Uncertainty, Decision, and Normal Form Games[1]

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Abstract

This paper offers a new equilibrium concept for finite normal form games motivated by the idea that players may have preferences which display uncertainty aversion. More specifically, it adopts the representation of preferences presented in Gilboa and Schmeidler (1989). Then an equilibrium with uncertainty aversion is defined and applied to a number of simple games. This equilibrium concept generalizes both Nash equilibrium and maxmin play. One interesting feature of the equilibrium is that it provides a new justification for some mixed strategy equilibria based on objectification. It also admits a natural channel through which some unmodelled aspects of the game can influence the analyst's choice of equilibrium. A refinement of equilibrium with uncertainty aversion incorporating the notion of common knowledge of rationality is introduced. The notion of weak admissibility is discussed and incorporated into the solution concept. *Journal of Economic Literature* Classification Numbers: C72, D81.

1 Introduction

Traditional decision theory and game theory have treated uncertainty (situations in which probabilities are unknown or subjective) with the same formalism as they have treated risk (situations where probabilities are known or objective); indeed, the word "uncertainty" is often used to describe both. This continues despite the fact that there is strong evidence which suggests that thoughtful decision-makers react to uncertainty differently than they react to risk.[1] The classic reference is the Ellsberg Paradox (Ellsberg 1961) a version of which may be demonstrated by the following choice situation:

	Bets		
	Black	Red	Yellow
1	\$100	\$0	\$0
2	\$0	\$100	\$0
3	\$0	\$0	\$100
4	\$100	\$0	\$100
5	\$100	\$100	\$0
6	\$0	\$100	\$100

An urn contains ninety balls, identical except for their color. Thirty of these balls are black. The remaining sixty are either red or yellow in unknown proportion. One ball will be drawn at random from the urn. You are asked to consider the above six bets, whose payoffs depend on the color of the drawn ball.

The preference ordering of many decision-makers when faced with these bets is $6 \succ 5 \sim 4$ and $1 \succ 2 \sim 3$. This ordering cannot be reconciled with any subjective probability assessment. Moreover, as Ellsberg (1961) recounts:

"The important finding is that, after rethinking all their 'offending' decisions in the light of the axioms, a number of people who are not only sophisticated but reasonable decide that they wish to persist in their choices. This includes many people who previously felt a

[1]Knight (1921) and Shackle (1949, 1949-50) were among earlier economists who argued for a distinction between uncertainty and risk.

'first-order commitment' to the axioms, many of them surprised and some dismayed to find that they wished, in these situations, to violate the Sure-thing Principle." [p. 656]

Further, the fact that many people do not change their behavior even when confronted with their violation of the standard axioms distinguishes this behavior from some other types of violations such as intransitivity in choice.[2] Although intransitivities are observed experimentally, when the violations of transitivity are pointed out subjects often wish to change their choices so as to make them transitive. I would argue that theories of reasoned or rational behavior as well as purely descriptive theories should try to incorporate those types of violations which persist. The fact that many thoughtful people are not convinced by the arguments for the standard axioms should cause us to at least question their predominance in economic analysis.

Fortunately, Schmeidler (first version 1982, 1989) (see also Gilboa 1987 and Gilboa and Schmeidler 1989) have recently developed axiomatic decision theories which allows for Ellsberg-type preferences. This paper adopts the multiple priors theory developed in Gilboa and Schmeidler (1989).[3] A common explanation for the Ellsberg preferences is that decision makers dislike uncertainty or ambiguity. This is consistent with the fact that bet 1 (which has a known probability of one-third of paying $100) is preferred to bets 2 and 3 and bet 6 (which pays $100 with probability two-thirds) is preferred to the uncertain bets 4 and 5.[4] Thus the Gilboa-Schmeidler theory allows for uncertainty aversion on the part of the decision maker. In the next section, I briefly review the Gilboa-Schmeidler theory. In the third section I present a new solution concept for normal form games in which players are Gilboa-Schmeidler (henceforth, G-S) decision makers. This section also contains examples to which the concept is applied. The fourth section presents a characterization of common knowledge of rationality in the sense of G-S preferences and uses this to present a refinement of the solution concept and some more examples. The fifth section reconsiders

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[2]For experimental evidence on this point see e.g. Slovic and Tversky (1974).

[3]Some alternative theories and further experimental evidence are described in the survey paper by Camerer and Weber (1992).

[4]One important question which Ellsberg's example does not address is whether this uncertainty aversion is more than lexicographic. In other words, would a decision maker be willing to give up anything to avoid uncertainty? Ellsberg himself (1961, p.664) provides evidence for this when he reports that many subjects maintain the above preferences even after one black ball is removed from the urn. Many subsequent studies (cited in Camerer and Weber (1992)) have found ambiguity premia which are strictly positive and are typically around $10-20\%$ in expected value terms.

the theory in the light of an admissibility criterion and proposes a modification which is then applied to games. The sixth section provides a comparison to some related literature on games with uncertainty aversion, with a focus on the papers of Lo (1995a) and Dow and Werlang (1994). The seventh section concludes. An appendix contains some proofs.

2 The Gilboa- Schmeidler Decision Theory

First some notation. The basic framework is one of "lottery-acts" (Anscombe and Aumann 1963). Let X be a set of "prizes" (e.g. cash rewards). Let Y be the set of probability measures over X with finite support. Elements of Y are called lotteries. Let S be a set of "states of the world" and let Σ be an algebra (of "events") on S. Let F be the set of bounded, measurable functions from S to Y. Preferences are defined over F, the set of "acts." To avoid technicalities, we limit discussion to the set L_0 of finite step functions in F. G-S propose six axioms on preferences (axioms A.1-A.6 are provided in the Appendix for reference). The main result of G-S is the following representation theorem:

Theorem 1 *(Gilboa and Schmeidler 1989)*

Let $\succeq$ be a binary relation on L_0. Then the following are equivalent,

(1) $\succeq$ satisfies A.1 - A.5 for $L = L_0$

(2) $\exists$ an affine function $u : Y \to \mathcal{R}$ and a non-empty, closed, convex set C of finitely additive probability measures on Σ such that $\forall f, g \in L_0$, $f \succeq g$ if and only if $\min_{p \in C} \int u \circ f dp \geq \min_{p \in C} \int u \circ g dp$.

Furthermore, the function u is unique up to a positive affine transformation and, if and only if A.6 holds, the set C is unique.

The reader is referred to the paper for the proof.[5] G-S do not interpret the set of measures, C, which appears in the representation. For reasons which will become clear

[5] The first step in the proof is to observe that, as constant acts may be identified with the choice set in the von Neumann-Morgenstern setting, axioms A.1-A.3 applied to constant acts give the function u through the von Neumann-Morgenstern expected utility theorem. I interpret this theory as implying that a decision maker behaves as an expected utility maximizer in situations where the probabilities are objective (i.e. where only risk but not uncertainty is present). This is not the only possible interpretation of the Gilboa-

later on, I would like to interpret C as the closure of the convex hull of the set of "possible" subjective probability distributions from the decision maker's point of view. Observe that C is the closure of the convex hull of exactly those probability measures which are used in place of an objective probability measure in valuing some subset of mappings from events to payoffs. In what sense then is C the set of "possible" subjective probability measures? In the standard theory of decision under uncertainty, a probability measure q is said to be the subjective probability measure of a decision maker when she behaves as if she were maximizing the expectation of her affine utility function on lotteries (elicited using standard techniques and objective probabilities) where the expectation is taken with respect to q. Here, a set C is the set of "possible" subjective probability measures when it is the closed, convex hull of those measures which are used to calculate expected utility for *some* acts and when the choice of which possible measure to use for which act is governed by which possible measure gives the minimum expected utility for that act. Notice that in expanding from a single measure to a set of measures I have had to specify a rule for assigning measures to acts. This is very important in obtaining a notion of *the* (i.e. unique) set of possible measures. If one were to allow both the set of measures and the assignment rule to vary, then there would be different sets of "possible" measures for different assignment rules even though the preferences being represented were not changing. Thus we must keep in mind that my interpretation of "possible" is contingent on the adoption of the G-S assignment rule.

Although the main focus of this paper is on the consequences for game theory, it is worthwhile to briefly explore some decision theoretic concerns. Does the G-S theory resolve the Ellsberg paradox? Is uncertainty aversion related to a preference for mixtures? Recall

Schmeidler theory. Other researchers (see Quiggin 1982, Yaari 1987, Wakker 1990, among others; Fishburn 1988, chapt. 2 has a survey) have interpreted representations which are special cases of the related non-additive representation of Schmeidler (1989) as models of decision making under risk where the probabilities are distorted by the decision maker. As the representation considered above is isomorphic to the Schmeidler (1989) representation under certain conditions, such an interpretation could also be applied here. However, although these interpretations are attractive from a descriptive point of view (e.g. are consistent with the Allais Paradox (Allais 1953)), they do not seem normatively compelling as they operationally require a decision maker to take perfectly known, objective probabilities and distort them before using them to weight outcomes. This seems much more objectionable than allowing that, in situations where probabilities are not known, the decision maker may not act as if he subjectively "knows" the probabilities (i.e. has a unique subjective probability distribution in mind). More importantly for our purposes, perhaps, this interpretation does not allow for Ellsberg-type behavior.

the thought experiment described above. Suppose that the decision maker in the experiment is uncertainty averse and thus is unwilling to use a unique probability measure in situations where uncertainty is present. In the experiment as described, there is certainly substantial uncertainty in that the decision maker is given no information about the relative proportions of the three colors aside from the fact that one-third of the balls are black. As long as the decision maker's set of possible probability measures includes at least one in which Prob(red) > Prob(yellow) and another in which the reverse is true (assuming that all assign one-third to black), the decision maker will display the Ellsberg preferences. Thus, in this very clear sense, it is the uncertainty about whether there are more red balls or yellow balls combined with the individual's dislike of uncertainty which results in the Ellsberg behavior.

Some observers have argued that the Ellsberg Paradox simply points out the need to teach people to obey the axioms of Savage-Anscombe-Aumann decision theory by presenting them with compelling examples that will persuade them that treating subjective probability differently than objective probability is a mistake. In a comment (Raiffa 1961) published along with Ellsberg's original article, Raiffa uses an example similar to ones offered by Ellsberg to make his argument.[6] I will use it to argue that a G-S decsion maker may have a preference for randomization. In Raiffa's example, two questions are asked of a decision maker. First, the decision maker is asked to consider an urn containing fifty red balls and fifty black balls and to name the dollar amount that he would pay to be allowed to name a color and receive one hundred dollars if a ball drawn at random is of the named color. Raiffa reports that the amounts given clustered around thirty dollars (thus displaying risk aversion). These same decision makers were then asked to say how much they would pay for the same opportunity with an urn which contains red and black balls in unknown proportion. The answers in this case typically involved much smaller dollar amounts, thus violating the standard axioms. In subsequent discussion, Raiffa finds that the following argument convinces people to change their answer to the second question so that it matches their answer to the first question: Suppose that in the second setting you draw a ball at random and do not examine its color. Then flip a coin and say "red" if heads and "black" if tails. Notice that this results in an objective probability of winning of one-half independent of the true proportions of red and black balls. Certainly, it should not matter whether the ball is drawn before or after the coin is flipped since the processes are

[6]See also chapter 5 in Raiffa 1968.

physically independent. Thus the second option (unknown proportions) should always be worth at least as much as the first option (known 50-50) since a coin flip can transform the second into the first.[7]

I find this argument compelling, and fully agree that an individual who values the second option less than the first is not acting rationally in the sense that once she thinks the problem through carefully (and either discovers or has pointed out to her the strategy of flipping a coin to decide) she will revise her decision. Note however that this argument does not contradict the results of Ellsberg in a similar experiment. In Ellsberg's version the set-up is the same but individuals are not given as much freedom in that they are asked to make specific pairwise comparisons between bets. Thus many individuals say they prefer betting on red in the known urn to betting on red in the unknown one and prefer betting on black in the known urn to black in the unknown urn. Notice that these responses clearly violate the standard axioms but cannot be remedied by randomizing since subjects are asked to compare two fixed bets, whereas Raiffa is asking them to compare two betting environments. Specifically, the reader can check that a decision maker whose preferences are consistent with the Gilboa-Schmeidler axioms will *always* value Raiffa's unknown urn option at least as much as they value the known urn option, and may at the same time prefer any fixed bet on the known urn to the same bet on the unknown one.

The reason for this is simply that randomization between bets which pay off in different states of the world (here, black or red drawn from the urn) helps to reduce uncertainty by spreading the utility over more states, which is exactly what an uncertainty averse decision maker would like to do. Thus it is possible for such an individual to strictly prefer the randomization over the two bets on the unknown urn to either of the bets themselves. This feature of uncertainty aversion will play an important role in our discussion of game theory.

We note that such a preference for randomization raises the issue of dynamic inconsistency, in the sense of wanting to randomize again once the outcome of the original randomization is known.[8] However, Machina (1989) surveys many such dynamic inconsistency objections to non-expected utility theories and argues that this notion of

[7] Throughout this paper, as in decision theory and game theory generally, it is assumed that participants have costless access to independent, privately-observable randomizing devices.

[8] For some deeper and separate issues concerning dynamic consistency and updating G-S preferences in the presence of new information see the brief remarks in the concluding section and the references there.

consistency is inappropriate for such decision makers. The flavor of his argument can be expressed here by the notion that strictly preferring a randomization over two acts, A and B, includes the preference for act A over any mixture *conditional* on having borne a risk of B. Thus if the result of the randomization was A, the individual would indeed be willing to perform A. A similar argument is made for B. The reader who is unconvinced by Machina's arguments may want to think of the decision and game situations we will look at as situations in which the participants have available some means of committing to a mixed strategy. For example, they may be giving instructions to agents.

Now that we have discussed the preferences, some game theoretic notions can be explored.

3 Game Theory with Uncertainty Aversion

Game theoretic situations are rife with uncertainty.[9] Almost never can a player publicly commit to playing a given strategy. Thus, from the point of view of his opponent(s), there will often be great uncertainty about what this strategy will be. Much of game theory can be viewed as the search for concepts which try to narrow this uncertainty in convincing ways. Nash equilibrium, the leading solution concept for non-cooperative games, does this by combining two fundamental ideas. First, it borrows from decision theory the idea that rational players will choose a strategy which is the most preferred given their beliefs about what other players will do. Second, it imposes the consistency condition that all players' beliefs are, in fact, correct. One major criticism of Nash equilibrium has been the strength of the consistency condition. In many settings it is far from clear that players will have exactly correct beliefs about each other.[10] Moreover, even if it is common knowledge that all players in a game believe that Nash equilibrium is the proper concept to use in determining their beliefs about play, the problem of multiple Nash equilibria remains. In a game with multiple Nash equilibria, even if the players themselves accept (and are

[9] The only empirical work exploring whether players beliefs reflect uncertainty aversion in games, as opposed to decision-theoretic settings, that I am aware of is that of Camerer and Karjalainen (1994). Across their four experiments they find that between 40 and 65 percent of subjects display some degree of uncertainty aversion, though this degree is often modest. Thus they offer some support for the notion that uncertainty aversion exists in the context of games.

[10] For recent work which has investigated conditions under which this will be true in particular repeated game settings see e.g. Fudenberg and Levine 1993 and Kalai and Lehrer 1993.

commonly known to accept) the Nash solution concept they may still face substantial uncertainty about the play of their opponents. Thus, there is wide scope for players' behavior under uncertainty to affect the conclusions of game theory. In this vein, I propose a solution concept for normal form games which generalizes the notion of Nash equilibrium by relaxing the consistency condition and allowing for players whose preferences can be represented as in theorem 1 above.[11]

Fix a finite normal form game G (i.e. a finite set of players $\{1, 2, \ldots, I\}$, a pure strategy space S_i for each player i such that $S = \times_i S_i$ is finite, and payoff functions $u_i : \times_i S_i \to \mathcal{R}$ which give player i's von Neumann-Morgenstern utility for each profile of pure strategies).

Definition: An *equilibrium with uncertainty aversion* of G is a $2 * I$-vector $(\sigma_1, \ldots, \sigma_I, B_1, B_2, \ldots, B_I)$ where $\sigma_i \in \Sigma_i$ (the set of mixed strategies for player i, i.e. the set of probability distributions over S_i) and the B_i are closed, convex subsets of P_{-i} (the set of probability distributions over $\times_{k \neq i} S_k$) such that, for all i,

(1) σ_i satisfies $\min_{p \in B_i} \sum_s u_i(s_i, s_{-i}) \sigma_i(s_i) p(s_{-i}) \geq \min_{p \in Bi} \sum_s u_i(s_i, s_{-i}) \sigma'_i(s_i) p(s_{-i})$ for all $\sigma'_i \in \Sigma_i$, and

(2) $\prod_{k \neq i} \sigma_k(s_k) \in B_i$.

Condition (2) relaxes the consistency condition imposed by Nash equilibrium. It says that each player's beliefs must not be mistaken, in the sense that they contain the truth. More specifically, the truth must be contained in the closed, convex hull of the set of possible subjective probability distributions over the strategies of the other players. Condition (1) simply says that each player's strategy is optimal given her beliefs, assuming that her preferences can be represented as in theorem 1.

One important thing to note about the sets of beliefs is that elements of these sets may allow for correlation between the strategies of the other players even though we require the true strategies to be independent. To see how this might arise, consider a three player game where each player may move either right or left. Player one might well believe that either two and three will both play right or two and three will both play left (because, for example, one knows that two and three grew up with the same social norm but one does not know what that norm is). Any convex combination of these two priors could only

[11] The relation to some alternative solution concepts, notably those of Dow and Werlang (1994) and of Lo (1995a), is discussed in section 6.

arise from correlation between two and three. In this way, one's uncertainty introduces subjective correlation into his beliefs even though he knows that only independent mixing is allowed.

Two polar special cases of this definition – when all the B_i's are singletons and when all the B_i's equal P_{-i} – yield familiar concepts as we observe in the following theorem.

Theorem 2 *(a) In any finite normal form game, a strategy profile σ is part of an equilibrium with uncertainty aversion where B_i is a singleton for all i if and only if σ is a Nash equilibrium profile.*

(b) In any finite normal form game, a strategy profile σ is part of an equilibrium with uncertainty aversion where $B_i = P_{-i}$ for all i if and only if, for all i, σ_i is a maximin strategy (i.e. a strategy which maximizes i's minimum payoff given that any opponents' play is possible).

Proof: (a) and (b) follow directly from the definition of equilibrium with uncertainty aversion. *QED*

Theorem 2 shows that equilibrium with uncertainty aversion spans the continuum between all players playing maximin strategies, a criterion often advocated in situations of complete ignorance, and Nash equilibrium where all players behave as if they had perfect knowledge of their opponents' strategies. Exactly when preferences in Theorem 1 coincide with subjective expected utility maximization, equilibrium with uncertainty aversion coincides with Nash equilibrium. Existence of an equilibrium with uncertainty aversion follows from the existence of a Nash equilibrium (Nash 1950).

The next observation shows that equilibria with uncertainty aversion are not often unique. This is to be expected as they are, by construction, very dependent on beliefs.

Observation 1 *A finite normal form game has a unique equilibrium with uncertainty aversion only if it has a unique Nash equilibrium and that Nash equilibrium consists of each player playing their unique maximin strategy.*

Proof: Any Nash equilibrium is an equilibrium with uncertainty aversion. From theorem 2, each player playing a maximin strategy is an equilibrium with uncertainty aversion. *QED*

		Player 2		
		X	Y	Z
	A	1,2	4,3	1,4
Player 1	B	1,2	3,3	3,1
	C	2,2	4,1	2,1

figure 1

Note that the converse is false, as is shown by the game in figure 1.

In this game, the unique Nash equilibrium is (C, X), which is also the unique maximin profile. However, (B, Y) is an equilibrium with uncertainty aversion if player 1 has a belief set consisting of all distributions over Y and Z, while player 2 has a belief set consisting of all distributions over A and B.

The best way to see the implications of this definition is through some examples. To keep things simple I will focus on 2 x 2 games. Consider the pure coordination game in figure 2.

	L	R
U	2,2	0,0
D	0,0	1,1

figure 2

This game has three Nash equilibria, (U, L), (D, R), and (1/3 U, 2/3 D; 1/3 L, 2/3 R). Let us focus on the mixed equilibrium. Notice that in the Nash setting, each player is indifferent between any pure or mixed strategy given their beliefs. Thus there is no affirmative reason to mix with these proportions. This need not be true with uncertainty aversion. For example, if each player's belief set, B_i, consists of all mixtures over their opponents' pure strategies (as it would, for instance, if players are uncertainty averse and

	L	R
U	9,9	0,8
D	8,0	7,7

figure 3

their sets of possible subjective probability measures include the Nash beliefs) then each player will strictly prefer to play the mixed strategy. This is true because by equalizing the payoff to, say, U and D under any distribution over L and R, the uncertainty is eliminated and the maximin payoff is achieved. In fact, as long as player 1 has some belief which assigns Prob(L) $< 1/3$ and some belief which assigns Prob(L) $> 1/3$, the mixed strategy is his strict best response. Similarly, if player 2 has some belief which assigns Prob(U) $< 1/3$ and one which has Prob(U) $> 1/3$, the mixed strategy is the strict best response. Thus equilibrium with uncertainty aversion can justify mixing as a response to strategic uncertainty. In contrast with Harsanyi's (1973) view of mixed equilibria as the limits of pure strategy equilibria in perturbed games, our setting allows common knowledge of payoffs to be taken seriously. Another advantage of this view of mixed strategies is that it can provide information about the likelihood or robustness of a mixed strategy outcome. Consider the game in figure 3, which has been commented on extensively in the literature.

In this game, unlike the game in figure 2, a mixed strategy will never be strictly preferred in equilibrium. This can be seen by noting that if any subjective distribution gives weight more than 1/8 to D (or R) then the best response is to play R (or D) and if all distributions give weight less than 1/8 to D (or R) then the best response is L (or U). The only beliefs for which mixing can occur in equilibrium are those which include giving weight 1/8 to D(or R) and possibly include some distributions which give weight less than 1/8 to D(or R). However mixing is not strictly preferred for these beliefs.

These two examples suggest that equilibrium with uncertainty aversion highlights mixed equilibria in some games but not in others. We would like to understand what it is about the game in figure 2 which leads to the possibility of a strict mixed equilibrium. Observe that, for player 1, U does better if 2 plays L while D does better if 2 plays R. Thus, as long as U does not weakly dominate D or vice-versa, a mixture over U and D will do better than D against L and will do better than U against R. Since an uncertainty averse individual cares about the minimum expected utility over her belief set, it is easy to see that mixing can raise this minimum as compared to either pure strategy for some beliefs.

In the game in figure 3, however, both U and D do better if 2 plays L. In this case, since both pure strategies are lower under R than under L, a mixed strategy will never raise the minimum expected utility compared to each of the pure strategies. More generally, if the expected payoffs to any two pure strategies are minimized (over B_i) by the same distribution $p \in B_i$, a mixture of the two will never be strictly preferred to each pure strategy by an uncertainty averse decision maker. This condition is only sufficient, however. This is easily seen by considering one strategy which strictly dominates another, but which is not minimized by the same distribution as the other. No mixing involving the dominated strategy will ever be preferred to the undominated strategy, yet these two strategies are not minimized by the same distribution. The following theorem gives necessary and sufficient conditions for not strictly preferring a mixture of two strategies to each strategy itself. In other words, these conditions characterize exactly when there is no gain to hedging between two strategies.

Theorem 3 *Fix a player i and two strategies σ_i and σ_i' such that $\sigma_i \succeq \sigma_i'$ (i.e. the minimum expected utility of σ_i is at least as big as the minimum expected utility of σ_i'). No mixture over σ_i and σ_i' will be strictly preferred to both σ_i and σ_i' if and only if there exists some $q \in B_i$ such that q minimizes the expected utility of σ_i and such that $\sum_s u_i(s_i, s_{-i})\sigma_i(s_i)q(s_{-i}) \geq \sum_s u_i(s_i, s_{-i})\sigma_i'(s_i)q(s_{-i})$.*

Proof: (sufficiency) Let there be such a q. Then the minimum expected utility of $\sigma_i = \sum_s u_i(s_i, s_{-i})\sigma_i(s_i)q(s_{-i}) \geq \sum_s u_i(s_i, s_{-i})\sigma_i'(s_i)q(s_{-i}) \geq$ minimum expected utility of σ_i'. Therefore $\sum_s u_i(s_i, s_{-i})(\alpha\sigma_i(s_i) + (1-\alpha)\sigma_i'(s_i))q(s_{-i}) \leq \sum_s u_i(s_i, s_{-i})\sigma_i(s_i)q(s_{-i}) =$ minimum expected utility of σ_i. This implies that $\sigma_i \succeq \alpha\sigma_i + (1-\alpha)\sigma_i'$ for all $\alpha \in (0,1)$.

(necessity) Assume that no mixture is strictly preferred and suppose, to the contrary, that for all $q \in B_i$ such that q minimizes the expected utility of σ_i it is true that $\sum_s u_i(s_i, s_{-i})\sigma_i(s_i)q(s_{-i}) < \sum_s u_i(s_i, s_{-i})\sigma_i'(s_i)q(s_{-i})$. Then for any such q, $\sum_s u_i(s_i, s_{-i})(\alpha\sigma_i(s_i) + (1-\alpha)\sigma_i'(s_i))q(s_{-i}) > \sum_s u_i(s_i, s_{-i})\sigma_i(s_i)q(s_{-i})$ for all $\alpha \in (0,1)$. Now consider any $q^* \in B_i$ that does not minimize the expected utility of σ_i. By uniform continuity, there exists a $\delta > 0$ such that if $\|q^* - q\| < \delta$ for a q which minimizes the

expected utility of σ_i, then $\sum_s u_i(s_i, s_{-i})\sigma_i(s_i)q^*(s_{-i}) < \sum_s u_i(s_i, s_{-i})\sigma_i'(s_i)q^*(s_{-i})$. For any such q^* (i.e. one within δ of a minimizer),
$\sum_s u_i(s_i, s_{-i})(\alpha\sigma_i(s_i) + (1-\alpha)\sigma_i'(s_i))q^*(s_{-i}) > \sum_s u_i(s_i, s_{-i})\sigma_i(s_i)q(s_{-i})$ for all $\alpha \in (0, 1)$.

By definition of a minimizer, there exists an $\epsilon > 0$ such that for any $q^* \in B_i$ such that $\|q^* - q\| \geq \delta$ for all q which minimize the expected utility of σ_i it is true that
$\sum_s u_i(s_i, s_{-i})\sigma_i(s_i)q^*(s_{-i}) > \sum_s u_i(s_i, s_{-i})\sigma_i(s_i)q(s_{-i}) + \epsilon$. Thus, for $\underline{\alpha}$ such that
$\underline{\alpha}\epsilon + (1-\underline{\alpha})(\min_{p\in B_i} \sum_s u_i(s_i, s_{-i})\sigma_i'(s_i)p(s_{-i}) - \sum_s u_i(s_i, s_{-i})\sigma_i(s_i)q(s_{-i})) = 0$, (which exists and is strictly less than one since the first term is positive and the second term is non-positive), it is true that for all $\alpha > \underline{\alpha}$, $\alpha\sigma_i + (1-\alpha)\sigma_i' \succ \sigma_i \succeq \sigma_i'$. This contradicts the assumption that no mixture of σ_i and σ_i' is strictly preferred to both strategies. This proves necessity. *QED*

In applying Theorem 3, it is often easier to check the sufficient condition mentioned above and given in the following corollary.

Corollary 3.1 *Fix a player i and two strategies σ_i and σ_i'. No mixture over σ_i and σ_i' will be strictly preferred to both σ_i and σ_i' if there exists some $q \in B_i$ such that q minimizes the expected utility of both σ_i and σ_i'.*

Proof: Assume without loss of generality that $\sigma_i \succeq \sigma_i'$. Such a q then satisfies the conditions of Theorem 3. *QED*

This sufficient condition becomes even easier to check in 2 x 2 games, as reference to particular beliefs B_i can be omitted.

Corollary 3.2 *Fix a player i in a 2 x 2 game. If there is a pure strategy of i's opponent which minimizes the payoff to both of i's pure strategies, then i will never strictly prefer a mixed strategy to both of i's pure strategies.*

Proof: Call the pure strategies of i's opponent a and b. Suppose that a minimizes the payoff to both of i's pure strategies. No matter what i's set of beliefs is, each of i's pure strategies will have its expected utility minimized by the distribution in the belief set which puts the most weight on a. Therefore the existence of a q satisfying the conditions in Corollary 3.1 is guaranteed for any belief set. *QED*

In the special case of $\sigma_i \sim \sigma_i'$, the sufficient condition of Corollary 3.1 is also necessary.

Corollary 3.3 *Fix a player i and two strategies σ_i and σ_i' such that $\sigma_i \sim \sigma_i'$. No mixture over σ_i and σ_i' will be strictly preferred to both σ_i and σ_i' if and only if there exists some*

$q \in B_i$ such that q minimizes the expected utility of both σ_i and σ_i'.

Proof: If $\sigma_i \sim \sigma_i'$ then the minimum expected utilities of σ_i and σ_i' must be equal. The only way to satisfy $\sum_s u_i(s_i, s_{-i})\sigma_i(s_i)q(s_{-i}) \geq \sum_s u_i(s_i, s_{-i})\sigma_i'(s_i)q(s_{-i})$ for a q which minimizes the left-hand side is to have the same q also minimize the right-hand side. *QED*

We can illustrate Theorem 3 (and Corollary 3.2 in particular) by again considering the game in figure 2. Suppose we modify this game by increasing player 1's payoff by one util when 2 plays L and increasing player 2's payoff by one util when one plays U. The modified game is as in figure 4.

	L	R
U	3,3	0,1
D	1,0	1,1

figure 4

Noting that each players' pure strategies now have their payoffs minimized by the distribution placing the most weight on R (or D), Corollary 3.2 tells us that a mixed strategy will never be strictly preferred. This contrasts with the earlier analysis of the game in figure 2, in which mixed strategies were strictly optimal for a wide range of beliefs. In comparing the two games, the reader can check that not only are the Nash equilibria unchanged, but each player's best response correspondence is unchanged as well.[12] However the equilibria with uncertainty aversion are affected.

What has happened, intuitively, is that the change in payoffs has turned a game in which mixing helped hedge against uncertainty into one where it cannot play that role. On a more formal level, these changes have no effect in the standard theory because the independence axiom requires that preference between two acts (strategies) be preserved when they are mixed with a common third act. In the setting of Theorem 1 however, the independence axiom need only hold for mixing with constant acts, whereas adding one to

[12] I use best response correspondence in the standard sense of a player's optimal strategy as a function of the opponents' strategies. An alternative notion of best response correspondence, defined as a player's optimal strategy as a function of that player's *beliefs* about the opponents' strategies, would, of course, give different correspondences for the two games.

player 1's payoff if 2 plays L, for example, is mixing the existing acts with a *non-constant* act. Thus such a transformation may change behavior.

The notion that this generalization of the Nash concept may allow for a natural way of refining predictions about the outcome of a game is another advantage of this approach. I view this equilibrium notion as allowing sharper prediction in the sense that it allows the use of information about players' beliefs in a way that the Nash concept does not. For example, in figure 3, if the players were known to be uncertainty averse and there was no compelling unmodelled feature of their environment which would lead each to include the mixed Nash strategy in their beliefs but not include any distribution which puts more weight on R (or D) (than the mixed Nash strategy), I would be very reluctant to predict the mixed Nash equilibrium as the outcome of the game. Furthermore, in situations where the players are likely experiencing substantial uncertainty about others' play (for example, if they have never previously met their opponent and have not played the game before), I would be tempted to predict (D,R) as the outcome of the game in figure 3. The reasoning behind this is that greater uncertainty will be reflected in a larger set of beliefs, and thus (D, R) becomes more likely in the sense that if any belief assigns Prob(D) (or R) $> 1/8$, the player's best response switches to R (or D).[13] Thus a compelling feature of equilibria with uncertainty aversion is that "comparative statics" in uncertainty becomes possible in a well-defined sense.[14]

The set of equilibria with uncertainty aversion has been contained in the set of

[13] There are other reasons why (D, R) is an attractive prediction in this game. Both the risk-dominance criterion of Harsanyi and Selten (1988) and Aumann's (1990) argument that pre-play communication is not likely to assist in coordination on (U, L) also lead to a prediction of (D, R). Note that the notion of risk-dominance in 2x2 games shares some of the flavor of uncertainty aversion but differs in important ways. Risk-dominance always produces a unique prediction, while equilibria with uncertainty aversion depend on players beliefs and uncertainty aversion. Furthermore, although figure 3 and heuristic considerations might lead one to think that the risk-dominant equilibrium is always the same as the equilibrium with uncertainty aversion when there is maximal uncertainty (or ignorance), this is not true. In figure 2, (U, L) is risk-dominant while the mixed strategy pair is picked out under ignorance and uncertainty aversion.

[14] This aspect of the theory could conceivably be tested in an experimental setting. After assessing subjects' utility functions (using objective probabilities) and using examples like those of Ellsberg to detect aversion to uncertainty, the experimenter would have the subjects play simple games. The level of uncertainty in their beliefs about their opponent could be manipulated by, say, providing or not providing a record of past games the opponent played; allowing or not allowing pre-play discussion or face-to-face contact etc. Subjects might also be asked to explicitly describe (ex-ante or ex-post) their beliefs about their opponent's play.

rationalizable[15] outcomes in the examples we have seen so far. This is not necessarily the case. Consider the game in figure 5.

	L	R
U	3,0	1,2
D	0,4	0,-100

figure 5

In this game the unique Nash equilibrium is (U, R). This outcome can also be found by iterated strict dominance and is thus the unique rationalizable outcome as well. However, if any of player 2's subjective probability measures assigns weight at least 1/53 to D, then (U, L) will be an equilibrium with uncertainty aversion. In fact, letting 2's payoff from (D, R) approach $-\infty$, 2 will have to put an arbitrarily high minimum probability on U to be willing to play R.

This example makes several important points: (1) the set of equilibrium outcomes with uncertainty aversion is not in general contained in the set of rationalizable outcomes; (2) as Fudenberg and Tirole (1991, chapter 1) discuss, predicting (U,R) in a game like figure 5 relies crucially on the assumption that it is common knowledge that dominated (or non-rationalizable) strategies will never be used; and (3) to the extent that this common knowledge assumption is appropriate, the concept of equilibrium with uncertainty aversion may be too weak. In the next section, I pursue this line of reasoning by proposing a refinement of equilibrium with uncertainty aversion.

4 Adding Common Knowledge of Rationality

Consider the following definition that is motivated by the concept of correlated rationalizability (Pearce 1984, Brandenburger and Dekel 1987, Fudenberg and Tirole 1991, chapter 2). It is a natural generalization to the context where players can be described as

[15] For a definition of rationalizability see Bernheim (1984) and Pearce (1984) who introduced the concept, or Fudenberg and Tirole (1991), chapter 2.

in Theorem 1. The idea is to start from the whole set of strategies and eliminate, in each round of iteration, those strategies which are never a best response in the sense of Theorem 1 when the set of beliefs B_i is restricted to those beliefs which are compatible with the knowledge that other players only play best responses to the restricted sets of beliefs derived in the previous round. Thus, in the first round of iteration, those strategies which are never best responses to any beliefs are eliminated. In the second round, any strategies from the remaining set that are never best responses to any beliefs concentrated on that remaining set are eliminated, and so on. The successive rounds of iteration capture successive layers of knowledge of the rationality (in the sense of Theorem 1) of the players. Assume that all payoffs are common knowledge. Then the first iteration corresponds to the assumption that each player is rational. The second iteration corresponds to the assumption that each player is rational and knows that the other players are rational. The nth iteration corresponds to the assumption that each player is rational and knows that the other players know that the players know ... that the players are rational, where n-1 levels of knowledge are assumed.

Definition: Set $\Sigma_i^0 = \Sigma_i$, $P_{-i}^0 =$ the set of probability measures on $\times_{k \neq i} S_k$ such that for each $k \neq i$ the marginal distribution over S_k is an element of Σ_i. Recursively define for each integer $m > 0$:

$\Sigma_i^m = \{\sigma_i \in \Sigma_i^{m-1}$ such that there exists a closed, convex subset, B_i, of P_{-i}^{m-1} such that σ_i satisfies condition (1) in the definition of equilibrium with uncertainty aversion with Σ_i^{m-1} replacing $\Sigma_i.\}$, and

$P_{-i}^m =$ the set of probability measures on $\times_{k \neq i} S_k$ such that for each $k \neq i$ the marginal distribution over S_k is an element of the convex hull of Σ_k^m.

The *uncertainty aversion rationalizable strategies* for player i are $R_i = \bigcap_{m=0}^{\infty} \Sigma_i^m$.

The *uncertainty aversion rationalizable belief set* for player i is $Q_i = \bigcap_{m=0}^{\infty} P_{-i}^m$.

An alternate and often more useful characterization can be given in terms of iterated deletion of dominated strategies.[16]

Theorem 4 *In finite normal form games the uncertainty aversion rationalizable strategies*

[16] It is important that we are considering the mixed strategy space. Epstein (1995) considers rationalizability for G-S preferences restricted to pure strategies and finds non-equivalence with the expected utility case.

for player i, R_i, are exactly those strategies for player i which survive iterated deletion of strictly dominated strategies, (denoted by I_i).

Proof: The definition of uncertainty aversion rationalizable strategies is equivalent to that of correlated rationalizable strategies when the set B_i is restricted to be a singleton. Since the set of correlated rationalizable strategies for player i is identical to the set of strategies for player i which survive iterated strict dominance (see Fudenberg and Tirole's (1991, chapter 2) modification of a proof by Pearce (1984)), R_i is a superset of I_i. As no strictly dominated strategy is a best response in the sense of condition (1) of the definition of equilibrium with uncertainty aversion, R_i is a subset of I_i. *QED*

Interpreting Q_i as the beliefs which are not ruled out by common knowledge of procedural rationality (i.e. maximization given beliefs) when preferences are restricted to obey the axioms in Theorem 1, a refinement of equilibrium with uncertainty aversion is offered.

Definition: An equilibrium with uncertainty aversion is an *equilibrium with uncertainty aversion and rationalizable beliefs* if and only if B_i is a subset of Q_i for all players i.

Note that equilibrium with uncertainty aversion and rationalizable beliefs can be viewed as a refinement of correlated rationalizability (and thus, using a result of Brandenburger and Dekel (1987), of a posteriori equilibria) in that it takes the rationalizability restrictions and adds to them a consistency requirement (condition (2) in the definition of equilibrium with uncertainty aversion). Note that correlated rationalizability already requires a condition equivalent to (2) in the case $B_i = Q_i$. Imposing the consistency condition for beliefs which are subsets of Q_i allows for knowledge about the other players, besides knowledge of their rationality, to be reflected in beliefs, and thus in the equilibrium. Condition (2) is an appropriate consistency condition for equilibrium in the sense that it requires that players not rule out strategies incorrectly. The basic idea is that the Nash consistency condition makes sense if you are sure of the distribution over strategies (i.e. B_i is a singleton), but the idea of not being surprised (i.e. not ruling out the strategy profile that is played) is more general than this, in that knowledge that rules out some, but not all, other options can be incorporated. For example, suppose I am a baseball player and I know that the opposing pitcher does not know how to throw a split-fingered fastball. Any outcome in which the pitcher does, in fact, throw this pitch is surely not much of an equilibrium. On the other hand, I may be unable or unwilling to summarize my beliefs in the form of a single distribution over the remaining pitches. Thus, a slider or a curveball or any randomization between the two might not surprise me, and could be part of what might be reasonably called an equilibrium.

To see that the set of equilibria with uncertainty aversion and rationalizable beliefs can be strictly smaller than the set of rationalizable outcomes, consider the "battle-of-the-sexes" game depicted in figure 6.

	U	D
U	2,1	0,0
D	0,0	1,2

figure 6

In this game, (U, D) is rationalizable but is not an equilibrium with uncertainty aversion and rationalizable beliefs.[17] To see this, observe that player 1 plays U only if he has no subjective beliefs which assign weight less than 1/3 to 2 playing U. Similarly, 2 plays D only if she has no subjective beliefs which assign weight less than 1/3 to 1 playing D. These beliefs fail the consistency condition (2). Thus this condition shares some of the flavor of Rabin's (1989) point that we might not want to assign an outcome a higher probability then either of the players could given that they are playing best responses. Another example where the set of equilibria with uncertainty aversion and rationalizable beliefs is strictly larger than the set of Nash equilibria is given in figure 7.

This game is a modification of the "battle-of-the-sexes" game which makes D more attractive to 1 and U more attractive to 2 than before. (D, U) is an equilibrium with uncertainty aversion and rationalizable beliefs. Any sets of beliefs that include any measures which put weight greater than 1/2 on 2 playing D will lead player 1 to play D. Similarly, if player 2 has any measures which assign probability greater than 1/2 to 1 playing U then 2 will play U. However, (D, U) is not a Nash equilibrium.[18] In fact, if we

[17]In fact, (U, D) is not even an equilibrium with uncertainty aversion. This, together with the example in figure 5, makes it clear that there is no general containment relation between the set of rationalizable (or correlated rationalizable) outcomes and the set of outcomes of equilibria with uncertainty aversion.

[18]Note that (U, D), as in figure 6, is rationalizable but is not an equilibrium with uncertainty aversion

	U	D
U	2,1	0,0
D	1,1	1,2

figure 7

	U	D
U	1,1	0,1
D	0,0	0,2

figure 8

replace the payoff of (1, 1) with a payoff of (k, k) where $0 < k < 2$, (D, U) fails to be Nash but is an equilibrium with uncertainty aversion and rationalizable beliefs for an ever wider class of beliefs as k approaches 2. Of course the mixed Nash equilibrium does approach (D, U) as k approaches 2, but it seems that allowing for a wider range of beliefs is much more helpful in assessing which outcomes would be expected in which environments. From the point of view of equilibrium with uncertainty aversion, the mixed strategy Nash outcome for $0 < k < 2$ will never be strictly preferred.

5 Weak Admissibility

Consider the game in figure 8.

In this game no strategies are eliminated by iterated strict dominance, thus the restriction to rationalizable beliefs makes no difference. There are lots of Nash equilibria (a continuum in fact). Thus there are also many equilibria with uncertainty aversion. Notice,

and rationalizable beliefs. Thus the game in figure 7 demonstrates that the set of equilibria with uncertainty aversion and rationalizable beliefs can lie strictly between the set of rationalizable profiles and the set of Nash equilibria.

however, that as long as player 1 thinks that U is possible, player 1 should play U in response. Similarly, as long as player 2 thinks that D is possible, player 2 should play D in response. This reasoning leads one to think that in any equilibrium with uncertainty aversion where 1 plays D (or 2 plays U) all beliefs in the belief set must assign probability 0 to 2 playing U (1 playing D). That this is not true is easily seen by considering the case where both players have belief sets which contain all possible distributions. In this case, each player is indifferent between any two strategies since all strategies give a minimum expected utility of 0. This is one aspect in which I feel that the Gilboa-Schmeidler axioms are too weak.

A similar point can be made by reconsidering the Ellsberg example. Consider again the thought experiment of section I, specifically options 1 and 4. It would seem irrefutable that unless a decision maker is certain that yellow will not be drawn she should prefer 4 to 1. However if the set of measures C simply includes a measure which assigns zero weight to yellow, even if other measures in C do not, then a Gilboa-Schmeidler decision maker will be indifferent between 1 and 4 (assuming that all measures in C assign one-third to black). Under our interpretation of C, such a decision maker considers it possible that yellow may occur in the sense that she is willing to use a measure which implies that in evaluating some acts. The fact that such a decision maker is indifferent is therefore unreasonable. To remedy this I use an additional axiom that appears in Schmeidler (1989).

Definition: An event $E \in \Sigma$ is *null* if and only if $\forall f, g \in L$ such that $\forall s \in S/E, f(s) \sim g(s)$, it is true that $f \sim g$.

Definition: Denote the *set of non-null events* by NNE $= \{E \in \Sigma$ such that E not null$\}$.

B.1 (Weak Admissibility)

$\forall f, g \in L$, if for all $s \in S, f(s) \succeq g(s)$ then $f \succeq g$ and [$f \succ g$ if an only if for some $E \in$ NNE, $f(s) \succ g(s), \forall s \in E$].

Intuitively, a null event is a set of states which is never decisive. To be null in the context of the Gilboa-Schmeidler theory, an event must never be assigned positive probability by any measure in C. This can be seen by considering two acts, one of which gives utility 100 if E occurs while the other gives utility 0 if E occurs and both of which give utility 200 if E does not occur. The distribution in C used to evaluate each of these acts will be the one(s) which puts the most weight on E. Thus E has probability zero according to all measures in C if and only if the decision maker is indifferent between the two acts. So we conclude that a null event must be assigned probability zero by all probability measures in C. Furthermore, any event which is assigned zero probability by all measures in C is a null event. Weak admissibility says that state-by-state weak dominance (and indifference) holds on the set of events which are given positive probability by some measure in C. We obtain the following representation theorem:

Theorem 5 *Let $\succeq$ be a binary relation on L_0. Then the following are equivalent,*

(1) $\succeq$ satisfies A.1 - A.3, A.5 and B.1 for $L = L_0$.

(2) There exists an affine function $u : Y \to \mathcal{R}$ and a non-empty, closed, convex set C of finitely additive probability measures on Σ satisfying [$p(E) = 0$ if and only if $\forall p \in C, p(E) = 0$] such that $\forall f, g \in L_0$, $f \succeq g$ if and only if $\min_{p \in C} \int u \circ f dp \geq \min_{p \in C} \int u \circ g dp$.

Furthermore, the function u is unique up to a positive affine transformation and, if and only if A.6 holds, the set C is unique.

Proof: See Appendix.

The new representation is identical to that in Theorem 1 except for the additional condition that each event be given either zero probability by all measures in C or positive probability by all measures in C (i.e. the measures in C are mutually absolutely continuous). This condition serves to impose the weak admissibility axiom (B.1). However, this requirement seems too strong. It does not allow a decision maker to be uncertain about whether a given event will occur with positive probability. In a two player game, for instance, this representation would not allow a player to include both a pure strategy and any other strategy (mixed or pure) in her belief set. In order to permit this type of uncertainty while maintaining weak admissibility, A.3 (continuity) will be relaxed. The intuitive idea is that weak admissibility is a second-order criterion, in the sense that A.4 (monotonicity) ensures that weak admissibility is only used to break ties in the original representation, thus engendering a possibly discontinuous preference relation. Unfortunately, simply dropping continuity only when applying weak admissibility directly to break ties will not allow us to maintain A.1 (weak order), which is in many ways the most fundamental axiom. To get around this problem, we allow for a finite number of hierarchically ordered preference relations, while placing conditions on these relations.

Consider a finite set of preference relations on L: $\succeq_i, i = 1, \dots, N$.

Definition: The $\succeq_i$ *agree on* L_c if and only if $\forall y, z \in L_c$, $[y \succeq_1 z$ if and only if $y \succeq_2 z \dots$ if and only if $y \succeq_N z]$.

Definition: The $\succeq_i$ *display non-increasing valuation of certainty* if and only if $\forall f \in L, y \in L_c, [f \sim_i y$ implies $y \preceq_{i+1} f]$ holds for $i = 1, \dots, N-1$.

Now consider the following axiom on the preference relation $\succeq$ on L:

B.2 (N-Hierarchy)

There exist $N \geq 1$ preference relations on L: $\succeq_1, \succeq_2, \dots, \succeq_N$ such that, $\forall f, g \in L, f \succeq g \Leftrightarrow [g \succ_i f \Rightarrow \exists k < i$, such that $f \succ_k g]$. Furthermore, each $\succeq_i$ satisfies A.1-A.5, and the $\succeq_i$ agree on L_c and display non-increasing valuation of certainty.

Observe that any preference relation $\succeq$ which satisfies A.1-A.5 will satisfy B.2 for $N = 1$. Thus imposing A.1, A.2, A.4, A.5, B.1, and B.2 is certainly no stronger than imposing A.1-A.5, and B.1. B.2 limits the way in which continuity can be relaxed. It says that there are a finite number of preference relations which are combined lexicographically to represent $\succeq$.[19] Furthermore, each of these N relations must satisfy the original axioms A.1-A.5, must order constant acts the same way, and reward constant acts versus uncertain ones (weakly) less and less. Thus the decision maker has first-order G-S preferences, second-order G-S preferences, etc., and aversion to uncertainty is not as important in breaking ties as it is in the ordering where the ties occur. One can think of the decision maker "accounting for" uncertainty in the manner of Theorem 1 with her first-order preferences, and, given that prospects are equal by this measure, being willing to venture a tie-breaking decision on the basis of preferences which do not give as much weight to uncertainty, since this weight has, in some sense, already been given. This type of refinement could continue through several levels.

An alternate scenario would be to think that instead of reducing uncertainty aversion at each stage, the decision maker actually became more uncertainty averse in the case of ties. An important drawback to this case, however, is that weak admissibility would end up imposing precisely the conditions which we wanted to avoid in Theorem 5. For this reason, I work with the former case.

[19] The only restriction beyond weak order in this requirement is that N be finite. See Fishburn (1974) and Chipman (1971) for more details.

We obtain the following representation theorem:

Theorem 6 *Let $\succeq$ be a binary relation on L_0. Then the following are equivalent,*

(1) $\succeq$ satisfies B.1 and B.2 for $L = L_0$.

(2) $\exists$ an affine function $u : Y \to \mathcal{R}$ and $N \geq 1$ non-empty, closed, convex sets $C_i, i = 1, \ldots, N$, of finitely additive probability measures on Σ such that $\forall f, g \in L_0, f \succeq g$ if and only if $(\min_{p \in C_i} \int u \circ f dp)_{i=1}^{N} \geq_L (\min_{p \in C_i} \int u \circ g dp)_{i=1}^{N}$, where if $p(E) > 0$ for some $E \in \Sigma$, $p \in C_1$ then there exists an i such that $p(E) > 0$, for all $p \in C_i$, and where $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_N$.[20]

Furthermore, the function u is unique up to a positive affine transformation, and, if and only if A.6 holds, the set C_1 is unique.[21]

Proof: See Appendix.

The following corollary makes it clear that A.3 (continuity) is the only one of the G-S axioms which is being relaxed:

Corollary 6.1

$\succeq$ satisfies B.1 and B.2 implies $\succeq$ satisfies A.1, A.2, A.4, and A.5.

Proof: It is straightforward to verify that the representation in Theorem 6 satisfies A.1, A.2, A.4, and A.5. *QED*

This representation satisfies weak admissibility, while also allowing the set of possible probability measures, C_1, to include both measures that assign zero probability to an event and ones that give the event positive weight. An interpretation of the subsets C_2 through C_N is that the measures in C_k are considered infinitesimally more likely (or more important in terms of the decision) than the measures in C_{k-1}/C_k in the sense that if two

[20]For $a, b \in \mathcal{R}^{\mathcal{N}}$, $a \geq_L b \Leftrightarrow [b_i > a_i \Rightarrow \exists k < i$ such that $a_k > b_k]$.

[21]In a context where the independence axiom is assumed to hold for all acts (and thus uncertainty aversion is ruled out), Blume, Brandenburger, and Dekel (1991) obtain a similar lexicographic representation where the belief sets are singletons, $N \leq \#S$, and the superset relations are not required to hold. The added structure provided by independence allows a more attractive axiomatization than the one here, obviating the need to refer to a hierarchy of preference relations in the axioms. Unfortunately the properties of an ordered vector space which they use do not seem applicable here.

acts are equally ranked using C_{k-1} then the decision maker will use the ranking under C_k to attempt to further discriminate, but if two acts are strictly ranked under C_{k-1} then the ranking under C_k is irrelevant. Viewed in this way, weak admissibility requires only that any event which is given positive weight by some measure in C_1 be considered at least infinitesimally more likely to occur with positive probability than to occur with zero probability.

The definition of equilibrium with uncertainty aversion can be extended to the preferences described in Theorem 6.

Definition: An *equilibrium with uncertainty aversion* of G is a $(N+1)*I$-vector $(\sigma_1, \ldots, \sigma_I, B_{11}, \ldots, B_{1N}, B_{21}, \ldots, B_{2N}, \ldots, B_{I1}, \ldots, B_{IN})$ where $\sigma_i \in \Sigma_i$ (the set of mixed strategies for player i, i.e. the set of probability distributions over S_i) and the B_{in} are closed, convex subsets of P_{-i} (the set of probability distributions over $\times_{k\neq i} S_k$) satisfying $B_{i1} \supseteq B_{i2} \supseteq \ldots \supseteq B_{iN}$ and $[p(s_{-i}) > 0$ for some $p \in B_{i1} \Rightarrow p(s_{-i}) > 0$ for all $p \in B_{in}$ for some $n]$ such that, for all i,

(1) σ_i satisfies
$(\min_{p \in B_i} \sum_s u_i(s_i, s_{-i})\sigma_i(s_i)p(s_{-i}))_{n=1}^N \geq_L (\min_{p \in Bi} \sum_s u_i(s_i, s_{-i})\sigma_i'(s_i)p(s_{-i}))_{n=1}^N$ for all $\sigma_i' \in \Sigma_i$, and

(2) $\prod_{k\neq i} \sigma_k(s_k) \in B_{i1}$.

Using this definition, analogues of all of the theorems in sections 1.3 and 1.4 can be derived, although the results are not as clean as with the simpler definition. To apply the new definition, we return to the game in figure 8.

Recall that without weak admissibility, there was a great multiplicity in the equilibria with uncertainty aversion (with or without rationalizable beliefs) even when no strategy of the opponent was ruled out by all measures in the belief set. However, with this restriction, unless all of player 1's beliefs (the set B_{i1}) assign probability zero to U, 1 should play U. Similarly, unless all of player 2's beliefs assign probability zero to D, 2 should play D. Thus, if players are uncertainty averse and any degree of uncertainty exists in each of their minds, weak admissibility argues that (U, D) will be the outcome.

Note that (U, D) is also the outcome picked out by deletion of weakly dominated strategies. In general, however, weak admissibility is a much weaker condition than weak dominance. Weak admissibility allows the play of weakly dominated strategies when a player always assigns probability zero to the state(s) where the dominance is strict. The power of weak dominance in the Nash framework is precisely (and only) that it rules out the play of weakly dominated strategies even when the relevant states are assigned probability zero. I believe that weak admissibility is a more accurate formalization of the ideas which are often used to motivate weak dominance. If the reason that weakly dominated strategies should not be played is that players will almost never be in a situation where they can be sure that their opponent(s) will not play a particular strategy or strategies then that idea should be expressed directly, in terms of beliefs, rather than in a rule which is to be universally applied regardless of the beliefs in any particular situation. Weak admissibility makes clear this dependence on beliefs. For example, if only rationalizable beliefs are allowed, then strategies which would have been eliminated by weak dominance are allowed if they were strictly dominated only by those actions which rationalizable beliefs must assign probability zero. For example, consider the game in figure 9.

	X	Y	Z
A	1,1	0,1	1,2
B	0,0	0,2	1,1

figure 9

In this game, weak dominance eliminates B for player 1, whereas, since X is eliminated by iterated strict dominance, B is not eliminated by weak admissibility under the restriction to rationalizable beliefs.

6 Related Literature

Two closely related papers are Dow and Werlang (1994) and Lo (1995a) which I discuss in turn:

An alternative notion of equilibrium when players are uncertainty averse has been proposed by Dow and Werlang (1994). In the context of the G-S multiple priors model their *Nash Equilibrium under Uncertainty* is defined as follows.[22]

Definition: A *Nash Equilibrium under Uncertainty* is a pair (B_1, B_2), where B_i is a closed, convex subset of P_{-i} (the set of probability distributions over S_{-i}) such that, for $i = 1, 2$, there exists a set of pure strategies $A_i \subseteq S_i$ satisfying

(1) Each $s_i \in A_i$ satisfies $\min_{p \in B_i} \sum_s u_i(s_i, s_{-i}) p(s_{-i}) \geq \min_{p \in Bi} \sum_s u_i(s_i', s_{-i}) p(s_{-i})$ for all $s_i' \in S_i$,

(2) $p_{-i}(A_i) = 1$ for *at least one* $p_{-i} \in B_{-i}$ and,

(3) For all $C_i \subset A_i$, $p_{-i}(C_i) < 1$ for *all* $p_{-i} \in B_{-i}$.[23]

As the definition specifies beliefs, not behavior, one should interpret any play from the sets A_i as consistent with the equilibrium. Observe that condition (1) requires only that players optimize over the set of available *pure strategies*. As we have seen above, for G-S decision makers a mixed strategy may be a strictly better response to beliefs than any pure strategy. Since they consider only pure strategies, Dow and Werlang naturally do not consider the issues involving mixed strategies taken up in section 3.

Condition (2) is a consistency condition on beliefs. In the context of the restriction to pure strategies, it may be interpreted as saying that at least one of the measures in the set representing beliefs puts full weight on the pure strategies of the opponent that will be played in equilibrium. This is similar in spirit to the consistency condition offered in this paper for mixed strategies. Note that Dow and Werlang's definition is formulated for two player games only. N-player extensions and refinements have subsequently been proposed by Eichberger and Kelsey (1995) and Marinacci (1996).[24]

The two concepts have similar implications in some games, for instance, Dow and Werlang's Example 1, showing that non-rationalizable outcomes can occur with under their

[22]Dow and Werlang's definition is given in terms of the non-additive probability model of Schmeidler (1989). Under the assumption of uncertainty aversion, that model is nested in the G-S multiple priors model used in this paper. Thus the definition as stated here technically applies to a larger class of decision makers than the original.

[23]Note that, as stated, condition (3) is not needed in the definition. If there exists a set satisfying (1) and (2), then there also exists a subset of that set satisfying (1), (2), and (3). Nonetheless, to move from a specification of equilibrium to a specification of behavior, the identity, and not simply the existence, of such sets A_i matter and (3) imposes a restriction on these sets.

[24]Other refinements, with a focus on lack of knowledge of rationality, are offered in Mukerji (1994) and Lo (1995b).

solution concept, is similar to the example in figure 5 above. However, they focus on breaking down backward induction in a finitely repeated Prisoner's Dilemma game, while the focus here has been on mixed strategies, rationalizability, and admissibility in one-shot games.

A second paper closely related to this one is by Lo (1995a). He works in the G-S multiple priors framework and offers solution concepts which are refinements of the ones proposed here. Specifically,

Definition: A *Beliefs Equilibrium* is an I-vector $(B_1, B_2, \ldots, B_I)$ where the B_i are nonempty, closed, convex subsets of P_{-i} such that, for all $i, j \neq i$,

(1) For all $p \in B_j$, if σ_i is the marginal probability measure on S_i generated by p, then σ_i satisfies $\min_{p \in B_i} \sum_s u_i(s_i, s_{-i})\sigma_i(s_i)p(s_{-i}) \geq \min_{p \in Bi} \sum_s u_i(s_i, s_{-i})\sigma_i'(s_i)p(s_{-i})$ for all $\sigma_i' \in \Sigma_i$.

In other words, *every* measure in each player's set of beliefs must generate marginals that are (mixed strategy) best responses for the other players. Intuitively, this corresponds to a situation in which the players know each others beliefs and also know that each player is rational in the sense of a G-S maximizer. All there is left to be uncertain about is *which* best response each player is using (and possibly the correlation, if any, among players' strategies). These strong informational requirements are imposed by Lo to stay as close to Nash equilibrium as possible, thus reflecting "... solely the effects of uncertainty aversion."

In contrast, as Lo shows formally, the solution concepts proposed here require essentially no knowledge of other players (or, as in the case of *equilibria with uncertainty aversion and rationalizable beliefs* only common knowledge of rationality). The key word here is *require*. As was noted in section 3, *equilibria with uncertainty aversion* span the continuum between maxmin behavior and Nash equilibrium. Thus they are naturally compatible with levels of knowledge ranging from complete ignorance (maxmin) to common knowledge of beliefs and/or rationality. I view this as an advantage of the present theory, as it allows great flexibility in the type of knowledge and beliefs that can be assumed/represented. Due to its more stringent requirements, it is easy to see that many of the phenomena discussed above will not occur under *Beliefs Equilibria*. For example, whenever a player has a strict best response to her beliefs, the other players must act as if they know this response in any *Beliefs Equilibrium*.

7 Conclusion

The goal of this paper has been to explore some of the consequences for game theory of an attractive broadening in the decision theory used to describe the players which also allows a weakening of the strong consistency requirements of Nash equilibrium. The concept of equilibrium with uncertainty aversion with or without a restriction to rationalizable beliefs turns out to have some nice features in normal form games. First, it provides a new justification based on reducing uncertainty for some equilibria involving mixed strategies. Second, the flexibility of belief structure points out certain equilibria which I argue would not make very good predictions unless very definite information about beliefs were available. Third, this framework allows for oft-mentioned unmodelled features of the game environment, such as social norms, past experience of the players, and knowledge of equilibrium concepts to be incorporated in a natural way through their effects on the uncertainty which uncertainty averse players experience. When rationalizable beliefs are imposed, this solution concept can be viewed as a refinement of correlated rationalizability which is not as restrictive as Nash equilibrium. Finally, the flexibility of beliefs helps make weak admissibility a relevant condition.

There is obviously much that needs to be done if these ideas are to form the basis of a complete theory. The biggest missing piece is an extension of these concepts to extensive form, and thus dynamic, games. One route to follow here is to develop a satisfactory notion of updating the sets of probability measures. Gilboa and Schmeidler (1993) have done some preliminary work on this front. One procedure which they suggest which seems potentially appealing is, after an event occurs, to rule out some of the measures and update the rest by applying Bayes' rule to each one. However, it is known (see Epstein and Le Breton (1993)) that no update rule for sets of measures in the G-S framework guarantees dynamically consistent preferences. Klibanoff (1995) responds to this by axiomatizing an alternative, explicitly dynamically consistent, representation of uncertainty aversion. Using such a theory to analyze dynamic games is a topic of future research.

Another thing missing from the present work is a discussion of games of incomplete information. However, it should not be difficult to apply a slightly adapted version of the present theory to such games. The basic change would involve an enlargement of the state space of player i, S_{-i}, to $S_{-i} \times \Theta$ where Θ is the space of unknown parameters.[25] On a

[25]Recently, Epstein and Wang (1995) have provided a formal framework to justify a "type-space" approach

more applied level, it would be good to develop a full-blown application using these equilibrium concepts.

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with non-Bayesian preferences.

A G-S Axioms

Define F_c to be the set of constant acts (i.e. acts that yield the same lottery y in each state of the world. Note that constant acts involve no uncertainty. Consider the following axioms on the preference relation $\succeq$ on F:

A.1 (Weak Order) (a) $\forall f, g \in F, f \succeq g$ or $g \succeq f$ or both.

(b) $\forall f, g, h \in F, \{f \succeq g$ and $g \succeq h\} \Rightarrow f \succeq h$.

A.2 (Certainty Independence)

$\forall f, g \in F$ and $h \in F_c$ and $\alpha \in (0,1)$, $f \succ g \Leftrightarrow \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$.

A.3 (Continuity)

$\forall f, g, h \in F$, if $f \succ g$ and $g \succ h$ then $\exists \alpha, \beta \in (0,1)$ such that

$\alpha f + (1-\alpha)h \succ g$ and $g \succ \beta f + (1-\beta)h$.

A.4 (Monotonicity)

$\forall f, g \in F$, if $f(s) \succeq g(s)$ on S then $f \succeq g$.

A.5 (Uncertainty Aversion)

$\forall f, g \in F$ and $\alpha \in (0,1)$, if $f \sim g$ then $\alpha f + (1-\alpha)g \succeq f$.

A.6 (Non-degeneracy)

Not for all $f, g \in F, f \succeq g$.

The only non-standard axioms are Certainty Independence and Uncertainty Aversion. Note that Certainty Independence is a strict weakening of the traditional Independence axiom when applied to the lottery-acts framework, as it requires that strict preference be preserved only under mixtures with constant acts.

B Proof of Theorems 5 and 6

Theorem 6 will be proved first, as it will be used to prove Theorem 5.

Theorem 6 *Let $\succeq$ be a binary relation on L_0. Then the following are equivalent,* *(1) $\succeq$ satisfies B.1 and B.2 for $L = L_0$.*

(2) $\exists$ an affine function $u : Y \to \mathcal{R}$ and $N \geq 1$ non-empty, closed, convex sets $C_i, i = 1, \ldots, N,$ of finitely additive probability measures on Σ such that $\forall f, g \in L_0, f \succeq g$ if and only if $(\min_{p \in C_i} \int u \circ f dp)_{i=1}^N \geq_L (\min_{p \in C_i} \int u \circ g dp)_{i=1}^N$, where if $p(E) > 0$ for some $E \in \Sigma$, $p \in C_1$ then there exists an i such that $p(E) > 0$, for all $p \in C_i$, and where $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_N$.[26]

Furthermore, the function u is unique up to a positive affine transformation, and, if and only if A.6 holds, the set C_1 is unique.

Proof of Theorem 6:

We will first prove that (1) implies (2), then that the uniqueness properties of the representation in (2) are satisfied, and finally that (2) implies (1). The proof of (1) implies (2) is the most involved. We will use theorem 1 applied to each $\succeq_i$ and B.2 to derive the basic form of the representation. Then, to show that the superset relations between the sets of beliefs hold, we will appeal to a construction of suitable sets C_i in Chateauneuf (1991). Finally we use a lemma and B.1 to show that the measures in the C_i must satisfy the conditions stated in (2).

(1) $\Rightarrow$ (2): From B.2 we know that the representation is lexicographic in the $\succeq_i$. Applying theorem 1 to each $\succeq_i$ we have that $f \succeq_i g$ if and only if $\min_{p \in C_i} \int u_i \circ f dp \geq \min_{p \in C_i} \int u_i \circ g dp$, for a non-empty, closed, convex set C_i and an affine $u_i : Y \to \mathcal{R}$ which is unique up to a positive affine transformation. As B.2 requires all $\succeq_i$ to agree on constant acts, we can take $u_i = u_1, i = 1, \ldots, N$. Thus we have the basic representation.

Now we prove the superset condition holds. Consider the space B of all bounded, Σ-measurable real functions on S. By Lemmas 3.1-3.4 of Gilboa and Schmeidler (1989), there exists, for each i, $I_i : B \to \mathcal{R}$ such that $I_i(u \circ y^*) = u(y)$ for $y^* \in L_c$ with outcome $y \in Y$; $f \succeq_i g$ if and only if $I_i(u \circ f) \geq I_i(u \circ g)$ for $f, g \in L_0$; I_i monotonic, superadditive, homogeneous of degree 1, and C-independent. Thus I_i satisfies the conditions of the

[26]For $a, b \in \mathcal{R}^{\mathcal{N}}$, $a \geq_L b \Leftrightarrow [b_i > a_i \Rightarrow \exists k < i$ such that $a_k > b_k]$. This is the reflexive relation induced by lexicographic ordering.

Fundamental lemma[27] in Chateauneuf (1991), and thus, by his constructive proof, we can take C_i to be the set $\{p|p$ is an additive probability measure on Σ; $\int bdp \geq I_i(b), \forall b \in B$ such that $I_i(b) > 0\}$. As B.2 requires $f \succeq_{i+1} y^*$ if $f \sim_i y^*$, $I_k(u \circ f) \geq I_i(u \circ f)$ if $k > i$. Thus for all $b \in B$, $I_k(b) \geq I_i(b)$ if $k > i$. From the definition of C_i, we see that this implies $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_N$. To complete the proof of (1) $\Rightarrow$ (2) we make use of the following result:

Lemma 6.1

An event $E \in NNE \Leftrightarrow p(E) > 0$ *for some* $p \in C_1$.

Proof of Lemma 6.1: $(\Rightarrow)$: $p(E) = 0, \forall p \in C_1$ implies $p(E) = 0, \forall p \in C_i$, which implies E null.

$(\Leftarrow)$: Consider $f, g \in L_0$ such that $u(f(s)) = u(g(s)) = k$ on S/E and $k > u(f(s)) > u(g(s))$ on E. $\min_{p \in C_i} \int u \circ f dp \neq \min_{p \in C_i} \int u \circ g dp$ if and only if $p(E) > 0$ for some $p \in C_i$. Thus $p(E) > 0$ for some $p \in C_1$ implies E not null. *QED*

For any $E \in NNE$, Lemma 6.1 tells us that $p(E) > 0$ for some $p \in C_1$. For any such E, consider f, g such that $u(f(s)) = u(g(s)) = k$ on S/E and $u(f(s)) > u(g(s)) > k$ on E. For each C_i, if there exists $p \in C_i$ such that $p(E) = 0$ then $\min_{p \in C_i} \int u \circ f dp = \min_{p \in C_i} \int u \circ g dp$. Since B.1 requires that $f \succ g$, there must be some $i \in \{1, \ldots, N\}$ such that $p(E) > 0$, for all $p \in C_i$.

Uniqueness: that u is unique up to a positive affine transformation follows directly from the vNM representation theorem (von Neumann and Morgenstern, 1947). If A.6 fails

[27] This lemma says that for $I : V \rightarrow \mathcal{R}$, where V is the set of all Σ-measurable functions from S to the positive reals, the following two conditions are equivalent:

Condition 1. I satisfies:

(i) for all $\alpha \geq 0, \beta \geq 0, x \in V : I(\alpha x + \beta 1^*) = \alpha I(x) + \beta$, where 1^* is a function which takes on the value 1 in all states.

(ii) $x, y \in V \Rightarrow I(x + y) \geq I(x) + I(y)$.

(iii) If $x \geq y$ on S, then $I(x) \geq I(y)$.

Condition 2.

There exists a unique closed, convex set C of additive probabilities on Σ, such that

(iv) $I(x) = \min_{p \in C} \int xdp$, for all $x \in V$.

To apply this lemma to I_i we can simply rescale u so that u takes on only positive values and consider the restriction of I_i to V. Monotonicity, superadditivity, homogeneity, and C-independence ensure Condition 1 is satisfied.

then any closed, convex set C_1 will do in combination with a constant u. Suppose A.6 holds. We adapt an argument of Gilboa-Schmeidler (1989) to our setting. Assume there exist $C_1' \neq C_1''$, non-empty, closed, and convex such that $(\min_{p \in C_i'} \int u \circ f dp)_{i=1}^N$, for some C_i', $i = 2, \ldots, N$ such that $C_1' \supseteq C_2' \supseteq \ldots \supseteq C_N'$ and $(\min_{p \in C_i''} \int u \circ f dp)_{i=1}^N$, for some C_i'', $i = 2, \ldots, N$ such that $C_1'' \supseteq C_2'' \supseteq \ldots \supseteq C_N''$ represent $\succeq$ on L_0 in the manner of the theorem. Without loss of generality, assume there exists $p' \in C_1'/C_1''$. By a separation theorem [Dunford and Schwartz, 1957, V.2.10], there exists $a \in B$ such that $\int a dp' < \min_{p \in C_1''} \int a dp$. Without loss of generality assume $a = u \circ f$ for some $f \in L_0$. Let $y \in Y$ be such that $u(y) = \min_{p \in C_1''} \int u \circ f dp$. Since $C_1'' \supseteq C_2'' \supseteq \ldots \supseteq C_N''$, this implies that $f \succeq y^*$ where y^* is the constant act which results in y. But $u(y) = \min_{p \in C_1''} \int u \circ f dp > \min_{p \in C_1'} \int u \circ f dp$, which implies $y^* \succ f$, a contradiction. Thus C_1 is unique if and only if A.6 holds.

(2) $\Rightarrow$ (1): We define $f \succeq_i g \Leftrightarrow \min_{p \in C_i} \int u \circ f dp \geq \min_{p \in C_i} \int u \circ g dp$. B.2 is then easily verified (recall that $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_N$). The fact that $p(E) > 0$ for some $E \in \Sigma$, $p \in C_1$ implies there is an i such that $p(E) > 0, \forall p \in C_i$, means that all non-null events are given positive weight in some element of $(\min_{p \in C_i} \int u \circ f dp)_{i=1}^N, \forall f \in L_0$. Suppose that $u(f(s)) \geq u(g(s)), \forall s \in S$. Since f and g are Σ-measurable, $u \circ f - u \circ g$ is Σ-measurable and thus $\{s : u(f(s)) - u(g(s)) > 0\} \in \Sigma$. $f \succ g$ if and only if $\{s : u(f(s)) - u(g(s)) > 0\}$ is not null. Therefore B.1 holds. *QED*

Theorem 5 *Let $\succeq$ be a binary relation on L_0. Then the following are equivalent,*

(1) $\succeq$ satisfies A.1 - A.3, A.5 and B.1 for $L = L_0$.

(2) There exists an affine function $u : Y \to \mathcal{R}$ and a non-empty, closed, convex set C of finitely additive probability measures on Σ satisfying $[p(E) = 0$ if and only if $\forall p \in C, p(E) = 0]$ such that $\forall f, g \in L_0, f \succeq g$ if and only if $\min_{p \in C} \int u \circ f dp \geq \min_{p \in C} \int u \circ g dp$.

Furthermore, the function u is unique up to a positive affine transformation and, if and only if A.6 holds, the set C is unique.

Proof of Theorem 5: First note that B.1 implies A.4 (Monotonicity).

(1) $\Rightarrow$ (2): A.1-A.5 imply B.2 with $N = 1$ by Theorem 1. B.1 and B.2 with $N = 1$ imply (2) by Theorem 6.

Uniqueness: Follows by the same arguments (vNM theorem, separation theorem) as uniqueness in Theorems 1 and 6.

(2) $\Rightarrow$ (1): (2) implies $\succeq$ satisfies A.1-A.5 by Theorem 1. (2) implies (by Lemma 6.1) that for all $p \in C$, $[p(E) > 0, \forall$ non-null $E \in \Sigma]$ which implies B.1 (weak admissibility) since $\{s : u(f(s)) - u(g(s)) > 0\}$ is Σ-measurable. *QED*

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