm{Put}\left(X, K, \sigma\left(X, K\right)\right) dK + \int_{X}^{X+\ell} \mathrm{Call}\left(X, K, \sigma\left(X, K\right)\right) dK$$

and

$$\mathrm{V}_{\ell} \equiv \int_{X-\ell}^{X} \frac{\mathrm{Put}\left(X, K, \sigma\left(X, K\right)\right)}{K^2} dK + \int_{X}^{X+\ell} \frac{\mathrm{Call}\left(X, K, \sigma\left(X, K\right)\right)}{K^2} dK$$

- Provide theoretical study $\longrightarrow$ Volatility indexes might respond to movements in the forward, possibly going beyond those relating to the fundamentals

Numerical example: Skew is constant but indexes change



<u>Left panel</u>: BP expected volatility vs BP ATM volatility. <u>Right panel</u>: Percentage expected volatility.

Resilience to jumps

- Assume

$$\frac{dX_\tau}{X_\tau} = -\left(\mathbb{E}_\tau^{Q^N}\left(e^{j(\tau)} - 1\right)\eta\left(\tau\right)\right)d\tau + \sigma_\tau \cdot dW\left(\tau\right) + \left(e^{j(\tau)} - 1\right)dJ\left(\tau\right),$$

where $J(\tau)$ is a Cox process under Q^N with intensity equal to $\eta(\tau)$, and $j(\tau)$ is the logarithmic jump size.

- Indexes are,

$$\mathrm{VX}_{Jt}^{\mathrm{bp}}\left(T\right) = \mathrm{VX}_t^{\mathrm{bp}}\left(T\right) \text{ (\textit{basis point})}$$

$$K_{J,Y} \equiv K_Y - 2\mathbb{E}_t^{Q^N}\left[\int_t^T \left(e^{j(\tau)} - 1 - j\left(\tau\right) - \tfrac{1}{2}j^2\left(\tau\right)\right)dJ\left(\tau\right)\right] \text{ (\textit{perc.})}$$

- Example: jumps distribution collapses to $\bar{j} < 0$ with constant intensity $\bar{\eta}$. Then,

$$\mathrm{VX}^{\mathrm{p}}_{Jt}(T) = \sqrt{\mathrm{VX}^{\mathrm{p}}_{t}(T) + 2\bar{\eta}\mathcal{J}}, \quad \mathcal{J} \equiv -(e^{\bar{j}} - 1 - \bar{j} - \frac{1}{2}\bar{j}^{2}) > 0$$

2/4 Interest rate swap markets

Highlights

- CBOE-SRVX
- Match market practice—Basis point
  - Key insight in the previous part: link BP variance to Quadratic contracts, and then price Q-contracts à la Carr-Madan
- Variance swap hedging: a component of it involves replicating $\int \frac{dR_t}{R_t}$
  - While $\frac{dR_t}{R_t}$ is not a return from any traded asset, the forward swap rate R_t can be dynamically replicated

The dynamics of the forward swap rate

- **Notation**
  - Let $R_t(T_1, \cdots, T_n)$ be the *Forward Swap Rate* prevailing at t, i.e. the fixed rate such that the value of a forward starting swap (at $T \equiv T_0$, with reset dates $T_0, \cdots, T_{n-1}$, tenor length $T_n - T$, and payment periods $T_1 - T_0, \cdots, T_n - T_{n-1}$) is zero at t

- **Assumption**
  - We assume $R_t(T_1, \cdots, T_n)$ is a *diffusion*,

$$dR_\tau(T_1, \cdots, T_n) = R_\tau(T_1, \cdots, T_n)\,\sigma_\tau(T_1, \cdots, T_n) \cdot dW_\tau, \quad \tau \in [t, T],$$

    where W_τ is a Brownian Motion under the annuity probability, and $\sigma_\tau(T_1, \cdots, T_n)$ is adapted to W_τ

- **Basis Point variance**
  - Define $V_n^{\mathrm{BP}}(t,T)$, as the *Basis Point* realized variance of the forward swap rate arithmetic changes in the time interval $[t,T]$,

$$V_n^{\mathrm{BP}}(t,T) \equiv \int_t^T R_\tau^2 (T_1,\cdots,T_n) \left\| \sigma_\tau (T_1,\cdots,T_n) \right\|^2 d\tau$$

- **Extension to Jumps**
  - Basis Point variance is:

$$V_n^{J,\mathrm{BP}}(t,T) \equiv V_n^{\mathrm{BP}}(t,T) + \int_t^T R_\tau^2 (T_1,\cdots,T_n) \left(e^{j_n(\tau)} - 1 \right)^2 dN_\tau,$$

  where N_τ is a Cox process under the annuity probability with intensity equal to η_τ, and $j_n(\tau)$ is the logarithmic jump size
  - **Jumps irrelevance results:** The price of BP variance is resilient to the presence of jumps as explained in Part I

Swap transaction risks

- **The annuity factor, or the PVBP**
  - The value of a fixed rate payer swap is:

$$\mathrm{SWAP}_T\left(K;T_1,\cdots,T_n\right)=\mathrm{PVBP}_T\left(T_1,\cdots,T_n\right)\left[R_T\left(T_1,\cdots,T_n\right)-K\right],$$

- **Swaptions**
  - Payer's and receiver's payoffs, $\mathrm{PVBP}_T\left(T_1,\cdots,T_n\right)\left[R_T\left(T_1,\cdots,T_n\right)-K\right]^+$ and $\mathrm{PVBP}_T\left(T_1,\cdots,T_n\right)\left[K-R_T\left(T_1,\cdots,T_n\right)\right]^+$
- Options on equities relate to a single source of risk, the stock price. Swaps (and swaptions), instead, are tied to two sources of risk:
  - the forward swap rate
  - the swap's PVBP realized at time T

  This adds complexity to defining and pricing swap market volatility

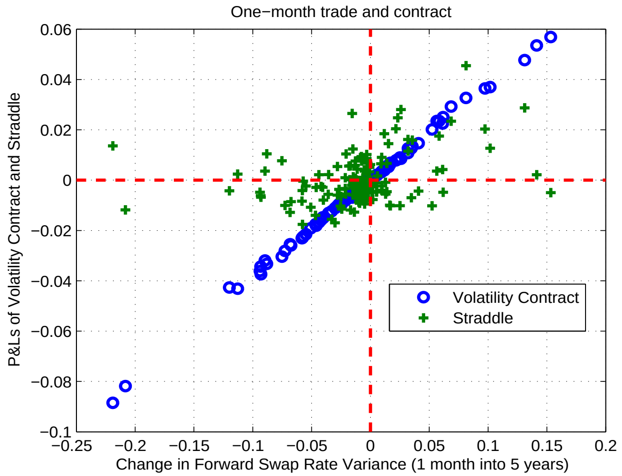
Option-based volatility trading

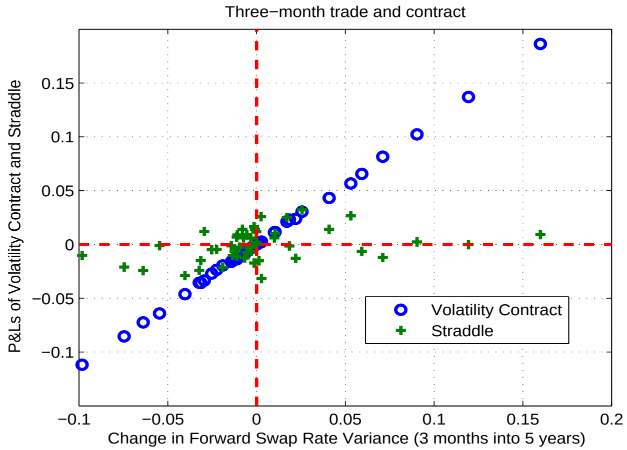
- Affected by both **price dependency** and the randomness of the **annuity factor**
- Noticed previously by banks while structuring notes linked to interest rate volatility $\longrightarrow$ addressed in an ad hoc way
- **Theoretically**,

$$\mathrm{P\&L}_T^{\mathrm{straddle}} \approx \sum_{t=1}^{T} \Gamma_t^{\$} \cdot \left[\left(\sigma_t^2 - \mathrm{IV}_0^2 \right) \mathrm{PVBP}_T \right] + \sum_{t=1}^{T} \mathrm{Straddle}_t \cdot \mathrm{Vol}_t \left(\mathrm{PVBP}_t \right) \cdot \frac{\widetilde{\Delta R_t}}{R_t}$$

  - First term: familiar price-dependency term (El Karoui, Jeanblanc-Picqué and Shreve, 1998)
  - Second: new. Links to randomness of the PVBP_t and the shocks affecting the swap rate, $\frac{\widetilde{\Delta R_t}}{R_t}$

- An empirical experiment





- Straddles P&L has the same sign as the variance risk-premium 62% of the time for one-month trades, and 65% of the time for three-month trades
- Corr(straddle P&L,variance risk-premium) = 33% (one-month), and 28% (three-month)
- When delta-hedged, these trades lead to higher correlations—with average corrs equal to 63% (Jiang, 2011)

Use of methodology in Part 1

- **We use the PVBP as a multiplier** as it is appealing and easy to calculate as explained

Contract design — Three "Gaussian" contracts

- **I: Interest Rate Variance (IRV) Forward Agreement**

At time t, counterparty A promises to pay B the annuity-factor adjusted BP variance realized over $[t, T]$, i.e.

$$V_n^{\mathrm{BP}}(t, T) \times \mathrm{PVBP}_T(T_1, \cdots, T_n).$$

The price B shall pay A at time t is the *Interest Rate Variance (IRV) Forward Rate*, and is denoted as $\mathbb{F}_{\mathrm{var},n}(t, T)$

– Links to Constant Maturity Swaps—see below
– Useful for implementing variance trading strategies—see Chapter 3 of the book + CBOE technical white paper

- **II: IRV Swap**

Counterparty A agrees to pay B the following difference at time T:

$$\text{Var-Swap}_n(t,T) \equiv V_n^{\text{BP}}(t,T) \times \text{PVBP}_T(T_1, \cdots, T_n) - \mathbb{P}_{\text{var},n}(t,T), \quad (1)$$

where $\mathbb{P}_{\text{var},n}(t,T)$, the *IRV Swap Rate*, is a fixed variance swap rate determined at time t, which makes the current value of $\text{Var-Swap}_n(t,T)$ in Eq. (1) equal to zero

- **III: Standardized IRV Swap**

  Counterparty A agrees to pay B the following difference at time T:

$$\text{Var-Swap}_n^*(t,T) \equiv \left[V_n^{\text{BP}}(t,T) - \mathbb{P}_{\text{var},n}^*(t,T)\right] \times \text{PVBP}_T(T_1,\cdots,T_n), \tag{2}$$

  where $\mathbb{P}_{\text{var},n}^*(t,T)$, the *Standardized IRV Swap Rate*, is a fixed variance swap rate determined at t, which makes the current value of $\text{Var-Swap}_n^*(t,T)$ in Eq. (2) equal to zero

- CBOE-SRVX[SM] is based on $\mathbb{P}_{\text{var},n}^*(t,T)$

Pricing

- One approximation to $\mathbb{F}_{\mathrm{var},n}(t,T)$ based on a finite number of swaptions,

$$\mathbb{F}_{\mathrm{var},n}(t,T) = 2\left[\sum_{i:K_i<R_t} \mathrm{SWPN}_t^{\mathrm{R}}(K_i,T;T_n)\,\Delta K_i + \sum_{i:K_i\geq R_t} \mathrm{SWPN}_t^{\mathrm{P}}(K_i,T;T_n)\,\Delta K_i\right],$$

and,

$$\mathbb{P}_{\mathrm{var},n}(t,T) = \frac{\mathbb{F}_{\mathrm{var},n}(t,T)}{P_t(T)}, \quad \text{and} \quad \mathbb{P}^*_{\mathrm{var},n}(t,T) = \frac{\mathbb{F}_{\mathrm{var},n}(t,T)}{\mathrm{PVBP}_t(T_1,\cdots,T_n)}$$

- **BP Index,**

$$Index_t(T,n) = \sqrt{\frac{1}{T-t}\mathbb{P}^*_{\mathrm{var},n}(t,T)}, \qquad \text{CBOE-SRVX}^{\text{SM}} = Index_t(1Y,10Y)$$

- Marking to market expressions are in the book

Constant weightings, Gamma exposure interpretation

- *Equity* options portfolio vega is insensitive to stock price with $\frac{1}{K^2}$ weightings in a lognormal market (Demeterfi, Derman, Kamal and Zou, 1999).

- The BP swap counterpart to equity is this:
  - Gaussian market,
  $$dR_\tau\left(T_1,\cdots,T_n\right)=\sigma_{\mathrm{n}}\cdot dW_\tau$$
  - OTM & ATM swaptions portfolio with weigthings $\omega\left(K\right)$ and value,
  $$\pi_t\left(R_t,T,\sigma_{\mathrm{n}}\right)\equiv\int\omega\left(K\right)\mathcal{O}_t\left(R_t,K,T,\sigma_{\mathrm{n}}\right)dK$$
  - Portfolio vega is insensitive to forward swap rate if and only if the weightings are independent of K,
  $$\frac{\partial}{\partial R}\left(\frac{\partial\pi_t\left(R,T,\sigma\right)}{\partial\sigma}\right)=0\Longleftrightarrow\omega\left(K\right)=\mathrm{const.}\Longleftrightarrow\text{Const. }\Gamma\text{ exposure}$$

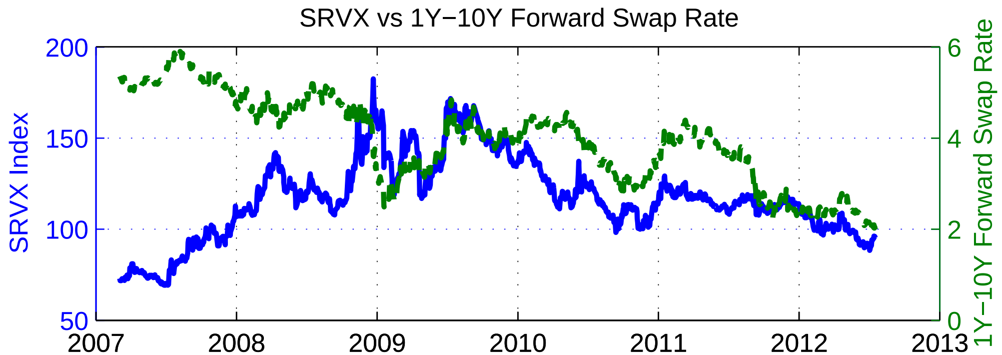
Homogeneity

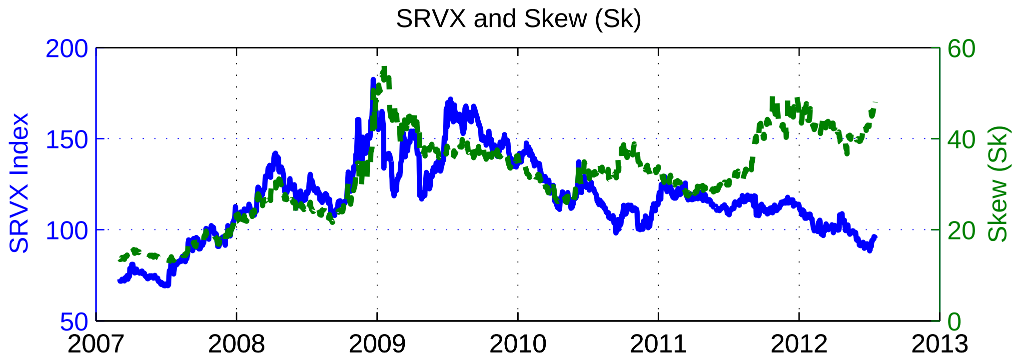
- Part I: *Suppose sticky smile. Then,* $\exists \xi_n (t, T)$ *independent of* R_t *s.t.,*

$$Index_t (T, n) = R_t \times \sqrt{\frac{1}{T - t} \xi_n (t, T)}$$

- Useful to interpret historical behavior of the index (Mele, Obayashi and Shalen, 2013)

R and skew


SRVX vs 1Y−10Y Forward Swap Rate
SRVX Index
1Y−10Y Forward Swap Rate
200
150
100
50
6
4
2
0
2007
2008
2009
2010
2011
2012
2013


SRVX and Skew (Sk)
SRVX Index
Skew (Sk)
200
150
100
50
60
40
20
0
2007
2008
2009
2010
2011
2012
2013

Hedging variance swaps

- Replicating BP-denominated contracts requires positioning in *quadratic contracts*, i.e. those delivering $R_T^2 - R_t^2$, rescaled by the PVBP.

- Consider, e.g., the *Standardized IRV Swap*.

Replication of the payoff relating to the Standardized BP-IRV contract. ZCB and OTM stand for zero coupon bonds and out-of-the-money

Portfolio	Value at t	Value at T
(i) short self-financed portfolio of ZCB	0	$\left[V_n^{\mathrm{BP}}(t,T) - (R_T^2 - R_t^2)\right] \times \mathrm{PVBP}_T$
(ii) long swaps and long OTM swaptions	$-\mathbb{F}_{\mathrm{var},n}(t,T)$	$(R_T^2 - R_t^2) \times \mathrm{PVBP}_T$
(iii) borrow basket of ZCB for $\mathbb{P}^*_{\mathrm{var},n}(t,T) \times \mathrm{PVBP}_t$	$+\mathbb{F}_{\mathrm{var},n}(t,T)$	$-\mathbb{P}^*_{\mathrm{var},n}(t,T) \times \mathrm{PVBP}_T$
Net cash flows	0	$\left[V_n(t,T) - \mathbb{P}^*_{\mathrm{var},n}(t,T)\right] \times \mathrm{PVBP}_T$

Links to Constant Maturity Swaps

- CMS are known to relate to the entire swaption skew since at least Hagan (2003) and Mercurio and Pallavicini (2006).

- We make a further step: CMS are actually a basket of BP-IRV forwards

- **Notation**. A party pays a counterparty the *spot* swap rate with a fixed tenor over a sequence of dates with legs set in advance, i.e. $R_{T_0+j\kappa}(T_1+j\kappa,\cdots,T_n+j\kappa)$ at times $T_0+(j+1)\kappa$, $j=0,\cdots,N-1$,
  - Approximately, the price of a CMS is:

$$\mathrm{CMS}_N(t) \equiv \sum_{j=0}^{N-1} P_t\left(\tau_j+k\right) R_t\left(T_1+j\kappa,\cdots,T_n+j\kappa\right)$$

$$+\sum_{j=0}^{N-1} G'\left(R_t\left(T_1+j\kappa,\cdots,T_n+j\kappa\right)\right)\mathbb{F}_{\mathrm{var},n}\left(t,T_0+j\kappa\right), \quad \text{for some function } G(\cdot)$$

- So IR variance swaps can be used to hedge against CMS

3/4 Government bonds and time-deposits

The numéraire

- Government bonds (GB) and time deposits (TD) share the same *numéraire*, a zero coupon bond, such that
  - the market probability is the forward probability,

$$\left.\frac{dQ_{F^T}}{dQ}\right|_{\mathbb{F}_T} = \frac{e^{-\int_t^T r_s ds}}{P_t(T)}$$

- Nevertheless GB & TD variance swaps have distinct features justifying separate treatments

- Recently launched CBOE/CBOT VXTYN on 10 Treas note vol

Begin with GB $\longrightarrow$

Pitfalls arising while applying standard VIX methodology to rates

Two introductory counter-examples

(i) No-spanning

(ii) Cash options

- We'll provide a third counter-example at the end of this part ("maturity mismatch")

Example 1—No-spanning

- Price of a zero expiring at T,

$$\frac{dP_\tau(T)}{P_\tau(T)} = r_\tau d\tau + \sigma_\tau(T) \cdot dW_\tau, \quad t \leq \tau \leq T, \tag{1}$$

where W_τ is a multid. BM under Q

- Variance swap has payoff that *occurs precisely at* T,

$$\int_t^T \|\sigma_\tau(T)\|^2 d\tau - \mathbb{P}(t,T),$$

where $\mathbb{P}(t,T)$ is the fair value of the strike

- Fair value is:

$$\mathbb{P}(t,T) = \frac{1}{P_t(T)}\mathbb{E}_t\left(e^{-\int_t^T r_\tau d\tau}\int_t^T \|\sigma_\tau(T)\|^2 d\tau\right)$$
$$= -2\mathcal{R}_{t,T} + \frac{2}{P_t(T)}\mathbb{E}_t\left(e^{-\int_t^T r_\tau d\tau}\int_t^T \frac{dP_\tau(T)}{P_\tau(T)}\right),$$

  where the log-return on the zero is

$$\mathcal{R}_{t,T} \equiv -\ln P_t(T) = \frac{1-P_t(T)}{P_t(T)} - \int_{P_t(T)}^1 (1-K)\frac{1}{K^2}dK$$

- It isn't model-free because $P_\tau(T)$ is not a martingale under the forward prob Q_{F^T}—naturally, the forward price *is*
- The "spanned part," $-2\mathcal{R}_{t,T}$, is actually negative so the unspanned part has a lot of economic meaning but it's model-dependent

Example 2—Cash options

- Suppose we are given a number of quotes re: ATM and OTM bond options settled on cash, not futures, as it is customary in OTC markets
- Aggregate these option prices following standard equity methodology to create a VIX-like index
- Would the resulting index necessarily reflect the fair value of a variance swap?
  - The answer is in the negative

- We have

$$\mathbb{P}\left(t,T,\mathbb{T}\right)=\mathbb{P}_{\mathrm{VIX}}\left(t,T,\mathbb{T}\right)+\mathrm{Bias}\left(t,T\right),$$

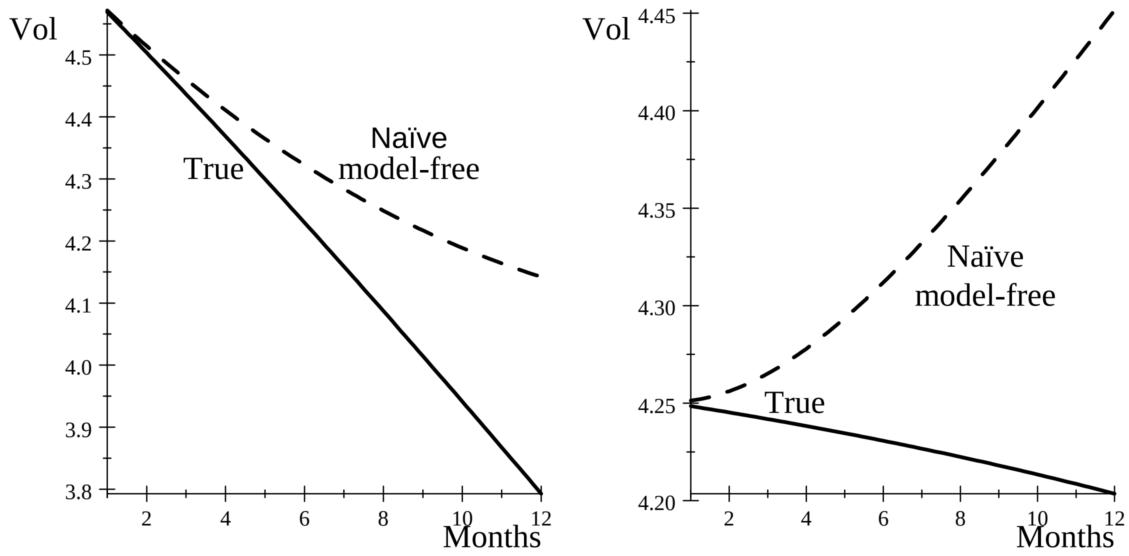
where

$$\mathbb{P}_{\mathrm{VIX}}\left(t,T,\mathbb{T}\right)$$

$$\equiv\frac{2}{P_t\left(T\right)}\left(\int_0^{F_t^o(T,\mathbb{T})}\mathrm{Put}_t\left(K,T,\mathbb{T}\right)\frac{1}{K^2}dK+\int_{F_t^o(T,\mathbb{T})}^{\infty}\mathrm{Call}_t\left(K,T,\mathbb{T}\right)\frac{1}{K^2}dK\right)$$

$$\mathrm{Bias}\left(t,T\right)\equiv 2\int_t^T\left[\mathbb{E}_t^{Q_{FT}}\left(r_\tau\right)-\mathbb{E}_t^{Q_{F\tau}}\left(r_\tau\right)\right]d\tau$$

and $F_t^o\left(T,\mathbb{T}\right)\equiv\frac{P_t(\mathbb{T})}{P_t(T)}$ is the "shadow price" of a forward contract



Biases arising while estimating expected volatility with standard model-free methodology in a hypothetical Vasicek's (1977) market (right panel obtained with a lower persistence than the left)

Basis asset

- Coupon bearing bond issued at time T_0, and paying off coupons $\frac{C_i}{n}$ over T_i, $i = 1, \cdots, N$, $\mathbb{T} \equiv T_N$, where n is the frequency of coupon payments, and $T_i - T_{i-1} = \frac{1}{n}$

- Price of coup bearing bond is $B_t(\mathbb{T})$. Forward bond price is $F_t(T, \mathbb{T}) = \frac{B_t(\mathbb{T})}{P_t(T)}$, and satisfies,

$$\frac{dF_\tau(T, \mathbb{T})}{F_\tau(T, \mathbb{T})} = v_\tau(T, \mathbb{T}) \cdot dW_{FT}(\tau), \quad \tau \in (t, T),$$

where $W_{FT}(\tau)$ is BM under the fwd prob, $v_\tau(T, \mathbb{T}) \equiv \sigma_\tau^B(\mathbb{T}) - \sigma_\tau(T)$, and $\sigma_\tau^B(\mathbb{T})$ is the vector of inst. vols of the coupon bearing bond

Pricing

- Apply methodology in Part I to obtain,
  - *Percentage*

$$\text{GB-VI}\,(t,T,\mathbb{T}) \equiv 100 \times \sqrt{\frac{\mathbb{P}\,(t,T,\mathbb{T})}{T-t}},$$

where $\mathbb{P}\,(t,T,\mathbb{T})$ is the fair value of a perc. variance swap,

$$\mathbb{P}\,(t,T,\mathbb{T}) = \frac{2}{P_t\,(T)}\left(\int_0^{F_t(T,\mathbb{T})} \text{Put}_t\,(K,T,\mathbb{T})\,\frac{1}{K^2}dK + \int_{F_t(T,\mathbb{T})}^{\infty} \text{Call}_t\,(K,T,\mathbb{T})\,\frac{1}{K^2}dK\right)$$

– *Basis point*

$$\text{GB-VI}^{\text{bp}}(t,T,\mathbb{T}) \equiv 100 \times 100 \times \sqrt{\frac{\mathbb{P}^{\text{bp}}(t,T,\mathbb{T})}{T-t}},$$

where $\mathbb{P}^{\text{bp}}(t,T,\mathbb{T})$ is the fair value of a BP variance swap,

$$\begin{aligned}&\mathbb{P}^{\text{bp}}(t,T,\mathbb{T})\\ &= \frac{2}{P_t(T)}\left(\int_0^{F_t(T,\mathbb{T})} \text{Put}_t(K,T,\mathbb{T})\,dK + \int_{F_t(T,\mathbb{T})}^{\infty} \text{Call}_t(K,T,\mathbb{T})\,dK\right)\end{aligned}$$

Getting the right volatility with the wrong model?

- Consider the following stochastic volatility extension of the Ho and Lee (1986) model,

$$\begin{cases} dr_\tau &= \theta_\tau d\tau + v_\tau dW_1(\tau) \\ dv_\tau^2 &= \xi v_\tau dW_2(\tau) \end{cases}$$

where ξ is a "volatility of variance" parameter, θ_τ is an "infinite-dimensional" parameter we need, to fit the initial yield curve at $\tau = t$ without error, and W_i are BM under the RN prob

- Zero coupon bond price is,

$$P_\tau\left(r_\tau, v_\tau^2, T\right) \equiv e^{-\int_\tau^T (s-T)\,\theta_s ds - (T-\tau)\,r_\tau + C_T(\tau)\,v_\tau^2},$$

where

$$\theta_\tau = \frac{\partial f_\$ (t, \tau)}{\partial \tau} + \frac{\partial^2 C_\tau (t)}{\partial \tau^2} v_t^2,$$

and $C_T (\tau)$ is the solution to Riccati's equation,

$$\dot{C}_T (\tau) = -\frac{1}{2} (T - \tau)^2 - \frac{1}{2} \xi^2 C_T^2 (\tau), \quad C_T (T) = 0$$

- In this market, the fair value of a GB variance swap is (the square of),

$$\text{GB-VI} (v_t; t, T, \mathbb{T}) \equiv \sqrt{\frac{1}{T - t} \int_t^T \bar{\phi}_\tau (t, T, \mathbb{T}) \, d\tau} \cdot v_t,$$

for some deterministic function $\bar{\phi}_\tau (t, T, \mathbb{T})$

- Under the misspecified assumption that the same GB variance swap is priced with the "wrong probability," the risk-neutral,

$$\widehat{\text{GB-VI}\,(v_t; t}, T, \mathbb{T}) \equiv \sqrt{\frac{1}{T-t}\int_t^T \phi_\tau\,(T, \mathbb{T})\,d\tau \cdot v_t},$$

for some deterministic function $\phi_\tau\,(T, \mathbb{T}) \neq \bar{\phi}_\tau\,(t, T, \mathbb{T})$.

- It's an arbitrage in a frictionless capital market

- Consider for example a future of squared index
  - Theoretically future has the same price as the square index, $\text{GB-VI}\,(v_t; t, T, \mathbb{T})$, because v_τ^2 is a martingale
  - But according to $\widehat{\text{GB-VI}\,(v_t; t}, T, \mathbb{T})$, there might be contango or backwardation according to parameter values

Basis point / duration-based yield volatility

- It is market practice to publish measures of "basis point yield volatility"
  - The level of volatility of yields consistent with a given ATM option price
  - This measure is model-dependent by nature
  - For example, you may identify yield volatility with the amount of a parallel shift in the yield curve, required to make a pricer consistent with the given option price
    $\longrightarrow$ Model dependency relates to the assumption of a parallel shift in the yield curve, or a weaker assumption of a shift predicted by a multifactor model
- Propose a model-free approach to measure basis point yield volatility
- We rely on a notion of "certainty equivalent prices"

Certainty equivalent prices

- What is the guaranteed price of the coupon bearing bond at time T, say $\mathcal{B}(t, T, \mathbb{T})$, such that the BP volatility index is the same as an hypothetical BP volatility index in this certainty equivalent market?

- Clearly, such an hypothetical market is also one where the forward is constant and equal to $\mathcal{B}(t, T, \mathbb{T})$, such that,

$$\begin{aligned}
&\text{GB-VI}^{\text{bp}}(t, T, \mathbb{T}) \\
&= \sqrt{\frac{1}{T-t}\frac{1}{P_t(T)}\mathbb{E}_t\left(e^{-\int_t^T r_\tau d\tau}\int_t^T \mathcal{B}^2(t, T, \mathbb{T})\left\|v_\tau(T, \mathbb{T})\right\|^2 d\tau\right)}
\end{aligned}$$

That is,

$$\text{GB-VI}^{\text{bp}}(t, T, \mathbb{T}) = \mathcal{B}(t, T, \mathbb{T}) \times \text{GB-VI}(t, T, \mathbb{T})$$

- Nothing new here really, it's simply a new representation of the BP implied volatility index

- We now build on $\mathcal{B}(t, T, \mathbb{T})$ to construct a model-free measure of yield volatility based on both the BP and the percentage implied vols, $\text{GB-VI}^{\text{bp}}(t, T, \mathbb{T})$ and $\text{GB-VI}(t, T, \mathbb{T})$

- First, a few words on the interpretation of $\mathcal{B}(t, T, \mathbb{T})$

- We show that $\mathcal{B}(t,T,\mathbb{T})$ is an average of the squared forward,

$$\mathcal{B}(t,T,\mathbb{T}) = \sqrt{\int_t^T \omega_\tau \mathbb{E}_t^{Q_{v^\tau}}\left[F_\tau^2(T,\mathbb{T})\right] d\tau}, \quad \text{with } \int_t^T \omega_\tau d\tau = 1,$$

where the "realized variance probability", Q_{v^τ}, has Radon-Nikodym derivative,

$$\rho(\tau;T) = \left.\frac{dQ_{v^\tau}}{dQ_{F^T}}\right|_{\mathcal{F}_\tau} = \frac{\left\|v_\tau(T,\mathbb{T})\right\|^2}{\mathbb{E}_t^{Q_{F^T}}\left[\left\|v_\tau(T,\mathbb{T})\right\|^2\right]}$$

- It distorts the fwd prob by giving more weight to the paths of $F_\tau(T,\mathbb{T})$ that have higher chances of experiencing episodes of high volatility

Back to BP yield volatility

Duration based

- Define

$$y_{\mathcal{B}}(t,T,\mathbb{T}) : \mathcal{B}(t,T,\mathbb{T}) = \frac{\text{GB-VI}^{\text{bp}}(t,T,\mathbb{T})}{\text{GB-VI}(t,T,\mathbb{T})} = \hat{P}\left(y_{\mathcal{B}}(t,T,\mathbb{T})\right),$$

and $D_{\mathcal{B}}(t,T,\mathbb{T})$, the modified duration of the guaranteed price $\mathcal{B}(t,T,\mathbb{T})$,

$$D_{\mathcal{B}}(t,T,\mathbb{T}) \equiv \frac{1}{1+\frac{y_{\mathcal{B}}(t,T,\mathbb{T})}{n}}\left(\sum_{i=1}^{N}\omega_i\frac{i}{n}+\hat{\omega}_N\frac{N}{n}\right)$$

$$\omega_i \equiv \frac{\frac{C_i}{n}\Big/\left(1+\frac{y_{\mathcal{B}}(t,T,\mathbb{T})}{n}\right)^i}{\mathcal{B}(t,T,\mathbb{T})}, \quad \hat{\omega}_N \equiv \frac{100/\left(1+\frac{y_{\mathcal{B}}(t,T,\mathbb{T})}{n}\right)^N}{\mathcal{B}(t,T,\mathbb{T})}$$

- *Model-free measure of duration-based yield volatility* is,

$$\text{GB-VI}^{\text{bp}}_{\text{Yd}}(t,T,\mathbb{T}) = 100 \times \frac{\text{GB-VI}(t,T,\mathbb{T})}{D_{\mathcal{B}}(t,T,\mathbb{T})},$$

or, using the definitions of $\hat{P}$ and $D_{\mathcal{B}}$,

$$\text{GB-VI}^{\text{bp}}_{\text{Yd}}(t,T,\mathbb{T}) = \frac{100 \times \left(1+\frac{1}{n}\hat{P}^{-1}\left[\frac{\text{GB-VI}^{\text{bp}}(t,T,\mathbb{T})}{\text{GB-VI}(t,T,\mathbb{T})}\right]\right) \times \text{GB-VI}^{\text{bp}}(t,T,\mathbb{T})}{\sum_{i=1}^{N}\frac{C_i}{n}\left(1+\frac{1}{n}\hat{P}^{-1}\left[\frac{\text{GB-VI}^{\text{bp}}(t,T,\mathbb{T})}{\text{GB-VI}(t,T,\mathbb{T})}\right]\right)^{-i}\frac{i}{n} + 100\left(1+\frac{1}{n}\hat{P}^{-1}\left[\frac{\text{GB-VI}^{\text{bp}}(t,T,\mathbb{T})}{\text{GB-VI}(t,T,\mathbb{T})}\right]\right)^{-N}\frac{N}{n}}$$

where $\hat{P}^{-1}$ denotes the inverse function of $\hat{P}$

- We also extend this gauge to the "post-issuance case," where the maturity T of the forward is higher than the date of issuance of the bond

American corrections

- Some data might rely on American not European options
- Design algorithms to convert American vols into Europeans
  - Resulting index calculation is model-dependent
- Main idea—three steps

(i) Calibrate a pricing kernel to the *market* price of American options on futures
(ii) Use the American-implied pricing kernel to derive implications on the (unobservable) price of European options on forwards
(iii) Feed vol indexes through *model*-based data obtained in step (ii)

Time-deposits

- Only focus on LIBOR variance contracts for reasons of space—*Rates* not prices
- **Notation** $l_t(\Delta)$ is the simply comp IR on a deposit from t to $t+\Delta$—say referenced to LIBOR
- Fwd contract is one where the payoff at T is $100 \times (1 - l_T(\Delta)) - Z_t(T, T+\Delta)$, where the forward price, $Z_t(T, T+\Delta)$ at t is $Z_t(T, T+\Delta) = 100 \times (1 - f_t(T, T+\Delta))$, where $f_t(T, T+\Delta)$ is the forward LIBOR
- $Z_t(T, T+\Delta)$ is a martingale under Q_{FT},

$$\frac{dZ_\tau(T, T+\Delta)}{Z_\tau(T, T+\Delta)} = v_\tau^z(T, \Delta)\, dW_{FT}(\tau)$$

- Therefore,

$$\frac{df_\tau(T,T+\Delta)}{f_\tau(T,T+\Delta)} = v_\tau^f(T,\Delta)\,dW_{FT}(\tau)$$
$$v_\tau^f(T,\Delta) \equiv \left(1 - f_\tau^{-1}(T,T+\Delta)\right)v_\tau^z(T,\Delta)$$

- Consider the BP case. The BP LIBOR integrated variance is,

$$V_t^{f,\mathrm{bp}}(T,\Delta) \equiv \int_t^T f_\tau^2(T,T+\Delta)\left\|v_\tau^f(T,\Delta)\right\|^2 d\tau$$

- The fair value of the time deposit *rate*-variance swaps at time t, is

$$\mathbb{P}_f^{\text{bp}}(t,T,\Delta) = \frac{2}{P_t(T)}\left(\int_0^{f_t(T,T+\Delta)} \text{Put}_t^f(K_f,T,\Delta)\,dK_f + \int_{f_t(T,T+\Delta)}^{\infty} \text{Call}_t^f(K_f,T,\Delta)\,dK_f\right),$$

where

$$\text{Put}_t^f\left(K_f,T,\Delta\right) = \frac{\text{Call}_t^z\left(100\left(1-K_f\right),T,\Delta\right)}{100}, \quad \text{Call}_t^f\left(K_f,T,\Delta\right) = \frac{\text{Put}_t^z\left(100\left(1-K_f\right),T,\Delta\right)}{100},$$

and $\text{Call}_t^z(\cdot,T,\Delta)$ and $\text{Put}_t^z(\cdot,T,\Delta)$ are the OTM options on the forward LIBOR price Z

- Index is,

$$\text{TD-VI}_f^{\text{bp}}(t,T,\Delta) \equiv 100^2 \times \sqrt{\frac{\mathbb{P}_f^{\text{bp}}(t,T,\Delta)}{T-t}}$$

Pitfalls: maturity mismatch

- Suppose the option maturity is, say, one month, and the underlying is a forward expiring in five years
- We show model-free indexes cannot exist in this case
  - Need to apply a model-dependent correction term
- Intuitively, let T be the maturity of the option and S be the maturity of the forward, with $T \leq S$
  - "Option spanning" operates under the T-fwd prob
  - Forward risk and, then, its volatility, are defined under the S-fwd prob
  - Unless $T = S$, we would price the fwd risk volatility with the "wrong" probability
  - reminiscent of convexity problems arising in fixed income security evaluation

- Illustrate the government bonds case only

- **Notation** $F_t(S, \mathbb{T})$ is the fwd price at t, for delivery at S, of the coupon bearing bond expiring at $\mathbb{T}$. It satisfies,

$$\frac{dF_\tau(S, \mathbb{T})}{F_\tau(S, \mathbb{T})} = v_\tau(S, \mathbb{T}) \cdot dW_{F^S}(\tau), \quad \tau \in (t, S)$$

- Only mention the percentage integrated variance,

$$V_t(T, S, \mathbb{T}) \equiv \int_t^T \|v_\tau(S, \mathbb{T})\|^2 \, d\tau$$

- Fair value of the variance swap referenced to $V_t(T, S, \mathbb{T})$ is,

$$\mathbb{P}(t, T, S, \mathbb{T}) = \mathbb{E}_t^{Q_{FT}}(V_t(T, S, \mathbb{T}))$$

- By a standard argument, for $\tau \in (t, T)$,

$$\frac{dF_\tau(S, \mathbb{T})}{F_\tau(S, \mathbb{T})} = v_\tau(S, \mathbb{T})(v_\tau(S, \mathbb{T}) - v_\tau(T, \mathbb{T}))\, d\tau + v_\tau(S, \mathbb{T}) \cdot dW_{F^T}(\tau),$$

where,

$$dW_{F^T}(\tau) = dW_{F^S}(\tau) - (v_\tau(S, \mathbb{T}) - v_\tau(T, \mathbb{T}))\, d\tau,$$

is a multidimensional BM under Q_{F^T}

- Therefore, by Itô's lemma,

$$-\mathbb{E}_t^{Q_F T}\left(\ln \frac{F_T(S,\mathbb{T})}{F_t(S,\mathbb{T})}\right) = -\mathbb{E}_t^{Q_F T}\left(\tilde{\ell}(t,T,S,\mathbb{T})\right) + \frac{1}{2}\mathbb{P}(t,T,S,\mathbb{T})$$

$$\tilde{\ell}(t,T,S,\mathbb{T}) \equiv \int_t^T v_\tau(S,\mathbb{T})\left(v_\tau(S,\mathbb{T}) - v_\tau(T,\mathbb{T})\right) d\tau$$

- On the other hand, a Carr-Madan expansion leaves

$$-\mathbb{E}_t^{Q_F T}\left(\ln \frac{F_T(S,\mathbb{T})}{F_t(S,\mathbb{T})}\right) = 1 - \mathbb{E}_t^{Q_F T}\left(e^{\tilde{\ell}(t,T,S,\mathbb{T})}\right) + \frac{1}{2}\mathbb{P}_{\text{vix}}(t,T,S,\mathbb{T})$$

- Summing up,

$$\mathbb{P}(t,T,S,\mathbb{T}) = \mathbb{P}_{\text{vix}}(t,T,S,\mathbb{T}) + 2\left(1 - \mathbb{E}_t^{Q_F T}\left(e^{\tilde{\ell}(t,T,S,\mathbb{T})} - \tilde{\ell}(t,T,S,\mathbb{T})\right)\right),$$

- Volatility index is,

$$\text{GB-VI}(t,T,\mathbb{T}) = \sqrt{\frac{1}{T-t}\mathbb{P}(t,T,S,\mathbb{T})}$$

It's model-dependent as $\mathbb{P}(t,T,S,\mathbb{T})$ is

4/4 Credit

Issues of methodology

- Introduce credit variance swaps on *loss-adjusted forward default swap spreads*
  - Model-free price, expressed in terms of traded CDS index option prices
  - Percentage & BP indexes

- Need concept of *survival contingent probability* to account for default risk which is absent from our previous derivations, or equity
  - Market numéraire: $defaultable$ annuity

The underlying risk

- CDS index
  - Buyer pays periodic premium—the CDS index spread
  - Seller insures losses from defaults by any of the index's constituents during the term of the contract
  - If a constituent defaults, the defaulted obligor is removed from the index, and the index continues to be traded with a prorated notional amount
- Options on a CDS index are European-style, to buy (payers) or sell (receivers) protection at the strike spread upon option expiry

Assumptions

- Credit events may occur over a sequence of regular intervals (T_{i-1}, T_i) with length $\frac{1}{b}$, for $i = 1, \cdots, bM$
  - M is the no of years the index runs, T_0 is the time of the index origination
    - E.g., $b = 4 \longrightarrow$ quarterly intervals

- We assume that

(i) Loss-given-default, LGD, is constant
(ii) The short-term rate r_τ is a diffusion process
(iii) Default arrives as a Cox process with intensity λ adapted to r

CDS indexes

- n is the initial number of names in the index decided at time $t \equiv T_0$, each constituent has notional value $\frac{1}{n}$, the same LGD, and the same intensity λ

- The number of names survived up to T_i is,

$$\mathcal{S}(T_i) \equiv \sum_{j=1}^{n} \left(1 - \mathbb{I}_{\{\tau_j \leq T_i\}}\right),$$

where τ_j is the time at which name j defaults, and the outstanding notional is,

$$\mathcal{N}(\tau) = \frac{1}{n}\mathcal{S}(\tau), \quad \mathcal{N}(t) \equiv 1$$

- Loss & premiums
  - Index *loss* at τ_j should obligor j default is $\mathrm{LGD}\frac{1}{n}\mathbb{I}_{\{t\leq\tau_j\leq T_{bM}\}}$
  - *Premium* at T_i is $\frac{1}{b}\overline{\mathrm{CDX}}_t\left(M\right)\times\frac{1}{n}\mathcal{S}\left(T_i\right)$—constant premium determined at t times the outstanding notional at time T_i, $\frac{1}{n}\mathcal{S}\left(T_i\right)$
  - Value of protection leg minus premium leg is:

$$\mathrm{DSX}_t=\mathrm{LGD}\cdot v_{0t}-\frac{1}{b}\overline{\mathrm{CDX}}_t\left(M\right)\cdot v_{1t},$$

where

$$v_{0,t}\equiv\mathbb{E}_t\left[e^{-\int_t^{\tau*}r(s)ds}\mathbb{I}_{\{t\leq\tau*\leq T_{bM}\}}\right],\ v_{1t}\equiv\sum_{i=1}^{bM}\mathbb{E}_t\left[e^{-\int_t^{T_i}r(\tau)d\tau}\cdot\mathbb{I}_{\{\mathrm{Surv}_*\text{ at }T_i\}}\right],$$

and τ_* is the default time for an hypothetical representative firm with default intensity λ

- v_{0t} = value at t of \$1 paid off at the time of default of the repr firm, provided default occurs $\leq$ index expiry

- v_{1t} = value at t of an annuity of \$1 paid at $T_1, \cdots, T_{bM}$, until default of the representative firm or the expiry of the index, whichever occurs first
  - Or, value at t of a basket of defaultable bonds with zero recovery value issued by a representative obligor—a *defaultable annuity*

Forward starting indexes and credit default options

- A *forward starting index* is an index decided at t and running at T
- A *CDS index payer* $\longrightarrow$ option to enter a CDS index as a protection buyer with strike spread K
- A forward starting index does not protect from any losses occurring $\leq$ index begins
  - Upon exercise the protection buyer would also receive a *front-end protection* arising from losses occurring $\leq$ index begins

- *Loss-adjusted forward default index* is,

$$\mathrm{DSX}_{t,T}^{L}(\tau) = \frac{1}{b}\boxed{\mathcal{N}(\tau)\, v_{1\tau}}\left(\mathrm{CDX}_{\tau}(M) - \overline{\mathrm{CDX}}_{t}(M)\right),$$

where $\mathrm{CDX}_{\tau}(M)$ is the value of $\overline{\mathrm{CDX}}_{\tau}(M)$: newly issued forwards are worthless, viz $\mathrm{DSX}_{\tau,T}^{L}(\tau) = 0$,

$$\frac{1}{b}\mathrm{CDX}_{\tau}(M) = \mathrm{LGD}\frac{v_{0,\tau}}{v_{1\tau}} + \frac{v_{\tau}^{\mathrm{F}}}{\mathcal{N}(\tau)\, v_{1\tau}}$$

- Note, $\mathcal{N}(\tau)\, v_{1\tau}$is the natural numéraire in this market

- Indeed, $\mathrm{CDX}_t(M)$ is a martingale under the "survival contingent probability" Q_{sc} defined through

$$\left.\frac{dQ_{\mathrm{sc}}}{dQ}\right|_{\mathbb{F}_T^r} = e^{-\int_\tau^T r(u)du} \frac{\mathcal{N}(T)\, v_{1T}}{\mathcal{N}(\tau)\, v_{1\tau}},$$

where $\mathbb{F}_T^r$ denotes the information set at time T, which includes the path of the short-term rate only.

- Prices of a payer and receiver with strike K expiring at T, are, for any $\tau \in [t, T]$,

$$\mathrm{SW}_\tau^p(K, T; M) \equiv \mathcal{N}(\tau)\, v_{1\tau} \cdot \mathbb{E}_\tau^{\mathrm{sc}}\left[\left(\mathrm{CDX}_T(M) - K\right)^+\right],$$

$$\mathrm{SW}_\tau^r(K, T; M) \equiv \mathcal{N}(\tau)\, v_{1\tau} \cdot \mathbb{E}_\tau^{\mathrm{sc}}\left[\left(K - \mathrm{CDX}_T(M)\right)^+\right],$$

Credit variance contracts

- Assume that

$$\frac{d\mathrm{CDX}_\tau(M)}{\mathrm{CDX}_\tau(M)} = -\left(\mathbb{E}_\tau^{\mathrm{sc}}\left(e^{j(\tau;M)}-1\right)\eta(\tau)\right)d\tau + \sigma(\tau;M)\cdot dW^{\mathrm{sc}}(\tau) + \left(e^{j(\tau;M)}-1\right)dJ^{\mathrm{sc}}(\tau),$$

- *Percentage variance*,

$$V_M(t,T) \equiv \int_t^T \|\sigma(\tau;M)\|^2 d\tau + \int_t^T j^2(\tau;M)\, dJ^{\mathrm{sc}}(\tau)$$

- *Basis point variance,*

$$V_M^{\mathrm{bp}}(t,T) \equiv \int_t^T \mathrm{CDX}_\tau^2(M) \left\|\sigma(\tau;M)\right\|^2 d\tau + \int_t^T \mathrm{CDX}_\tau^2(M) \left(e^{j(\tau;M)} - 1\right)^2 dJ^{\mathrm{sc}}(\tau)$$

- Only provide the pricing of *Basis point variance* in this presentation for reasons of space

- Moreover, we only consider "standardized contracts"—Chapter 5 in the book considers three contacts just as Part II of this presentation

- **Standardized BP-Credit Variance Swap Rate** The Standardized BP-Credit Variance Swap rate is the fixed variance swap rate $\mathbb{P}^{*\text{bp}}_{\text{var},M}(t,T)$, which zeroes the current value of

$$\left[V^{\text{bp}}_{M}(t,T) - \mathbb{P}^{*\text{bp}}_{\text{var},M}(t,T)\right] \times \mathcal{N}(T)\, v_{1T}$$

- We have,

$$\mathbb{P}^{*\text{bp}}_{\text{var},M}(t,T)$$
$$= \frac{2}{v_{1t}} \left[\sum_{i:K_i<\text{CDX}_t(M)} \text{SW}^{r}_{t}(K_i,T;M)\,\Delta K_i + \sum_{i:K_i\geq\text{CDX}_t(M)} \text{SW}^{p}_{t}(K_i,T;M)\,\Delta K_i \right]$$

- Marking-to-market & replication issues in the book

Credit Volatility Index

- The BP Credit volatility index is,

$$\text{C-VI}_M^{\text{bp}}(t,T) \equiv 100 \times 100 \times \sqrt{\frac{1}{T-t}\mathbb{P}_{\text{var},M}^{*\text{bp}}(t,T)}$$